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Hardy Spaces on the Polydisk

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Abstract. In this paper we will study the boundary values properties of the functions in the Hardy spaces; generalize the F. and M. Riesz theorem to higher dimensions; discuss the existence of boundary values of the functions in $H^p(\mathbb{D}^n)$ on non-distinguished boundary $\partial\mathbb{D}^n \setminus \mathbb{T}^n$ and the intersection of the spaces $H_u^p(\mathbb{D}^n)$.

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1. Introduction

This paper basically consists of two parts. In the first part, consisting of Sections 2, 3 and 4, we study the properties of the functions on the classical Hardy spaces of n-harmonic functions and the Hardy spaces of holomorphic functions on the polydisk. In Section 2 we will show that the functions in the classical Hardy spaces can be restored by the Poisson integral of its radial limit. In Section 3 we will restate and prove the celebrated F. and M. Riesz theorem to higher dimensions. In Section 4 we will study the boundary values of the functions in $H^p(\mathbb{D})$ on the non-distinguished boundary, $\partial\mathbb{D}^n \setminus \mathbb{T}^n$.

The second part of this paper consists of Section 5. In this section we study the Poletsky–Stessin Hardy spaces $H_u^p(\mathbb{D}^2)$ on bidisk. We mainly establish two things - there are nontrivial Poletsky–Stessin Hardy spaces and the intersection of the Poletsky–Stessin Hardy spaces over all exhaustion functions is $H^\infty(\mathbb{D}^2)$, the space of bounded holomorphic functions on $\mathbb{D}^2$.

2. Hardy Spaces and Poisson Integral Formula

An n-harmonic function u on $\mathbb{D}^n$ is a function which is harmonic in each variable separately. Denote by $h^p(\mathbb{D}^n)$ the space of all n-harmonic functions satisfying

$$\sup_{0\le r<1}\int_{\mathbb{T}^n} |u_r(\zeta)|^p \, dm(\zeta) < \infty \tag{1}$$

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where $u_r(\zeta) = u(r\zeta)$ and dm is the normalized Lebesgue measure on $\mathbb{T}^n$. The p -th root of (1) defines a norm on $h^p(\mathbb{D}^n)$ when $p \geq 1$. With this norm $h^p(\mathbb{D}^n)$ is Banach.

We will use the following notations:

$$\begin{aligned} z =& (z_1, \ldots, z_n) \\ \zeta =& (\zeta_1, \ldots, \zeta_n) \\ P(z,\zeta) =& P(z_1,\zeta_1) \ldots P(z_n,\zeta_n) \end{aligned}$$

where $P(z,\zeta)$ is the Poisson kernel and

$$P(z_j,\zeta_j) = \mathbf{Re}\left(\frac{\zeta_j + z_j}{\zeta_j - z_j}\right) = \frac{1-|z_j|^2}{|\zeta_j - z_j|^2}, \quad j = 1, \ldots, n.$$

Theorem 1. *Let $u \in h^p(\mathbb{D}^n)$, $p > 1$. Then there exists a function $f \in L^p(\mathbb{T}^n)$ such that*

$$u(z) = \int_{\mathbb{T}^n} P(z,\zeta) f(\zeta)\, dm(\zeta).$$

Proof. Take $r_j \nearrow 1$. Then (1) implies that there is a weakly convergent subsequence of u_{r_j}. We will write the subsequence u_{r_j} just to avoid the sub-subscript. Hence for $g \in L^q(\mathbb{T}^n)$

$$g \mapsto \lim_{j\to\infty} \int_{\mathbb{T}^n} g(\zeta) u_{r_j}(\zeta)\, dm(\zeta)$$

is a linear functional on $L^q(\mathbb{T}^n)$. By Riesz theorem there exists an $f \in L^p(\mathbb{T}^n)$ such that

$$\lim_{j\to\infty} \int_{\mathbb{T}^n} g(\zeta) u_{r_j}(\zeta)\, dm(\zeta) = \int_{\mathbb{T}^n} g(\zeta) f(\zeta)\, dm(\zeta).$$

Now take $g(\zeta) = P(z,\zeta)$. Then

$$u(z) = \lim_{j\to\infty} u_{r_j}(z) = \lim_{j\to\infty} \int_{\mathbb{T}^n} P(z,\zeta) u_{r_j}(\zeta)\, dm = \int_{\mathbb{T}^n} P(z,\zeta) f(\zeta)\, dm(\zeta).$$

The second equality above follows from [7, Theorem 2.1.2]. □

What makes the above proof work is the duality of L^p spaces. Since L^∞ is the dual of L^1, the same result holds with the same proof for $p = \infty$. Of course we have to change the statement accordingly. But unfortunately L^1 is not dual of anything, we don't have the same result for $p = 1$. Instead, since the space of finite signed measures on $\mathbb{T}^n$ is dual of the space of continuous functions $C(\mathbb{T}^n)$ we have the following result from [7, Theorem 2.1.3, (e)].

Theorem 2. *If the hypothesis of Theorem 1 holds for $p = 1$ then there exists a finite signed measure μ on $\mathbb{T}^n$ with*

$$u(z) = \int_{\mathbb{T}^n} P(z,\zeta)\, d\mu(\zeta).$$

So the function $u \in h^p(\mathbb{D}^n)$, $p > 1$, is the Poisson integral of some function $f \in L^p(\mathbb{T}^n)$. Is there any other connection between u and f? We know, when $n = 1$, f is the boundary value function of u and when $n > 1$ the following theorem [7, Theorem 2.3.1] answers this question.

Theorem 3. *If $f \in L^1(\mathbb{T}^n)$, if σ is a measure on $\mathbb{T}^n$ which is singular with respect to dm, and if $u = P[f + d\sigma]$, then $u^*(\zeta) = f(\zeta)$ for almost every $\zeta \in \mathbb{T}^n$.*

Recall that $u^*(\zeta) = \lim_{r\to 1} u(r\zeta)$ is the radial limit. Thus any n-harmonic function satisfying the growth condition (1) for $p > 1$ can be restored by the Poisson integral of its boundary value function.

For $p = 1$ we just saw in Theorem 2 that $u(z) = P[d\mu](z)$. By the Lebesgue decomposition theorem

$$d\mu = f\,dm + d\sigma$$

where σ is singular with respect to m and $f \in L^1(\mathbb{T}^n)$. Hence we have $u^*(\zeta) = f(\zeta)$ but u can not be restored by the Poisson integral of its boundary value function unless, of course, $P[d\sigma] = 0$.

Also in [7] it has been proved that if $f \in L^p(\mathbb{T}^n)$, $1 \le p < \infty$, and $u = P[f]$ then u_r converges to f in the L^p-norm as $r \to 1$, i.e. $\lim_{r\to 1} \|u_r - f\|_{L^p} = 0$. But when $p = 1$ we have the weak-$*$ convergence.

Theorem 4. *Let $f(z) = P[d\mu](z)$ with μ a finite signed measure on $\mathbb{T}^n$. Then $f_r\,dm \to d\mu$ weak-$*$ as $r \to 1$.*

Proof. Let $\varphi \in C(\mathbb{T}^n)$. Then

$$\begin{aligned}
\left| \int_{\mathbb{T}^n} \varphi(\zeta) f_r(\zeta)\,dm(\zeta) - \int_{\mathbb{T}^n} \varphi(\zeta)\,d\mu(\zeta) \right| &= \left| \int_{\mathbb{T}^n} \varphi(\zeta) \left(\int_{\mathbb{T}^n} P(r\zeta,\eta)\,d\mu(\eta) \right) dm(\zeta) - \int_{\mathbb{T}^n} \varphi(\eta)\,d\mu(\eta) \right| \\
(\because P(r\zeta,\eta) = P(r\eta,\zeta)) &= \left| \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} P(r\eta,\zeta)\varphi(\zeta)\,dm(\zeta) \right) d\mu(\eta) - \int_{\mathbb{T}^n} \varphi(\eta)\,d\mu(\eta) \right| \\
&= \left| \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} P(r\eta,\zeta)\varphi(\zeta)\,dm(\zeta) - \varphi(\eta) \right) d\mu(\eta) \right| \\
&\to 0
\end{aligned}$$

because the inner integral goes to zero uniformly on η. Hence $f_r\,dm \to d\mu$ weak-$*$ as $r \to 1$. □

We define $H^p(\mathbb{D}^n)$, $0 < p < \infty$, to be the class of all holomorphic functions $f \in \mathbb{D}^n$ for which

$$\sup_{0\le r<1} \int_{\mathbb{T}^n} |f_r(\zeta)|^p\,dm < \infty$$

and $H^\infty(\mathbb{D}^n)$ is the space of all bounded holomorphic functions in $\mathbb{D}^n$.

Since $|f|^p$ is n-subharmonic, sup in the definition can be replaced by lim as $r \to 1$.

It is known that if $f \in H^p(\mathbb{D}^n)$, $0 < p < \infty$, then f has a non-tangential limit at almost all points of $\mathbb{T}^n$ [11, Ch. XVII, Theorem 4.8]. We denote this limit by f^* as in [7] and call it a boundary value function. Moreover, we have the following results from Rudin (see [7, Theorem 3.4.2 and 3.4.3]).

Theorem 5. *If $f \in H^p(\mathbb{D}^n)$, $0 < p < \infty$, then $f^* \in L^p(\mathbb{T}^n)$ and*

(i) $\lim_{r\to 1} \int_{\mathbb{T}^n} |f_r|^p \, dm = \int_{\mathbb{T}^n} |f^*|^p \, dm$

(ii) $\lim_{r\to 1} \int_{\mathbb{T}^n} |f_r - f^*|^p \, dm = 0.$

When $p \geq 1$ the function in $H^p(\mathbb{D}^n)$ can be represented by the Poisson integral of its boundary value function.

Theorem 6. *If $f \in H^1(\mathbb{D}^n)$, then*

$$f(z) = \int_{\mathbb{T}^n} P(z,\zeta) f^*(\zeta) \, dm.$$

(The case $n = 1$ can be found in [6, Theorem 17.11].)

Proof. Since $z \in \mathbb{D}^n$, $P(z,\zeta)$ is bounded on $\mathbb{T}^n$ and by *(ii)* of the theorem above

$$\left| \int_{\mathbb{T}^n} P(z,\zeta) f_r(\zeta) \, dm(\zeta) - \int_{\mathbb{T}^n} P(z,\zeta) f^*(\zeta) \, dm(\zeta) \right| \leq \int_{\mathbb{T}^n} P(z,\zeta) |f_r(\zeta) - f^*(\zeta)| \, dm(\zeta)$$
$$\to 0.$$

Now by [7, Theorem 2.1.2]

$$\begin{aligned} f(z) &= \lim_{r\to 1} f_r(z) \\ &= \lim_{r\to 1} \int_{\mathbb{T}^n} P(z,\zeta) f_r(\zeta) \, dm(\zeta) \\ &= \int_{\mathbb{T}^n} f^*(\zeta) \, dm(\zeta). \end{aligned}$$

□

3. The F. and M. Riesz Theorem

Now we want to generalize the F. and M. Riesz theorem.

Theorem 7. *Let μ be a complex Borel measure on $\mathbb{T}^n$. If*

$$\int_{\mathbb{T}^n} e^{i(k\theta)} \, d\mu(\theta) = 0$$

for $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with at least one k_j, $j = 1, 2, \ldots, n$ positive, where $(k\theta) = k_1\theta_1 + \ldots + k_n\theta_n$ then μ is absolutely continuous with respect to dm.

(When $n = 1$ see [6, Theorem 17.13].)

Proof. Define $f(z) = P[d\mu](z)$. Then, with the notations

$$\begin{aligned} z =& (z_1, \dots, z_n) \text{ with } z_j = r_j e^{i\theta_j}, j = 1, \dots, n \\ r^{|k|} =& r_1^{|k_1|} \dots r_n^{|k_n|} \\ (k \cdot \theta) =& k_1\theta_1 + \dots + k_n\theta_n \\ (k \cdot t) =& k_1 t_1 + \dots + k_n t_n \end{aligned}$$

and using the series representation for the Poisson kernel, we get

$$\begin{aligned} f(z) =& \int_{\mathbb{T}^n} P(z, e^{it})\, d\mu(t) \\ =& \int_{\mathbb{T}^n} \left(\sum_{k \in \mathbb{Z}^n} r^{|k|} e^{i(k\cdot\theta)} e^{-i(k\cdot t)} \right) d\mu(t) \\ =& \sum_{k \in \mathbb{Z}^n} \left(\int_{\mathbb{T}^n} e^{-i(k\cdot t)} d\mu(t) \right) r^{|k|} e^{i(k\cdot\theta)} \\ =& \sum_{k \in \mathbb{Z}^n_+} c_k z^k \end{aligned}$$

where $c_k = \int_{\mathbb{T}^n} e^{-i(k\cdot t)}\, d\mu(t)$ and $z_k = r^{|k|} e^{i(k\cdot\theta)}$. Notice that all other integrals in the above sum vanish by the hypothesis. Thus $f(z)$ is holomorphic.

For $0 \le r < 1$,

$$\begin{aligned} \int_{\mathbb{T}^n} |f_r(\zeta)|\, dm(\zeta) =& \int_{\mathbb{T}^n} \left| \int_{\mathbb{T}^n} P(r\zeta, \eta)\, d\mu(\eta) \right| dm(\zeta) \\ \le& \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} P(r\zeta, \eta)\, d|\mu|(\eta) \right) dm(\zeta) \\ =& \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} P(r\zeta, \eta)\, dm(\zeta) \right) d|\mu|(\eta) \\ =& \|\mu\|. \end{aligned}$$

Thus $f \in H^1(\mathbb{D}^n)$ and hence $f(z) = P[f^*](z)$, where $f^* \in L^1(\mathbb{T}^n)$. Now the uniqueness of the Poisson integral representation shows that

$$d\mu = f^* dm$$

and the proof is completed. □

4. Boundary Values

Do the boundary values of functions in $H^p(\mathbb{D}^n)$ exist on the non-distinguished boundary? Now we want to look into this question.

Let $\{j_1,\ldots,j_k\}$ and $\{i_1,\ldots,i_l\}$ be disjoint sets of indices such that their union is $\{1,\ldots,n\}$ where $j_1 < j_2 < \ldots < j_k$ and $i_1 < i_2 < \ldots < i_l$. Define the sections of $\mathbb{D}^n$ as follows

$$\mathbb{D}^n_{z_{j_1},\ldots,z_{j_k}} = \{(z_1,\ldots,z_n) \in \mathbb{D}^n : z_{j_1},\ldots,z_{j_k} \text{ are fixed}\}$$

and define $f_{z_{j_1},\ldots,z_{j_k}} = f|_{\mathbb{D}^n_{z_{j_1},\ldots,z_{j_k}}}$. We will write $f_{z_{j_1},\ldots,z_{j_k}}(z_{i_1},\ldots,z_{i_l})$ instead of $f_{z_{j_1},\ldots,z_{j_k}}(z_1,\ldots,z_n)$.

We will see below that for $f \in H^p(\mathbb{D}^n)$, $1 \le p < \infty$, the non-tangential limit of $f_{z_{j_1},\ldots,z_{j_k}}$ exists at almost all points of the distinguished boundary of the section $\mathbb{D}^n_{z_{j_1},\ldots,z_{j_k}}$ which is $\mathbb{T}^l$ and the function $f_{z_{j_1},\ldots,z_{j_k}}$ can be restored by the Poisson integral of this limit.

Theorem 8. *Let $f \in H^p(\mathbb{D}^n)$, $1 \le p < \infty$. Then $f_{z_{j_1},\ldots,z_{j_k}} \in H^p(\mathbb{D}^l)$.*

Proof. Without loss of generality we suppose that $\{j_1,\ldots,j_k\} = \{1,\ldots,k\}$. Let's use the following notations for the Poisson kernels

$$P_j(\zeta_j) = \begin{cases} P(z_j,\zeta_j) & j = 1,\ldots,k \\ P(r\xi_j,\zeta_j) & j = k+1,\ldots,n \end{cases}$$

where $|\xi_j| = 1$. Then, for $0 < r < 1$, by Theorem 6

$$f_{z_1,\ldots,z_k}(r\xi_{k+1},\ldots,r\xi_n) = \int_{\mathbb{T}^n} P_1(\zeta_1)\ldots P_n(\zeta_n) f^*(\zeta_1,\ldots,\zeta_n)\, dm_n.$$

By Hölder and Fubini

$$\begin{aligned}
\int_{\mathbb{T}^l} |f_{z_1,\ldots,z_k}(r\xi_{k+1},\ldots,r\xi_n)|^p dm_l &= \int_{\mathbb{T}^l} \left| \int_{\mathbb{T}^n} P_1(\zeta_1)\ldots P_n(\zeta_n) f^*(\zeta_1,\ldots,\zeta_n)\, dm_n \right|^p dm_l \\
&\le \int_{\mathbb{T}^l} \left(\int_{\mathbb{T}^n} P_1(\zeta_1)\ldots P_n(\zeta_n) |f^*(\zeta_1,\ldots,\zeta_n)|^p\, dm_n \right) dm_l \\
&= \int_{\mathbb{T}^n} P_1(\zeta_1)\ldots P_k(\zeta_k) |f^*(\zeta_1,\ldots,\zeta_n)|^p \\
&\quad \times \left(\int_{\mathbb{T}^l} P_{k+1}(\zeta_{k+1})\ldots P_n(\zeta_n)\, dm_l \right) dm_n \\
&\le \frac{2^k}{(1-|z_1|)\ldots(1-|z_k|)} \int_{\mathbb{T}^n} |f^*(\zeta_1,\ldots,\zeta_n)|^p dm_n.
\end{aligned}$$

The last quantity above is independent of r and is finite by Theorem 5. Thus the theorem is proved. □

The following corollary is immediate.

Corollary 1. *If $f \in H^p(\mathbb{D}^n)$, $1 \le p < \infty$, then the non-tangential limit $f^*_{z_{j_1},\ldots,z_{j_k}}$ of the function $f_{z_{j_1},\ldots,z_{j_k}}$ exists almost everywhere on $\mathbb{T}^l$ and belongs to $L^p(\mathbb{T}^l)$.*

The following theorems are the direct consequences of Theorems 5 and 6.

Theorem 9. *If $1 \le p < \infty$ and $f \in H^p(\mathbb{D}^n)$, then*

(i) $\lim_{r\to 1} \int_{\mathbb{T}^l} |(f_{z_{j_1},\ldots,z_{j_k}})_r|^p \, dm_l = \int_{\mathbb{T}^l} |f^*_{z_{j_1},\ldots,z_{j_k}}|^p \, dm_l$

(ii) $\lim_{r\to 1} \int_{\mathbb{T}^l} |(f_{z_{j_1},\ldots,z_{j_k}})_r - f^*_{z_{j_1},\ldots,z_{j_k}}|^p \, dm_l = 0$

where $(f_{z_{j_1},\ldots,z_{j_k}})_r(\zeta_{i_1},\ldots,\zeta_{i_l}) = f_{z_{j_1},\ldots,z_{j_k}}(r\zeta_{i_1},\ldots,r\zeta_{i_l})$.

Theorem 10. *If $f \in H^1(\mathbb{D}^n)$, then*

$$f_{z_{j_1},\ldots,z_{j_k}}(z_{i_1},\ldots,z_{i_l}) = \int_{\mathbb{T}^l} P(z_{i_1},\zeta_{i_1})\ldots P(z_{i_l},\zeta_{i_l}) f^*_{z_{j_1},\ldots,z_{j_k}}(\zeta_{i_1},\ldots,\zeta_{i_l})\, dm_l.$$

Theorem 11. *Let f be a holomorphic function in $\mathbb{D}^n$. If $1 \le p < \infty$ and*

$$\sup_{\substack{(z_{j_1},\ldots,z_{j_k}) \\ |z_{j_1}|=\ldots=|z_{j_k}|}} \|f_{z_{j_1},\ldots,z_{j_k}}\|_{H^p(\mathbb{D}^{n-k})} = M < \infty,$$

then $f \in H^p(\mathbb{D}^n)$.

Proof. For simplicity we take $\{j_1,\ldots,j_k\} = \{1,\ldots,k\}$. And, of course, this theorem makes sense only when $k > 0$. Now for $0 \le r < 1$,

$$\begin{aligned}
\int_{\mathbb{T}^n} |f(r\zeta_1,\ldots,r\zeta_n)|^p \, dm_n &= \int_{\mathbb{T}^k} \left(\int_{\mathbb{T}^{n-k}} |f(r\zeta_1,\ldots,r\zeta_n)|^p \, dm_{n-k} \right) dm_k \\
&\le \int_{\mathbb{T}^k} \left(\sup_{0\le t<1} \int_{\mathbb{T}^{n-k}} |f(r\zeta_1,\ldots,r\zeta_k, t\zeta_{k+1},\ldots,t\zeta_n)|^p \, dm_{n-k} \right) dm_k \\
&= \int_{\mathbb{T}^k} \|f_{r\zeta_1,\ldots,r\zeta_k}\|^p_{H^p(\mathbb{D}^{n-k})} dm_k \\
&\le M^p.
\end{aligned}$$

Thus $f \in H^p(\mathbb{D}^n)$. □

5. Poletsky–Stessin Hardy Spaces on the Bidisk

Let u be a negative continuous plurisubharmonic function on the bidisk

$$\mathbb{D}^2 = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$$

such that $u(z_1,z_2) \to 0$ as $(z_1,z_2) \to (\zeta_1,\zeta_2) \in \partial\mathbb{D}^2$. Following Demailly [2], for $r < 0$ we define

$$S_u(r) = \{(z_1,z_2) \in \mathbb{D}^2 : u(z_1,z_2) = r\}$$
$$B_u(r) = \{(z_1,z_2) \in \mathbb{D}^2 : u(z_1,z_2) < r\}.$$

For convenience we will write $z = (z_1,z_2)$. Associated with this u we define the positive measure $\mu_{u,r}$ called Monge-Ampère measures by

$$\mu_{u,r} = (dd^c u_r)^2 - \chi_{\mathbb{D}^2 \setminus B_u(r)}(dd^c u)^2$$

where $u_r = \max\{u,r\}$. These measures are supported by the level sets $S_u(r)$. Demailly has proved the following [2, Theorem 1.7].

Theorem 12 (Lelong–Jensen Formula). *For all $r < 0$ every plurisubharmonic function φ on $\mathbb{D}^2$ is $\mu_{u,r}$-integrable and*

$$\mu_{u,r}(\varphi) = \int_{B_u(r)} \varphi(dd^c u)^2 + \int_{B_u(r)} (r-u)(dd^c\varphi) \wedge (dd^c u).$$

Denote by $\mathscr{E}(\mathbb{D}^2)$ the set of all continuous negative plurisubharmonic functions u on $\mathbb{D}^2$ and equal to zero on $\partial\mathbb{D}^2$ whose Monge–Ampère mass is finite, i.e.

$$\int_{\mathbb{D}^2} (dd^c u)^2 < \infty$$

and denote by $\mathscr{E}_1(\mathbb{D}^2)$ the set of those $u \in \mathscr{E}(\mathbb{D}^2)$ for which $\int_{\mathbb{D}^2} dd^c u = 1$.

Following [3] we define, what we call, the Poletsky–Stessin Hardy space $H^p_u(\mathbb{D}^2)$, $p > 0$, as the space of all holomorphic functions on $\mathbb{D}^2$ for which

$$\limsup_{r \to 0^-} \mu_{u,r}(|f|^p) < \infty.$$

These new spaces are contained in the classical spaces, that is, $H^p_u(\mathbb{D}^2) \subset H^p(\mathbb{D}^2)$. Since $\mu_{u,r}(|f|^p)$ is an increasing function of r the lim sup in the definition can be replaced by lim. For $p \geq 1$

$$\|f\|^p_{H^p_u} = \lim_{r \to 0^-} \mu_{u,r}(|f|^p)$$

is a norm and with this norm $H^p_u(\mathbb{D}^2)$ is Banach [3, Theorem 4.1]. The Poletsky–Stessin Hardy spaces on the unit disk have been studied in detail in [1, 5, 8–10].

In [4] Poletsky has proved that the intersection of all Poletsky–Stessin Hardy spaces $H^p_u(D)$, $p \geq 1$, where D is a strongly pseudoconvex domain with C^2 boundary, is $H^\infty(D)$, the space of bounded holomorphic functions. Hence it immediately follows that the intersection of all $H^p_u(\mathbb{D})$ is $H^\infty(\mathbb{D})$. We will prove this result for the polydisk. It is enough to consider the bidisk.

Let $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$ and $\alpha = (\alpha_1, \alpha_2)$, $0 < \alpha_1, \alpha_2 < \pi/2$. Following [11] we define the approach region $T_\alpha(\zeta)$ as

$$T_\alpha(\zeta) = T_{\alpha_1}(\zeta_1) \times T_{\alpha_2}(\zeta_2)$$

where $T_{\alpha_j}(\zeta_j)$ is the Stolz angle at $\zeta_j \in \mathbb{T}$ with vertex angle $2\alpha_j$. Here we will consider only the congruent symmetric approach regions meaning that the Stolz angles are symmetric with respect to the radius to ζ_j and the vertex angles are equal, i.e. $\alpha_1 = \alpha_2$. Following [4] we define the Green ball of radius $0 < r < 1$ and center at w to be the set

$$G(w,r) = \{z \in \mathbb{D}^2 : g(z,w) < \log r\}$$

where $g(z,w)$ is the Green function for $\mathbb{D}^2$ with pole at w. The Green function for $\mathbb{D}^2$ is explicitly given by

$$g(z,w) = \log\max\left\{\left|\frac{z_1 - w_1}{1 - \overline{w_1} z_1}\right|, \left|\frac{z_2 - w_2}{1 - \overline{w_2} z_2}\right|\right\}.$$

Hence it follows that

$$G(w,r) = \left\{z_1 \in \mathbb{D} : \left|\frac{z_1 - w_1}{1 - \overline{w_1} z_1}\right| < r\right\} \times \left\{z_2 \in \mathbb{D} : \left|\frac{z_2 - w_2}{2 - \overline{w_2} z_2}\right| < r\right\}.$$

Lemma 1. *Let $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$ and $0 < r < 1$. For any $0 < t < 1$ there exists $0 < \alpha < \pi/2$ such that $G(t\zeta, r) \subset T_\alpha(\zeta)$ where $t\zeta = (t\zeta_1, t\zeta_2)$ and $T_\alpha(\zeta) = T_\alpha(\zeta_1) \times T_\alpha(\zeta_2)$.*

Proof. Observe that

$$\left\{z_j \in \mathbb{D} : \left|\frac{z_j - t\zeta_j}{1 - t\overline{\zeta_j} z_j}\right| < r\right\}$$

is the image of the disk $\{|w_j| < r\} \subset \mathbb{C}$ under the conformal map

$$w_j \mapsto \frac{w_j + t\zeta_j}{1 + t\overline{\zeta_j} w_j}$$

which is a disk contained in $\mathbb{D}$ with center at

$$\frac{t(1-r^2)}{1-r^2t^2}\zeta_j$$

and radius equal to

$$\frac{r(1-t^2)}{1-r^2t^2}.$$

The tangents to this disk that pass through ζ_j make an angle of

$$\alpha = \arcsin\left(\frac{r(1+t)}{1+tr^2}\right)$$

with the radius to ζ_j. Hence

$$\left\{z_j \in \mathbb{D} : \left|\frac{z_j - t\zeta_j}{1 - t\overline{\zeta_j} z_j}\right| < r\right\} \subset T_\alpha(\zeta_j)$$

for $j = 1, 2$ and $G(t\zeta, r) \subset T_\alpha(\zeta)$. Since for fixed $0 < r < 1$

$$t \mapsto \frac{r(1+t)}{1+tr^2}$$

is an increasing function of $t \in [0,1]$ we have

$$0 < \frac{r(1+t)}{1+tr^2} \leq \frac{2r}{1+r^2} < 1.$$

From this it follows that

$$0 < \alpha \leq \arcsin\left(\frac{2r}{1+r^2}\right) < \frac{\pi}{2}.$$

□

Remark 1. *For fixed* $0 < r < 1$*,*

$$t \mapsto \frac{r(1-t^2)}{1-r^2t^2}$$

is a decreasing function of $t \in [0,1]$ *that decreases to zero as* $t \to 1$*. Therefore we can make the size of the Green ball* $G(t\zeta, r)$ *as small as we want simply by choosing* t *close enough to* 1*.*

The plurisubharmonic envelope $E\phi$ of a continuous function ϕ on a domain $\Omega \subset \mathbb{C}^n$ is the maximal plurisubharmonic function on Ω less than or equal to ϕ. For a sequence of functions $\{u_j\} \subset \mathcal{E}(\mathbb{D}^2)$, we denote by $E\{u_j\}$ the envelope of $\inf\{u_j\}$. The following Lemma [4, Theorem 3.3] gives the estimate on the Monge–Ampère mass of the envelope.

Lemma 2. *If* Ω *is a strongly hyperconvex domain and continuous plurisubharmonic functions* $\{u_j\} \subset \mathcal{E}(\Omega)$*, then*

$$\int_\Omega (dd^c E\{u_j\})^n \leq \sum \int_\Omega (dd^c u_j)^n.$$

Theorem 13. *Let* f *be a holomorphic function on* $\mathbb{D}^2$*. Suppose that* f *has non-tangential limits at points* $\{\zeta_j\} \subset \mathbb{T}^2$ *and* $\lim_{j\to\infty} |f^*(\zeta_j)| = \infty$*. Then for any* $p \geq 1$ *there exists* $u \in \mathcal{E}_1(\mathbb{D}^2)$ *such that* $f \notin H^p_u(\mathbb{D}^2)$*.*

The proof that Poletsky gave to this theorem in [4] in the case when D is a strongly pseudoconvex domain with C^2 boundary also works when the domain is a polydisk. We will mimic his proof in our context.

Proof. Let us take a sequence $\{a_j\}$ of positive numbers such that

$$\sum_{j=1}^{\infty} a_j < \infty \text{ and } \sum_{j=1}^{\infty} a_j^2 |f^*(\zeta_j)|^p = \infty.$$

For $0 < t_j < 1$ we write $G_j = G(t_j\zeta_j, e^{-1})$. By Lemma 1 there exists $0 < \alpha_j < \pi/2$ such that $G_j \subset T_{\alpha_j}(\zeta_j)$. Now we inductively construct a sequence $\{t_k\}, 0 < t_k < 1$, satisfying certain conditions. Choose any $0 < t_1 < 1$. Suppose that $t_1, \dots, t_{k-1}$ have already been chosen. Now chose $0 < t_k < 1$ so that the following conditions are satisfied:

(i) $|f| > |f^*(\zeta_k)|/2$ on G_k

(ii) $G_k \cap G_j = \phi$

(iii) $g(z, t_k\zeta_k) > -a_j/2^{k+1}$ on G_j

(iv) $a_j g(z, t_j\zeta_j) > -a_k/2^{j+1}$ on G_k

for $1 \le j \le k-1$. The conditions (i) and (ii) can be achieved simply by taking t_k close enough to 1. Since G_j, $j < k$, and G_k are disjoint, $g(z, t_k\zeta_k) \to 0$ uniformly on G_j as $t_k \to 1$. Hence (iii) can be achieved for t_k close enough to 1. Since $g(z, t_j\zeta_j) = 0$ when $z \in \partial\mathbb{D}^2$, we can choose t_k so close to 1 that

$$G_k \subset \bigcap_{j=1}^{k-1} \left\{z \in \mathbb{D}^2 : a_j g(z, t_j\zeta_j) > -a_k/2^{j+1}\right\}.$$

Thus (iv) can be achieved.

Define

$$u_j(z) = a_j \max\{g(z, t_j\zeta_j), -2\}.$$

Note that if F is an open set in $\mathbb{D}^2$ containing $G(t_j\zeta_j, e^{-2})$ then

$$\int_F (dd^c u_j)^2 = a_j^2.$$

Let $u = E\{u_j\}$. Since the series $v = \sum_{j=1}^{\infty} u_j$ converges uniformly on $\overline{\mathbb{D}^2}$, $v \in \mathscr{E}(\mathbb{D}^2)$. So $u \ge v$ is a continuous plurisubharmonic function on $\mathbb{D}^2$ equal to 0 on $\partial\mathbb{D}^2$. By Lemma 2,

$$\int_{\mathbb{D}^2} (dd^c u)^2 \le \sum_{j=1}^{\infty} \int_{\mathbb{D}^2} (dd^c u_j)^2 = \sum_{j=1}^{\infty} a_j^2 < \infty.$$

Hence $u \in \mathscr{E}(\mathbb{D}^2)$.

Now we evaluate $\int_{G_k} (dd^c u)^2$. Observe that $u_k \ge u \ge v$ on $\mathbb{D}^2$. By the conditions on the choices of t_j, on ∂G_k we get

$$-a_k \ge u \ge -\sum_{j=1}^{k-1} \frac{a_k}{2^{j+1}} - a_k - \sum_{j=k+1}^{\infty} \frac{a_k}{2^{j+1}} \ge -\frac{3}{2}a_k.$$

Hence $u + 3a_k/2 \ge 0$ on ∂G_k and the set $F_k = \{6(u + \frac{3}{2}a_k) < u_k\}$ compactly belongs to G_k. Moreover, if $z \in \partial G(t_k\zeta_k, e^{-2})$ then

$$6\left(u(z) + \frac{3}{2}a_k\right) \le 6\left(u_k(z) + \frac{3}{2}a_k\right) = -3a_k < -2a_k = u_k(z).$$

Thus $G(t_k\zeta_k, e^{-2}) \subset F_k$. By the comparison principle

$$36\int_{G_k}(dd^c u)^2 = \int_{G_k}(dd^c 6(u(z)+\frac{3}{2}a_k))^2 \geq \int_{F_k}(dd^c u_k)^2 = a_k^2.$$

Hence by Lelong–Jensen formula

$$\|f\|^p_{H^p_u} \geq \int_{\mathbb{D}^2}|f|^p(dd^c u)^2 \geq \sum_{k=1}^{\infty}\int_{G_k}|f|^p(dd^c u)^2 \geq \frac{1}{36\cdot 2^p}\sum_{k=0}^{\infty}|f^*(\zeta_k)|^p a_k^2 = \infty.$$

Hence $f \notin H^p(\mathbb{D}^2)$. □

The following corollary shows the existence of nontrivial Poletsky–Stessin Hardy spaces on the bidisk.

Corollary 2. *For every $p \geq 1$ there exists a function $u \in \mathscr{E}_1(\mathbb{D}^2)$ such that $H^p_u(\mathbb{D}^2) \nsubseteq H^p(\mathbb{D}^2)$.*

Proof. Take $f \in H^p(\mathbb{D}^2)$ that is unbounded. Then the non-tangential limit f^* on $\mathbb{T}^2$ must be unbounded because otherwise

$$f(z) = \int_{\mathbb{T}^2} P(z,\zeta)f^*(\zeta)\,dm$$

would imply that $f(z)$ is bounded. So there exists a set of points $\{\zeta_j\} \in \mathbb{T}^2$ such that $\lim_{j\to\infty}|f^*(\zeta_j)| = \infty$. Hence the corollary follows from Theorem 13. □

Now we prove the most important theorem of this section.

Theorem 14. *Let $p \geq 1$. Then*

$$\bigcap_{u\in\mathscr{E}_1(\mathbb{D}^2)} H^p_u(\mathbb{D}^2) = H^\infty(\mathbb{D}^2).$$

Proof. Let $f \in \bigcap_{u\in\mathscr{E}_1(\mathbb{D}^2)} H^p_u(\mathbb{D}^2)$. Then the non-tangential limit f^* on $\mathbb{T}^2$ is bounded because otherwise by Theorem 13 there would exist a $u \in \mathscr{E}_1(\mathbb{D}^2)$ such that $f \notin H^p_u(\mathbb{D}^2)$. Thus, since f^* is bounded,

$$f(z) = \int_{\mathbb{T}^2} P(z,\zeta)f^*(\zeta)\,dm$$

implies that $f \in H^\infty(\mathbb{D}^2)$. □

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