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One-Parameter Planar Motions in Affine Cayley-Klein Planes

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Abstract. In 1956, W. Blaschke and H.R. Müller introduced the one-parameter planar motions and obtained the relation between absolute, relative, sliding velocities and accelerations in the Euclidean plane $\mathbb{E}^2$ [3]. A. A. Ergin [4] considering the Lorentzian plane $\mathbb{L}^2$, instead of the Euclidean plane $\mathbb{E}^2$, introduced the one-parameter planar motions in the Lorentzian plane $\mathbb{L}^2$ and also gave the relations between the velocities and accelerations in 1991. In addition to this, in 2013, M. Akar and S. Yüce [1] introduced the one-parameter motions in the Galilean plane $\mathbb{G}^2$ and gave same concepts stated above. In this paper, we will introduce one parameter planar motions in affine Cayley-Klein (CK) planes $\mathbb{P}_\epsilon$ and we will discuss the relations between absolute, relative, sliding velocities and accelerations.

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1. Introduction

The geometrical systems have a significant role in plane geometries. Cayley-Klein (CK) geometries, first introduced by Klein in 1871 and Cayley, are number of geometries including Euclidean, Galilean, Minkowskian and Bolyai-Lobachevsikan [8, 9]. Following Cayley and Klein, Yaglom distinguished these geometries with choosing one of three ways of measuring length (parabolic, elliptic, or hyperbolic) between two points on a line and one of the three ways of measuring angles between two lines (parabolic, elliptic, or hyperbolic) [14]. This gives nine ways of measuring lengths and angles and thus the nine plane geometries listed in Table 1.

A great deal of studies are conducted in CK-planes [5–7, 10–13]. There is a known (but not well-known) relationship between the plane geometries which have parabolic measure of distance: Euclidean, Galilean and Minkowskian (Lorentz) geometries. They are called affine CK-plane geometries [14].

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Table 1: Nine CK-geometries in the Plane

		Measure of length between two points		
		Elliptic	***Parabolic***	***Hyperbolic***
Measure of angles between two lines	***Elliptic***	Elliptic Geometry	Euclidean Geometry	Hyperbolic Geometry
	Parabolic (Euclidean)	co-Euclidean Geometry (Anti-Newton Hooke)	Galilean Geometry	co-Minkowskian Geometry (Newton-Hooke)
	Hyperbolic	co-Hyperbolic Geometry (Anti-De-Sitter)	Minkowskian Geometry	doubly-Hyperbolic Geometry (De-Sitter)

In kinematics, the one-parameter planar motions introduced by W. Blaschke and H.R. Müller and the relation between absolute, relative and sliding velocities (accelerations) are examined on the Euclidean plane $\mathbb{E}^2$ [3]. Then, the one-parameter planar motions on the Lorentzian (Minkowskian) plane $\mathbb{L}^2$ were given by [4]. In addition to this, same concept are investigated on the Galilean plane $\mathbb{G}^2$ by [1] and [2].

In this paper, we will introduce and focus on one parameter planar motions in affine CK-planes with generalizing the notations introduced by above scientists. Also, we will discuss the relations between absolute, relative and sliding velocities (accelerations).

2. Basic Notations of Affine CK-Planes

In this section, we will investigate the basic notations of affine CK-planes [6, 14]. These planes are denoted by $\mathbb{P}_\epsilon$. Let us consider $\mathbb{R}^2$ with the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle_\epsilon = x_1 y_1 + \epsilon x_2 y_2$$

where ϵ may be $1, 0$ or -1 and $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. The matrix of this bilinear form is given as below:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}.$$

For all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{P}_\epsilon$ we can write $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T B \mathbf{y}$. For $\epsilon = 1$ we have Euclidean plane $\mathbb{E}^2$, for $\epsilon = 0$ we have Galilean plane $\mathbb{G}^2$ and for $\epsilon = -1$ we have Lorentzian plane $\mathbb{L}^2$.

If $\langle \mathbf{x}, \mathbf{y} \rangle_\epsilon = 0$, then the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{P}_\epsilon$ are orthogonal. Self-orthogonal vectors are called isotropic.

The norm of the vector $\mathbf{x} = (x_1, x_2)$ in $\mathbb{P}_\epsilon$ is defined by

$$\|\mathbf{x}\|_\epsilon = \sqrt{\left|\langle \mathbf{x}, \mathbf{x} \rangle_\epsilon\right|} = \sqrt{\left|x_1^2 + \epsilon x_2^2\right|}.$$

The vector system $\left\{\mathbf{c_1} = (1,0), \mathbf{c_2} = (0,1)\right\}$ is orthonormal basis for $\mathbb{P}_\epsilon$. The distance between two points $A = \left(x_1, x_2\right)$ and $B = \left(y_1, y_2\right)$ is defined by

$$\|\mathbf{AB}\| = \sqrt{\left|\langle \mathbf{AB}, \mathbf{AB} \rangle_\epsilon\right|} = d_{AB} = \sqrt{\left|\left(y_1 - x_1\right)^2 + \epsilon\left(y_2 - x_2\right)^2\right|}.$$

For $\epsilon = 1$ only the zero vector, for $\epsilon = 0$ zero vectors and vertical vectors are isotropic and for $\epsilon = -1$ zero vectors and vectors parallel to $(\pm 1, 1)$ are isotropic [6].

A circle is the locus of points equidistant from a given fixed point, the center of the circle. The unit circle in $\mathbb{P}_\epsilon$ is the set of points with $\|\mathbf{P}\| = 1$, for all $\mathbf{P} \in \mathbb{P}_\epsilon$. The equation of the unit circle in $\mathbb{P}_\epsilon$ is $\mathbf{x}^2 + \epsilon \mathbf{y}^2 = \pm 1$. They are shown in the Figure 1.

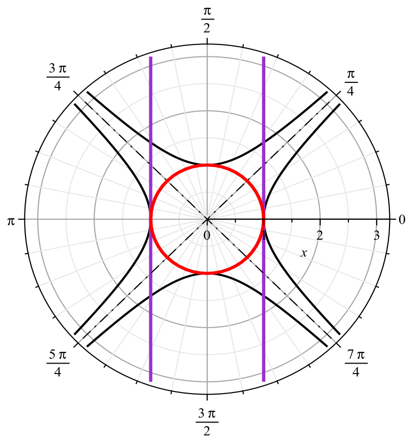

Figure 1: Unit Circles in $\mathbb{P}_\epsilon$

The linear transformation $J : \mathbb{P}_\epsilon \rightarrow \mathbb{P}_\epsilon$ with matrix, also denoted by J and given as below:

$$J = \begin{bmatrix} 0 & -\epsilon \\ 1 & 0 \end{bmatrix}.$$

This linear transformation converts any vector $\mathbf{x}$ to an orthogonal vector $J\mathbf{x}$. If $\mathbf{x}$ is a nonisotropic and $\mathbf{y}$ is orthogonal to $\mathbf{x}$, then it is written such that $\mathbf{y} = kJ\mathbf{x}$ for some real number k [6].

It is not difficult to verify directly from the definition of the matrix exponential as $e^{J\varphi} = \sum_{n=0}^{\infty} \frac{(J\varphi)^n}{n!}$ that

$$e^{J\varphi} = \cos_\epsilon \varphi + J \sin_\epsilon \varphi = \begin{bmatrix} \cos_\epsilon \varphi & -\epsilon \sin_\epsilon \varphi \\ \sin_\epsilon \varphi & cos_\epsilon \varphi \end{bmatrix}$$

where

$$\cos_{\epsilon}\varphi = \sum_{n=0}^{\infty} \frac{(-\epsilon^n)\varphi^{2n}}{(2n)!} \quad \sin_{\epsilon}\varphi = \sum_{n=0}^{\infty} \frac{(-\epsilon^n)\varphi^{2n+1}}{(2n+1)!}.$$

For $\epsilon = 1$ these are usual cosine and sine functions, for $\epsilon = -1$ they are hyperbolic cosine and sine functions, and for $\epsilon = 0$ they are just $\cos_0\varphi = 1$ and $\cos_0\varphi = \varphi$ for all φ.

In all cases, we obtain

$$\cos^2{}_{\epsilon}\varphi + \epsilon \sin^2{}_{\epsilon}\varphi = 1$$

and

$$\partial_{\varphi} \cos_{\epsilon}\varphi = -\epsilon \sin_{\epsilon}\varphi, \partial_{\varphi} \sin_{\epsilon}\varphi = \cos_{\epsilon}\varphi.$$

By equating corresponding entries of the matrix equation $e^{J(\varphi+\theta)} = e^{J\varphi}e^{J\theta}$, we can find the sum formulae [14] as follows:

$$\begin{array}{rcl} \cos_{\epsilon}(\varphi+\theta) & = & \cos_{\epsilon}\varphi \cos_{\epsilon}\theta - \epsilon \sin_{\epsilon}\varphi \sin_{\epsilon}\theta \\ \sin_{\epsilon}(\varphi+\theta) & = & \sin_{\epsilon}\varphi \cos_{\epsilon}\theta + \cos_{\epsilon}\varphi \sin_{\epsilon}\theta \end{array}.$$

3. One-Parameter Planar Motions in Affine CK-Planes

3.1. Derivative Formulae, Velocities and Pole Point Notation

In this section, we stated that the one-parameter planar motions in affine CK-planes is an extension of the one-parameter planar motions in the Euclidean plane, Lorentzian plane, and Galilean plane, respectively given [3], [4] and [1]. We will define the one-parameter planar motions in affine CK-planes and we will obtain the relations between velocities and accelerations of a point under these motions.

Let $\mathbb{P}_{\epsilon}$ and $\mathbb{P}'_{\epsilon}$ be moving and fixed affine CK-planes and $\{O; \mathbf{c_1}, \mathbf{c_2}\}$ and $\{O'; \mathbf{c'_1}, \mathbf{c'_2}\}$ be their orthonormal coordinate systems, respectively. Let us take the vector

$$\mathbf{OO'} = \mathbf{u} = u_1\mathbf{c_1} + u_2\mathbf{c_2} \text{ for } u_1, u_2 \in \mathbb{R}. \tag{1}$$

Let us define a transformation as given below:

$$\mathbf{x}' = \mathbf{x} - \mathbf{u}, \tag{2}$$

where $\mathbf{x}$, $\mathbf{x}'$ are coordinate vectors with respect to the moving and fixed rectangular coordinate system of a point $X = (x_1, x_2) \in \mathbb{P}_{\epsilon}$, respectively.

By the equation (2), *one-parameter planar motions in affine CK-planes* are defined. These motions denoted by $\mathbb{P}_{\epsilon}/\mathbb{P}'_{\epsilon}$.

Moreover, φ, the angle between the vectors $\mathbf{c_1}$ and $\mathbf{c'_1}$, is the *rotation angle* of the motions $\mathbb{P}_{\epsilon}/\mathbb{P}'_{\epsilon}$ and $\mathbf{x}$, $\mathbf{x}'$, $\mathbf{u}$ are continuously differentiable functions of a time parameter $t \in I \subset \mathbb{R}$. For $t = 0$, the coordinate systems are coincident. By taking $\varphi = \varphi(t)$, we can write

$$\left\{ \begin{array}{ll} \mathbf{c_1} = & \cos_{\epsilon}\varphi \mathbf{c'_1} + \sin_{\epsilon}\varphi \mathbf{c'_2} \\ \mathbf{c_2} = -\epsilon & \sin_{\epsilon}\varphi \mathbf{c'_1} + \cos_{\epsilon}\varphi \mathbf{c'_2} \end{array} \right. \tag{3}$$

We assume that

$$\dot{\varphi}(t) = \frac{d\varphi}{dt} \neq 0,$$

and $\dot{\varphi}(t)$ is called the angular velocity of the motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$.

By differentiating the equations (1) and (3) with respect to t, the *derivative formulae* of the motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$ are obtained as follows:

$$\begin{cases} \dot{\mathbf{c_1}} &= \dot{\varphi}\mathbf{c_2} \\ \dot{\mathbf{c_2}} &= -\epsilon\dot{\varphi}\mathbf{c_1} \\ \dot{\mathbf{u}} &= (\dot{u_1} - \epsilon\dot{\varphi}u_2)\mathbf{c_1} + (\dot{u_2} + \dot{\varphi}u_1)\mathbf{c_2}. \end{cases} \tag{4}$$

By using these derivative formulae, we will define velocities of a point $X = (x_1, x_2) \in \mathbb{P}_\epsilon$. The velocity of the point X with respect to $\mathbb{P}_\epsilon$ is called the *relative velocity* denoted by $\mathbf{V}_r$ and it is defined by $\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}}$:

$$\mathbf{V}_r = \dot{x_1}\mathbf{c_1} + \dot{x_2}\mathbf{c_2}. \tag{5}$$

Besides, the *absolute velocity* of the X with respect to $\mathbb{P}_\epsilon$ is obtained by differentiating the equation (2) with respect to t and using derivative formulae. It is denoted by $\mathbf{V}_a$ and obtained as follows:

$$\mathbf{V}_a = \frac{d\mathbf{x}'}{dt} = \{-\dot{u_1} + \epsilon\dot{\varphi}(u_2 - x_2)\}\mathbf{c_1} + \{\dot{u_2} + \dot{\varphi}(-u_1 + x_1)\}\mathbf{c_2} + \mathbf{V}_r. \tag{6}$$

By using equation (6), we get the sliding velocity vector as below:

$$\mathbf{V}_f = \{-\dot{u_1} + \epsilon\dot{\varphi}(u_2 - x_2)\}\mathbf{c_1} + \{\dot{u_2} + \dot{\varphi}(-u_1 + x_1)\}\mathbf{c_2}. \tag{7}$$

From equations (5), (6), and (7), the following theorem can be given.

Theorem 1. *Let X be a moving point on the plane $\mathbb{P}_\epsilon$ and $\mathbf{V}_r, \mathbf{V}_a$ and $\mathbf{V}_f$ be the relative, absolute and sliding velocities of X under the one-parameter planar motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$, respectively. Then, the relation between the velocities are given as below:*

$$\mathbf{V}_a = \mathbf{V}_f + \mathbf{V}_r.$$

Proof. The proof is obvious from the calculations of velocities given in the equations (5), (6), and (7). □

Now, we will investigate the points that does not move during the motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$ and the sliding velocity vector $\mathbf{V}_f$ is equal to zero for every $t \in [t_0, t_1]$. These points are called the *pole points* or the *instantaneous rotation pole centers*. If we use the equation (8) for a pole point $P = (p_1, p_2) \in \mathbb{P}_\epsilon$ of the motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$, we have

$$\begin{cases} -\dot{u_2} + \dot{\varphi}(-u_1 + x_1) = 0 \\ -\dot{u_1} + \epsilon\dot{\varphi}(u_2 - x_2) = 0 \end{cases} \tag{8}$$

So, we obtain the pole point from the solution of the system (8) as follows:

$$\begin{cases} p_1(t) = x_1(t) = u_1(t) + \frac{\dot{u_2}(t)}{\dot{\varphi}(t)} \\ \epsilon p_2(t) = \epsilon x_2(t) = \epsilon u_2(t) - \frac{\dot{u_1}(t)}{\dot{\varphi}(t)} \end{cases} \tag{9}$$

Therefore, the point P is instant in the plane $\mathbb{P}_\epsilon$.

Let us rearrange the sliding velocity vector (7) by using the equation (9):

$$\mathbf{V}_f = \{-\epsilon(x_2 - p_2)\mathbf{c_1} + (x_1 - p_1)\mathbf{c_2}\}\dot{\varphi}. \tag{10}$$

With reference the above equation, we can give the following corollaries:

Corollary 1. *During the one-parameter planar motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$ in affine CK-planes, the pole ray* **PX** *and the sliding velocity $\mathbf{V}_f$ are perpendicular vectors in the sense of affine CK-geometry, i.e., $\langle \mathbf{PX}, \mathbf{V}_f \rangle_\epsilon = 0$. Then, the focus of the point X of the motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$ is an orbit that its normal pass through the rotation pole P.*

Corollary 2. *Under the motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$, the affine CK-norm of the sliding velocity $\mathbf{V}_f$ is written below:*

$$\left\|\mathbf{V}_f\right\|_\epsilon = \|\mathbf{PX}\|_\epsilon \left|\dot{\varphi}\right|.$$

3.2. Accelerations and Acceleration Pole Point Notation

In this section, we will define relative, absolute, sliding and Coriolis acceleration vectors denoted by $\mathbf{b_r}, \mathbf{b_a}, \mathbf{b_f}$ and $\mathbf{b_c}$, respectively, during the one-parameter planar motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$ in affine CK-planes.

Let X be a moving point in $\mathbb{P}_\epsilon$. By differentiating the relative velocity vector according to t, we obtain the *relative acceleration* $\mathbf{b_r}$ as below:

$$\mathbf{b}_r = \dot{\mathbf{V}}_r = \ddot{\mathbf{x}} = \ddot{x_1}\mathbf{c_1} + \ddot{x_2}\mathbf{c_2}. \tag{11}$$

The acceleration of the point X with respect to $\mathbb{P}'_\epsilon$ is known as the *absolute acceleration* and it is defined by $\mathbf{b}_a = \frac{d\mathbf{V}_a}{dt} = \dot{\mathbf{V}}_a$. If we differentiate the equation (6) with respect to t and use the equations (4), we obtain the absolute acceleration as below:

$$\begin{aligned} \mathbf{b}_a =& \epsilon \left\{\dot{\varphi}\dot{p_2} - (\dot{\varphi})^2(x_1 - p_1) - \ddot{\varphi}(x_2 - p_2)\right\}\mathbf{c_1} + \left\{-\dot{\varphi}\dot{p_1} - \epsilon(\dot{\varphi})^2(x_2 - p_2) + \ddot{\varphi}(x_1 - p_1)\right\}\mathbf{c_2} \\ & + \ddot{x_1}\mathbf{c_1} + \ddot{x_2}\mathbf{c_2} + 2\dot{\varphi}(-\epsilon\dot{x_2}\mathbf{c_1} + \dot{x_1}\mathbf{c_2}). \end{aligned} \tag{12}$$

In the equation (12), the expression

$$\mathbf{b}_f = \epsilon \left\{\dot{\varphi}\dot{p_2} - (\dot{\varphi})^2(x_1 - p_1) - \ddot{\varphi}(x_2 - p_2)\right\}\mathbf{c_1} + \left\{-\dot{\varphi}\dot{p_1} - \epsilon(\dot{\varphi})^2(x_2 - p_2) + \ddot{\varphi}(x_1 - p_1)\right\}\mathbf{c_2} \tag{13}$$

is called the *sliding acceleration* and

$$\mathbf{b}_c = 2\dot{\varphi}(-\epsilon\dot{x_2}\mathbf{c_1} + \dot{x_1}\mathbf{c_2}). \tag{14}$$

is called the *Coriolis acceleration* of the one-parameter planar motion $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$.

Consequently, we can give the following theorem and corollary with using the equations (11), (12), (13), and (14).

Theorem 2. *Let X be a moving point on the plane $\mathbb{P}_\epsilon$ and $\mathbf{b}_r$, $\mathbf{b}_a$, $\mathbf{b}_f$ and $\mathbf{b}_c$ be the relative, absolute, sliding and Coriolis accelerations of X, respectively. Then, the relation between the accelerations under the one-parameter planar motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$ are given as below:*

$$\mathbf{b}_a = \mathbf{b}_f + \mathbf{b}_c + \mathbf{b}_r.$$

Corollary 3. *During the motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$, the Coriolis acceleration vector $\mathbf{b}_c$ and the relative velocity vector $\mathbf{V}_r$ are perpendicular to each other in the sense of affine CK-geometry,i.e. $\langle \mathbf{V}_r, \mathbf{b}_c \rangle_\epsilon = 0$.*

During the one-parameter planar motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$, the acceleration pole is characterized by $\mathbf{b}_f = \mathbf{0}$. Then, if we take the acceleration pole point $Q = (q_1, q_2) \in \mathbb{P}_\epsilon$ of the motions $\mathbb{P}_\epsilon/\mathbb{P}'_\epsilon$, we get the following equation system:

$$\begin{cases} \epsilon\left(\dot{\varphi}\right)^2\left(x_1 - p_1\right) + \epsilon\ddot{\varphi}\left(x_2 - p_2\right) = \epsilon\dot{\varphi}\dot{p}_2 \\ \ddot{\varphi}\left(x_1 - p_1\right) - \epsilon\left(\dot{\varphi}\right)^2\left(x_2 - p_2\right) = \dot{p}_1\dot{\varphi} \end{cases} \tag{15}$$

If $\epsilon\left(\dot{\varphi}\right)^4 + \left(\ddot{\varphi}\right)^2 \neq 0$, we obtain the pole point from the above system as follows:

$$\begin{cases} \left(\epsilon^2\left(\dot{\varphi}(t)\right)^4 + \epsilon\left(\ddot{\varphi}(t)\right)^2\right)q_1(t) = & \left(\epsilon^2\left(\dot{\varphi}(t)\right)^4 + \epsilon\left(\ddot{\varphi}(t)\right)^2\right)p_1(t) \\ & +\dot{\varphi}(t)\left(\epsilon\ddot{\varphi}(t)\dot{p}_1(t) + \epsilon^2\left(\dot{\varphi}(t)\right)^2\dot{p}_2(t)\right) \\ \left(\epsilon^2\left(\dot{\varphi}(t)\right)^4 + \epsilon\left(\ddot{\varphi}(t)\right)^2\right)q_2(t) = & \left(\epsilon^2\left(\dot{\varphi}(t)\right)^4 + \epsilon\left(\ddot{\varphi}(t)\right)^2\right)p_2(t) \\ & -\epsilon\dot{\varphi}(t)\left(\left(\dot{\varphi}(t)\right)^2\dot{p}_1(t) - \ddot{\varphi}(t)\dot{p}_2(t)\right) \end{cases}$$

So that the point Q is instant in the plane $\mathbb{P}_\epsilon$.

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