EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 9, No. 2, 2016, 240-243
ISSN 1307-5543 – www.ejpam.com

On the Exponential Diophantine Equation $(M_{pq})^x + (M_{pq}+1)^y = z^2$

Azizul Hoque

Department of Mathematics, Gauhati University, Guwahati, India-781014

Abstract. In this paper, we consider the number $M_{pq} = p^q - 1$, where $p > 0$ and $q > 1$ are integers, and the Exponential Diophantine equation $(M_{pq})^x + (M_{pq}+1)^y = z^2$, where x, y and z are positive integers. We find the solutions to the title equation expect the case only when both p and y are odd integers.

2010 Mathematics Subject Classifications: 11D61; 11D41

Key Words and Phrases: Exponential Diophantine Equation

1. Introduction

Sroysang [2] established that the Exponential Diophantine equation $31^x + 32^y = z^2$ has no non-negative solution. Recently, Sroysang [3] also showed that the Exponential Diophantine equation $7^x + 8^y = z^2$ has only one solution, that is $(x, y, z) = (0, 1, 3)$. Sroysang [3] introduced an open problem regarding the set of all solutions (x, y, z) for the Exponential Diophantine equation $p^x + (p+1)^y = z^2$, where x, y and z are non-negative integers.

In this paper, we consider the number $M_{pq} = p^q - 1$, where $p > 0$ and $q > 1$ are integers, and the Exponential Diophantine equation $(M_{pq})^x + (M_{pq}+1)^y = z^2$, where x, y and z are positive integers. We show that $(M_{pq}, x, y, z) = (7, 0, 1, 3)$ and $(M_{pq}, x, y, z) = (3, 2, 2, 5)$ are the only solutions to the above equation except the case when both p and y are odd integers.

2. Main Results

In this article, we use Catalan's conjecture [1], which states that the only solution in integers $a > 1$, $b > 1$, $x > 1$, $y > 1$ to the equation $a^x - b^y = 1$ is $(a, b, x, y) = (3, 2, 2, 3)$.

We shall now solve the exponential Diophantine equation $(M_{pq})^x + (M_{pq}+1)^y = z^2$, where x, y, z, p, q are non-negative integers and $M_{pq} = p^q - 1$ with $q > 1$. We exclude the case when both p and y are odd positive integers.

Email address: ahoque.ms@gmail.com

Theorem 1. *The Exponential Diophantine equation*

$$(M_{pq})^x + (M_{pq}+1)^y = z^2 \tag{1}$$

except the case when both p and y are odd positive integers, has only two solutions in non-negative integer, $(M_{pq}, x, y, z) = (7, 0, 2, 3)$ *and* $(M_{pq}, x, y, z) = (3, 2, 2, 5)$.

Proof. We divide the problem into two cases.

Case 1: Let p be an even positive integer. Then $M_{pq} \equiv 3(mod\ 4)$.

From Eq.(1) we observe that z must be odd, and thus $z^2 \equiv 1(mod\ 4)$ and $M_{pq}+1 \equiv 0(mod\ 4)$.

Sub-case 1.1: Let $x = 0$, then Eq. (1) becomes

$$1 + (M_{pq}+1)^y = z^2.$$

This gives $p^{qy} = z^2 - 1$ and thus $p^{qy} = (z+1)(z-1)$. Hence there exists non-negative integers m and n such that $p^m = z+1$ and $p^n = z-1$, where $m > n$ and

$$m + n = qy \tag{2}$$

Now we have,

$$p^n(p^{m-n} - 1) = p^m - p^n = (z+1) - (z-1) = 2.$$

This implies $p = 2$, $n = 1$ and $m = 2$. Thus Eq. (2) gives $qy = 3$ and hence either $q = 1$, $y = 3$ or $q = 3$, $y = 1$.

Since $q > 1$, so that $q = 3$, $y = 1$.

Now $z = p^n + 1 = 3$ and $M_{pq} = 7$. Hence $(M_{pq}, x, y, z) = (7, 0, 1, 3)$ is the only solution to the Eq. (1) in this sub-case.

Sub-case 1.2: Let $x \geq 1$.

Since $(M_{pq}+1)^y \equiv 0(mod\ 4)$ and $z^2 \equiv 1(mod\ 4)$, the Eq. (1) gives $(M_{pq})^x \equiv 1(mod\ 4)$. Again since $M_{pq} \equiv 3(mod\ 4)$, x must be even.

Let $x = 2k$ for some integer $k \geq 1$. Then Eq. (1) implies

$$(M_{pq})^{2k} + p^{qy} = z^2.$$

This gives $p^{qy} = z^2 - ({M_{pq}}^k)^2$ and thus $p^{qy} = (z + {M_{pq}}^k)(z - {M_{pq}}^k)$. Hence there exists non-negative integers i and j such that $p^i = z + {M_{pq}}^k$ and $p^j = z - {M_{pq}}^k$, where $i > j$ and

$$i + j = qy. \tag{3}$$

Now we have $p^j(p^{i-j} - 1) = p^i - p^j = 2(M_{pq})^k$.

Since p is even, let $p = 2t$ for some positive integer t. Then we have

$$2^{j-1} t^j (p^{i-j} - 1) = (M_{pq})^k. \tag{4}$$

If $t > 1$ then $t \mid (M_{pq})^k$ and hence $p \mid 2(M_{pq})^k$. Since $gcd(p, M_{pq}) = 1$, we have $p \mid 2$, a contradiction. Hence $t = 1$ and $p = 2$.

Now Eq. (4) gives, $j = 1$ and it becomes,

$$p^{i-1} - 1 = (M_{pq})^k \tag{5}$$

By using Catalan's Conjecture, the equation $p^{i-1} - (M_{pq})^k = 1$ has only one solution $(p, M_{pq}, i-1, k) = (3, 2, 2, 3)$ only when $i > 2$ and $k > 1$. But since $p = 2$, Eq. (5) has no solution only when $i > 2$ and $k > 1$.

It is now remaining to examine only when either $i \geq 2$ or $k \geq 1$. But we have $i > 1$, $q > 1$, $k \geq 1$ and Eq. (3) gives $i + 1 = qy$. Thus we get $i = 2$, $q = 3$, $y = 1$ or $k = 1$.

Now if $i = 2$, $q = 3$ and $y = 1$, then Eq.(5) gives,

$$p - 1 = (M_{pq})^k \Rightarrow 1 = (M_{pq})^k \Rightarrow k = 0$$

This contradicts to $k \geq 1$. Hence Eq. (1) has no solution in this case.

Again, if $k = 1$, then Eq. (5) gives

$$\begin{aligned} p^{i-1} - 1 =& M_{pq} \\ &\Rightarrow p^{i-1} - 1 = p^q - 1 \\ &\Rightarrow i - 1 = q \\ &\Rightarrow qy - 2 = q \\ &\Rightarrow q(y-1) = 2 \\ &\Rightarrow q = 2, y = 2. \end{aligned}$$

Thus we have $M_{pq} = 3$, $x = 2k = 2$, $y = 2$ and $z = p^j + (M_{pq})^k = 5$. Therefore $(M_{pq}, x, y, z) = (3, 2, 2, 5)$ is the only solution to Eq. (1) in this sub-case.

Case 2: Let p be an odd positive integer. Then $M_{pq} \equiv 0(mod\ 4)$. From Eq. (1) we observe that z must be odd, and thus $z^2 \equiv 1(mod\ 4)$.

Sub-case 2.1: Let $y = 0$. Then Eq. (1) becomes $(M_{pq})^x + 1 = z^2$. This implies $(M_{pq})^x = (z+1)(z-1)$ and thus there exists non-negative integers a, b such that $(M_{pq})^a = z+1$ and $(M_{pq})^b = z - 1$, where $a > b$ and $x = a + b$.

Now $(M_{pq})^b(M_{pq}^{a-b} - 1) = (M_{pq})^a - (M_{pq})^b = 2$. This gives $2 \equiv 0(mod\ 4)$, an absurdity.

Thus there is no solution to Eq. (1) in this sub-case.

Sub-case 2.2: Let $y \geq 1$ even integer and let $y = 2k$. Then Eq. (1) becomes

$$(M_{pq})^x + (M_{pq} + 1)^{2k} = z^2.$$

This equation implies

$$(M_{pq})^x = z^2 - (p^{kq})^2 = (z + p^{kq})(z - p^{kq}).$$

Thus there are non-negative integers c, d such that $(M_{pq})^c = z + p^{kq}$ and $(M_{pq})^d = z - p^{kq}$, where $c > d$ and $c + d = x$.

Now,

$$(M_{pq})^d(M_{pq}^{c-d} - 1) = (M_{pq})^c - (M_{pq})^d = 2p^{kq} = 2(M_{pq} + 1)^k.$$

This implies $0 \equiv 2(mod\ 4)$. This is an absurdity. Hence Eq. (1) has no solution in this case. □

ACKNOWLEDGEMENTS The Author acknowledges UGC for JRF Fellowship (No.GU/ UGC/VI(3)/JRF/2012 /2985).

References

[1] P. Mihailescu. *Primary cyclotomic units and a proof of Catalan's conjecture*, Journal für die reine und angewandte Mathematik, 27, 167-195. 2004.

[2] B. Sroysang. *On the Diophantine equation* $31^x + 32^y = z^2$, International Journal of Pure and Applied Mathematics, 81, 609-612. 2012.

[3] B. Sroysang. *On the Diophantine equation* $7^x + 8^y = z^2$, International Journal of Pure and Applied Mathematics, 84, 111-114. 2013.