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Some Properties of the p-adic Beta Function

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Abstract. In the present work we consider a p-adic analogue of the classical beta function by using Y. Morita's p-adic gamma function. We obtain some elementary properties of the p-adic beta function. We give some relations between the classical beta and the p-adic beta functions at the values of natural numbers.

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1. Introduction

Let p be fixed prime number. It is well known that the p—adic valuation of any $x \in \mathbb{Q}$, $x \neq 0$ is determined by the formula

$$x = p^{v_p(x)} . \frac{a}{b}$$

where $v_p(x) \in \mathbb{Z}$ and ab is not divided by p. The p-adic norm $|\cdot|_p$ is defined by

$$|x|_p = \begin{cases} p^{-v_p(x)}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

By $\mathbb{Q}_p$ we denote the completion of rational numbers field $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$. The ring of p-adic integers is the valuation ring

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}.$$

Note that every $x \in \mathbb{Z}_p$ can be written in the form

$$x = b_0 + b_1 p + b_2 p^2 + \ldots + b_n p^n + \ldots$$

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with $0 \leq b_i \leq p-1$; and also, every $x \in \mathbb{Q}_p$ can be written in the form

$$x = b_{-n_0}p^{-n_0} + \ldots + b_0 + b_1 p + b_2 p^2 + \ldots + b_n p^n + \ldots = \sum_{n \geq -n_0} b_n p^n$$

with $0 \leq b_i \leq p-1$ and $-n_0 = v_p(x)$ (for details see [12]).

The classical gamma function is an extension of the factorial function and is defined by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

for all $\text{Re}(x) > 0$ [1]. The basic properties of the classical gamma function are following:

(i) $\Gamma(n+1) = n!$ for all non negative integer n

(ii) $\Gamma(z+1) = z\Gamma(z)$ $(Re(z) > 0)$

(iii) $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ $(Re(z) > 0)$

(iv) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

It is well known that the classical beta function $B(x,y)$ is defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and it has the integral representation

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

for all $\text{Re}(x)$, $\text{Re}(y) > 0$. The basic properties of the classical beta function are the following:

(i) $B(x,y) = B(y,x)$

(ii) $B(x+1,y) = B(x,y)\frac{x}{x+y}$

(iii) $B(x,y+1) = B(x,y)\frac{y}{x+y}$

(iv) $B(x,y)B(x+y,1-y) = \frac{\pi}{x\sin(\pi y)}$

(v) $\binom{n}{k} = \frac{1}{(n+1)B(n-k+1,k+1)}$ $(n,k \in \mathbb{N}, k \leq n)$

(vi) $B(\frac{1}{2},\frac{1}{2}) = \pi$

(vii) $B(x+1,y)+B(x,y+1)=B(x,y)$

(viii) $B(x,y+1)=\frac{y}{x}B(x+1,y)=\frac{y}{x+y}B(x,y)$

(ix) $B(x,y)B(x+y,z)B(x+y+z,w)=\frac{\Gamma(x)\Gamma(y)\Gamma(z)\Gamma(w)}{\Gamma(x+y+z+w)}$

where $\operatorname{Re}(x), \operatorname{Re}(y), \operatorname{Re}(z), \operatorname{Re}(w)>0$. The p-adic analogue of the classical gamma function depends on the p-adic version of factorial function. The p-adic version of factorial function is defined by

$$(n!)_p := \prod_{\substack{1\le j\le n \\ (j,p)=1}} j$$

The function $f(n)=(-1)^{n+1}(n!)_p$ can be interpolated and the p-adic gamma function Γ_p is defined as follows:

Definition 1 ([10]). *The p-adic gamma function Γ_p is the continuous extension to $\mathbb{Z}_p$ of*

$$n \mapsto (-1)^n \prod_{\substack{1\le j< n \\ (j,p)=1}} j (n\ge 2).$$

Moreover, $\Gamma_p:\mathbb{Z}_p\to\mathbb{Q}_p$ function is defined by

$$\Gamma_p(x):=\lim_{n\to x}(-1)^n \prod_{\substack{1\le j< n \\ (j,p)=1}} j.$$

According the definition of p-adic factorial function we conclude that:

Corollary 1. $\Gamma_p(n+1)=(-1)^{n+1}(n!)_p$ $(n\in\mathbb{N})$.

To prove our results we use the following properties of p-adic gamma function:

Proposition 1 ([12]). *Let $p\ne 2$. Then Γ_p has the following properties:*

(i) For all $x\in\mathbb{Z}_p$

$$\Gamma_p(x+1)=h_p(x)\Gamma_p(x) \tag{1}$$

where

$$h_p(x):=\begin{cases}-x & \text{if } |x|_p=1\\ -1 & \text{if } |x|_p<1\end{cases}$$

(ii) $\Gamma_p(0)=1$, $\Gamma_p(1)=-1$, $\Gamma_p(2)=1$. *For all $x\in\mathbb{Z}_p$ we have* $\left|\Gamma_p(x)\right|_p=1$

(iii) For all $x,y\in\mathbb{Z}_p$

$$\left|\Gamma_p(x)-\Gamma_p(y)\right|_p\le\left|x-y\right|_p. \tag{2}$$

Also, the properties (i) and (ii) hold for $p = 2$, and the instead of (iii) the relations

$$\left|\Gamma_2(x) - \Gamma_2(y)\right|_2 \leq \left|x - y\right|_2 \ (x, y \in \mathbb{Z}_2, \left|x - y\right|_2 \neq \frac{1}{4})$$
$$\left|\Gamma_2(x) - \Gamma_2(y)\right|_2 \leq 2\left|x - y\right|_2 \ (x, y \in \mathbb{Z}_2, \left|x - y\right|_2 = \frac{1}{4})$$

hold.

Proposition 2 ([12])**.** *A formula for $\Gamma_p(-n)$ $(n \in \mathbb{N})$ is given by*

$$\Gamma_p(-n) = (-1)^{n+1-\left[\frac{n}{p}\right]}(\Gamma_p(n+1))^{-1}. \tag{3}$$

Proposition 3 ([7, 12])**.** *If $p \neq 2$ then*

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\ell(x)} \ (x \in \mathbb{Z}_p) \tag{4}$$

and for $p = 2$

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\sigma_1(x)+1} \ (x \in \mathbb{Z}_2) \tag{5}$$

where $\ell : \mathbb{Z}_p \to \{1, 2, \ldots, p\}$ assigns to $x \in \mathbb{Z}_p$ its residue $\in \{1, 2, \ldots, p\}$ modulo $p\mathbb{Z}_p$ and where σ_1 is defined by the formula

$$\sigma_1(\sum_{j=0}^{\infty} a_j 2^j) = a_1$$

Corollary 2 ([12])**.** *Let $p \neq 2$. We get*

$$\Gamma_p(\frac{1}{2})^2 = (-1)^{\ell(\frac{1}{2})} \tag{6}$$

Now $\ell(\frac{1}{2}) = \ell(\frac{1}{2}(p+1)) = \frac{1}{2}(p+1)$ so that

$$\Gamma_p(\frac{1}{2})^2 = \begin{cases} 1 & \text{if } p \equiv 3 \pmod 4 \\ -1 & \text{if } p \equiv 1 \pmod 4 \end{cases} \tag{7}$$

Proposition 4 ([12])**.** *Let $n \in \mathbb{N}$ and let s_n be sum of the digits of $n = \sum_{j=0}^{s} a_j p^j$ $(a_s \neq 0)$ in base p. Then*

(i) $\Gamma_p(n+1) = (-1)^{n+1} \frac{n!}{\left[\frac{n}{p}\right]! p^{\left[\frac{n}{p}\right]}}$

(ii) $\Gamma_p(p^n) = (-1)^p \frac{p^n!}{p^{n-1}! p^{p^{n-1}}}$

(iii) $n! = (-1)^{n+1-s}(-p)^{(n-s_n)/(p-1)} \prod_{j=0}^{n} \Gamma_p\left(\left[\frac{n}{p^j}\right] + 1\right)$

(iv) $p^n! = (-1)^p(-p)^{(p^n-1)/(p-1)} \prod_{j=0}^{n} \Gamma_p\left(p^j\right).$

The p-adic gamma function have been considered by many authors (see [3–10, 13]). We note that another p-adic analogue of classical gamma function was constructed by G. Overholtzer [11], but we consider Morita's p-adic gamma function.

In 1980 the p-adic beta function is used in Dwork cohomology and an cohomological interpretation of p-adic beta function is given by M. Boyarsky [4]. In 2006 F. Baldassarri [2] considered two constructions of the p-adic beta functions as the p-adic etale and p-adic crystalline beta functions. Also, some comparisons between the p-adic etale and p-adic crystalline beta functions with relations via Fontaine's periods are given.

In the present work we study a p-adic analogue of classical beta function by using Morita's p-adic gamma function, and we obtain some elemantary properties of the p-adic beta function.

2. Main Results

Naturally a p-adic analogue of the classical beta function can be defined as follows.

Definition 2. *The p-adic beta function $B_p : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Q}_p$ is defined by the formula*

$$B_p(x,y) := \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)}, x, y \in \mathbb{Z}_p. \tag{8}$$

We investigate some properties of the p-adic beta function. Now, we give basic properties of the p-adic beta function.

Theorem 1. *The p-adic beta function is symmetric. Namely,*

$$B_p(x,y) = B_p(y,x)$$

for $x, y \in \mathbb{Z}_p$.

Proof. From Definition 2, we can prove that the p-adic beta function is symmetric:

$$\begin{aligned} B_p(x,y) &= \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)} \\ &= \frac{\Gamma_p\left(y\right)\Gamma_p(x)}{\Gamma_p(y+x)} \\ &= B_p(y,x) \end{aligned}$$

for $x, y \in \mathbb{Z}_p$. □

Theorem 2. *For $x, y \in \mathbb{Z}_p$, then*

$$B_p(x,y)B_p(x+y,1-y) = \begin{cases} \frac{(-1)^{\ell(y)}}{h_p(x)}, & p \neq 2 \\ \frac{(-1)^{\sigma_1(y)+1}}{h_p(x)} & p = 2 \end{cases}$$

where

$$h_p(x) := \begin{cases} -x & \text{if } |x|_p = 1 \\ -1 & \text{if } |x|_p < 1 \end{cases}$$

and $\ell : \mathbb{Z}_p \to \{1,2,\ldots,p\}$ assigns to $x \in \mathbb{Z}_p$ its residue $\in \{1,2,\ldots,p\}$ modulo $p\mathbb{Z}_p$ and σ_1 is defined by the formula

$$\sigma_1\Big(\sum_{j=0}^{\infty} a_j 2^j\Big) = a_1$$

Proof. Let $p \neq 2$. From Definition 2 and Proposition 1 it follows that

$$\begin{aligned} B_p(x,y)B_p(x+y,1-y) &= \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)} \frac{\Gamma_p\left(x+y\right)\Gamma_p(1-y)}{\Gamma_p(x+1)} \\ &= \frac{\Gamma_p(x)\Gamma_p(y)\Gamma_p(1-y)}{\Gamma_p(x+1)} \\ &= \frac{\Gamma_p(x)\Gamma_p(y)\Gamma_p(1-y)}{\Gamma_p(x)h_p(x)} \\ &= \frac{\Gamma_p(y)\Gamma_p(1-y)}{h_p(x)}, \end{aligned}$$

and by Proposition 3 we obtain that

$$B_p(x,y)B_p(x+y,1-y) = \frac{(-1)^{\ell(y)}}{h_p(x)}.$$

In similar way, we can prove the theorem for $p = 2$. □

Theorem 3. *The equality*

$$B_p(x+1,y) = \frac{h_p(x)}{h_p(x+y)} B_p(x,y)$$

holds for all $x, y \in \mathbb{Z}_p$.

Proof. By using Definition 2 and Proposition 1 we have that

$$B_p(x+1,y) = \frac{\Gamma_p(x+1)\Gamma_p(y)}{\Gamma_p(x+1+y)}$$

$$
\begin{aligned}
&=\frac{\Gamma_p(x)h_p(x)\Gamma_p(y)}{\Gamma_p((x+y)+1)}\\
&=\frac{\Gamma_p(x)h_p(x)\Gamma_p(y)}{\Gamma_p(x+y)h_p(x+y)}\\
&=\frac{h_p(x)}{h_p(x+y)}\frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)}\\
&=\frac{h_p(x)}{h_p(x+y)}B_p(x,y).
\end{aligned}
$$

□

Theorem 4. *The equality*

$$B_p(x,y+1)=\frac{h_p(y)}{h_p(x+y)}B_p(x,y)$$

holds for all $x,y\in\mathbb{Z}_p$.

Proof. From Definition 2 and Proposition 1 we get

$$
\begin{aligned}
B_p(x,y+1)&=\frac{\Gamma_p(x)\Gamma_p(y+1)}{\Gamma_p(x+y+1)}\\
&=\frac{\Gamma_p(x)h_p(y)\Gamma_p(y)}{\Gamma_p((x+y)+1)}\\
&=\frac{\Gamma_p(x)h_p(y)\Gamma_p(y)}{\Gamma_p(x+y)h_p(x+y)}\\
&=\frac{h_p(y)}{h_p(x+y)}\frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)}\\
&=\frac{h_p(y)}{h_p(x+y)}B_p(x,y).
\end{aligned}
$$

□

Corollary 3. *The relation*

$$B_p(x+1,y)+B_p(x,y+1)=\frac{h_p(x)+h_p(y)}{h_p(x+y)}B_p(x,y)$$

holds for all $x,y\in\mathbb{Z}_p$.

Proof. According to Theorem 3 and Theorem 4 we have

$$B_p(x+1,y)+B_p(x,y+1)=\frac{h_p(x)}{h_p(x+y)}B_p(x,y)+\frac{h_p(y)}{h_p(x+y)}B_p(x,y)$$

$$= \frac{h_p(x)+h_p(y)}{h_p(x+y)} B_p(x,y).$$

□

Corollary 4. *For all $x, y \in \mathbb{Z}_p$, the equality*

$$B_p(x,y+1) = \frac{h_p(y)}{h_p(x)} B_p(x+1,y)$$

holds.

Proof. It follows from Theorem 3 that

$$B_p(x,y) = \frac{h_p(x+y)}{h_p(x)} B_p(x+1,y). \tag{9}$$

Using (9) in Theorem 4 we obtain that

$$\begin{aligned} B_p(x,y+1) =& \frac{h_p(y)}{h_p(x+y)} \frac{h_p(x+y)}{h_p(x)} B_p(x+1,y) \\ =& \frac{h_p(y)}{h_p(x)} B_p(x+1,y). \end{aligned}$$

□

Theorem 5. *The equality*

$$B_p(x+1,y+1) = \frac{h_p(x)h_p(y)}{h_p(x+y+1)h_p(x+y)} B_p(x,y)$$

holds for all $x, y \in \mathbb{Z}_p$.

Proof. In similar way, we obtain that

$$\begin{aligned} B_p(x+1,y+1) =& \frac{\Gamma_p(x+1)\Gamma_p(y+1)}{\Gamma_p(x+1+y+1)} \\ =& \frac{\Gamma_p(x)h_p(x)\Gamma_p(y)h_p(y)}{\Gamma_p((x+y+1)+1)} \\ =& \frac{\Gamma_p(x)h_p(x)\Gamma_p(y)h_p(y)}{\Gamma_p(x+y+1)h_p(x+y+1)} \\ =& \frac{h_p(x)h_p(y)}{h_p(x+y+1)} \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p((x+y)+1)} \end{aligned}$$

$$=\frac{h_p(x)h_p(y)}{h_p(x+y+1)}\frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)h_p(x+y)}$$
$$=\frac{h_p(x)h_p(y)}{h_p(x+y+1)h_p(x+y)}B_p(x,y).$$

□

Corollary 5. *For all* $x, y, z \in \mathbb{Z}_p$

$$B_p(x,y)B_p(x+y,z)B_p(x+y+z,w)=\frac{\Gamma_p(x)\Gamma_p(y)\Gamma_p(z)\Gamma_p(w)}{\Gamma_p(x+y+z+w)}$$

Proof. It is clear from Definition 2 that

$$B_p(x,y)B_p(x+y,z)B_p(x+y+z,w)=\frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)}\frac{\Gamma_p(x+y)\Gamma_p(z)}{\Gamma_p(x+y+z)}\frac{\Gamma_p(x+y+z)\Gamma_p(w)}{\Gamma_p(x+y+z+w)}$$
$$=\frac{\Gamma_p(x)\Gamma_p(y)\Gamma_p(z)\Gamma_p(w)}{\Gamma_p(x+y+z+w)}.$$

□

Theorem 6. *The equality*

$$B_p(x,1-x)=\begin{cases}(-1)^{\ell(x)+1} & \text{if } p\neq 2\\ (-1)^{\sigma_1(y)+2} & \text{if } p=2\end{cases}$$

holds for all $x, y \in \mathbb{Z}_p$.

Proof. Note that $\Gamma_p(1)=-1$. By Definition 2 we get

$$B_p(x,1-x)=\frac{\Gamma_p(x)\Gamma_p(1-x)}{\Gamma_p(x+1-x)}$$
$$=\frac{\Gamma_p(x)\Gamma_p(1-x)}{\Gamma_p(1)}.$$

By Proposition 3, if $p\neq 2$ then

$$B_p(x,1-x)=-(-1)^{\ell(x)}=(-1)^{\ell(x)+1}$$

and, if $p=2$ then,

$$B_p(x,1-x)=-(-1)^{\sigma_1(y)+1}=(-1)^{\sigma_1(y)+2}.$$

□

It is well known that the classical beta function can be defined as binomial coefficient indices. We can give a similar formula for the p-adic beta function.

Theorem 7. *The equality*

$$\binom{n}{k}_p B_p(n-k+1,k+1) = \frac{-1}{h_p(n+1)}$$

holds for all $n,k \in \mathbb{N}, k \leq n$*. Here, the notation* $\binom{n}{k}_p$ *is defined by*

$$\binom{n}{k}_p = \frac{(n!)_p}{((n-k)!)_p(k!)_p}.$$

Proof. It is well known that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

for $n,k \in \mathbb{N}, k \leq n$ and

$$(n!)_p = (-1)^{n+1}\Gamma_p(n+1).$$

Hence, we can write

$$\begin{aligned}\binom{n}{k}_p B_p(n-k+1,k+1) &= \frac{(n!)_p}{(k!)_p((n-k)!)_p}\frac{\Gamma_p\,(n-k+1)\,\Gamma_p(k+1)}{\Gamma_p(n+2)} \\ &= \frac{(-1)^{n+1}\Gamma_p\,(n+1)}{(-1)^{k+1}\Gamma_p(k+1)(-1)^{n-k+1}\Gamma_p(n-k+1)}\frac{\Gamma_p\,(n-k+1)\,\Gamma_p(k+1)}{\Gamma_p(n+2)} \\ &= \frac{-\Gamma_p\,(n+1)}{\Gamma_p(n+2)}.\end{aligned}$$

Thus, by Proposition 1, we obtain that

$$\begin{aligned}\binom{n}{k}_p B_p(n-k+1,k+1) &= \frac{-\Gamma_p\,(n+1)}{\Gamma_p(n+1)h_p(n+1)} \\ &= \frac{-1}{h_p(n+1)}.\end{aligned}$$

□

Now, we analyze the relationship between the classical beta and the p-adic beta function at the values of natural numbers.

Theorem 8. *The equality*

$$B(n+1,m+1) = -B_p(n,m)\frac{h_p(n)h_p(m)}{h_p(m+n)(m+n+1)}\frac{\left[\frac{n}{p}\right]!\left[\frac{m}{p}\right]!}{\left[\frac{m+n}{p}\right]!}p^{\left[\frac{n}{p}\right]+\left[\frac{m}{p}\right]-\left[\frac{m+n}{p}\right]}$$

holds for all $m,n \in \mathbb{N}$.

Proof. It follows from the definition of classical beta function and main proposition of classical gamma function that

$$\begin{aligned}B(n+1,m+1) &= \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)}\\ &= \frac{\Gamma(n+1)\Gamma(m+1)}{(n+m+1)\Gamma(m+n+1)}\\ &= \frac{n!m!}{(n+m+1)(m+n)!}.\end{aligned}$$

By Proposition 4(i) we have

$$B(n+1,m+1) = \frac{(-1)^{n+1}\Gamma_p(n+1)\left[\frac{n}{p}\right]!p^{\left[\frac{n}{p}\right]}(-1)^{m+1}\Gamma_p(m+1)\left[\frac{m}{p}\right]!p^{\left[\frac{m}{p}\right]}}{(n+m+1)(-1)^{m+n+1}\Gamma_p(m+n+1)\left[\frac{m+n}{p}\right]!p^{\left[\frac{m+n}{p}\right]}}.$$

Then, by Proposition 1 we obtain

$$B(n+1,m+1) = (-1)\frac{\Gamma_p(n)\Gamma_p(m)h_p(n)h_p(m)}{\Gamma_p(n+m)h_p(n+m)}\frac{\left[\frac{n}{p}\right]!\left[\frac{m}{p}\right]!}{\left[\frac{m+n}{p}\right]!}\frac{p^{\left[\frac{n}{p}\right]+\left[\frac{m}{p}\right]-\left[\frac{m+n}{p}\right]}}{(n+m+1)}.$$

Thus, using Definition 2 we complete the proof of the theorem

$$B(n+1,m+1) = -B_p(n,m)\frac{\left[\frac{n}{p}\right]!\left[\frac{m}{p}\right]!}{\left[\frac{m+n}{p}\right]!}\frac{p^{\left[\frac{n}{p}\right]+\left[\frac{m}{p}\right]-\left[\frac{m+n}{p}\right]}}{(n+m+1)}\frac{h_p(n)h_p(m)}{h_p(n+m)}.$$

□

Theorem 9. *The equality*

$$B(n+1,m+1) = B_p(n+1,m+1)\frac{\left[\frac{n}{p}\right]!\left[\frac{m}{p}\right]!}{\left[\frac{m+n+1}{p}\right]!}p^{\left[\frac{n}{p}\right]+\left[\frac{m}{p}\right]-\left[\frac{m+n+1}{p}\right]}$$

holds for all $m,n \in \mathbb{N}$.

Proof. In similar way, using the definitions and Proposition 4(i) we can obtain that

$$\begin{aligned}B(n+1,m+1) &= \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)}\\ &= \frac{n!m!}{(n+m+1)!}\\ &= \frac{(-1)^{n+1}\Gamma_p(n+1)\left[\frac{n}{p}\right]!p^{\left[\frac{n}{p}\right]}(-1)^{m+1}\Gamma_p(m+1)\left[\frac{m}{p}\right]!p^{\left[\frac{m}{p}\right]}}{(-1)^{m+n+2}\Gamma_p(m+n+2)\left[\frac{m+n+1}{p}\right]!p^{\left[\frac{m+n+1}{p}\right]}}\end{aligned}$$

$$=\frac{\Gamma_p(n+1)\Gamma_p(m+1)}{\Gamma_p(m+n+2)}\frac{\left[\frac{n}{p}\right]!\left[\frac{m}{p}\right]!}{\left[\frac{m+n+1}{p}\right]!}p^{\left[\frac{n}{p}\right]+\left[\frac{m}{p}\right]-\left[\frac{m+n+1}{p}\right]}$$
$$=B_p(n+1,m+1)\frac{\left[\frac{n}{p}\right]!\left[\frac{m}{p}\right]!}{\left[\frac{m+n+1}{p}\right]!}p^{\left[\frac{n}{p}\right]+\left[\frac{m}{p}\right]-\left[\frac{m+n+1}{p}\right]}.$$

□

Theorem 10. *The equality*

$$B(p^n+1,p^m+1)=B_p(p^n,p^m)\frac{(p^{n-1})!(p^{m-1})!}{(p^{n-1}+p^{m-1})!}\frac{1}{h_p(p^n+p^m)(p^n+p^m+1)}$$

holds for all $m,n\in\mathbb{N}$.

Proof. From the definition of classical beta function and main proposition of classical gamma function follow that

$$B(p^n+1,p^m+1)=\frac{\Gamma(p^n+1)\Gamma(p^m+1)}{\Gamma(p^n+p^m+2)}=\frac{p^n!p^m!}{(p^n+p^m+1)(p^n+p^m)!}$$

By Proposition 4 (i) and (ii) we get

$$B(p^n+1,p^m+1)=\frac{\Gamma_p(p^n)(-1)^p(p^{n-1})!p^{p^{n-1}}\Gamma_p(p^m)(-1)^p(p^{m-1})!p^{p^{m-1}}}{(p^n+p^m+1)\Gamma_p(p^n+p^m+1)(-1)^{p^n+p^m}\left[\frac{p^n+p^m}{p}\right]!p^{\left[\frac{p^n+p^m}{p}\right]}}$$

Hence, we obtain

$$B(p^n+1,p^m+1)=\frac{\Gamma_p(p^n)\Gamma_p(p^m)}{\Gamma_p(p^n+p^m)}\frac{(p^{n-1})!(p^{m-1})!p^{p^{n-1}}p^{p^{m-1}}}{h_p(p^n+p^m)(p^{n-1}+p^{m-1})!p^{p^{n-1}+p^{m-1}}(p^n+p^m+1)}$$
$$=B_p(p^n,p^m)\frac{(p^{n-1})!(p^{m-1})!}{(p^{n-1}+p^{m-1})!}\frac{1}{h_p(p^n+p^m)(p^n+p^m+1)}.$$

□

Corollary 6. *If* $p\neq 2$ *then*

$$B_p(\frac{1}{2},\frac{1}{2})=\begin{cases}-1 & \text{if } p\equiv 3 \pmod 4\\ 1 & \text{if } p\equiv 1 \pmod 4\end{cases}$$

Proof. Using Corollary 2 and Proposition 1, we have

$$B_p(\frac{1}{2},\frac{1}{2})=\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{2})}{\Gamma_p(1)}$$

$$=(-1)^{\ell(\frac{1}{2})+1}, \ \ell(\frac{1}{2}) = \ell(\frac{1}{2}(p+1)) = \frac{1}{2}(p+1)$$
$$=(-1).\begin{cases} 1 & \text{if } p \equiv 3 \pmod 4 \\ -1 & \text{if } p \equiv 1 \pmod 4 \end{cases}$$
$$= \begin{cases} -1 & \text{if } p \equiv 3 \pmod 4 \\ 1 & \text{if } p \equiv 1 \pmod 4 \end{cases}.$$

□

Now we prove that the p-adic beta function has the following properties for negative integers.

Theorem 11. *If $n, m \in \mathbb{N}$, then*

$$B_p(-n,-m) = (-1)^{\left[1+\left[\frac{n+m}{p}\right]-\left[\frac{n}{p}\right]-\left[\frac{m}{p}\right]\right]} \frac{h_p(n+m)}{h_p(n)h_p(m)} \frac{1}{B_p(n,m)}$$

Proof. By Definition 2 and Proposition 2 we get

$$B_p(-n,-m) = \frac{\Gamma_p(-n)\Gamma_p(-m)}{\Gamma_p(-n-m)}$$
$$= \frac{(-1)^{n+1-\left[\frac{n}{p}\right]}(\Gamma_p(n+1))^{-1}(-1)^{m+1-\left[\frac{m}{p}\right]}(\Gamma_p(m+1))^{-1}}{(-1)^{n+m+1-\left[\frac{n+m}{p}\right]}(\Gamma_p(n+m+1))^{-1}},$$

and by Proposition 1(i) we have

$$B_p(-n,-m) = (-1)^{1+\left[\frac{n+m}{p}\right]-\left[\frac{n}{p}\right]-\left[\frac{m}{p}\right]} \frac{\Gamma_p(n+m+1)}{\Gamma_p(n+1)\Gamma_p(m+1)}$$
$$= (-1)^{1+\left[\frac{n+m}{p}\right]-\left[\frac{n}{p}\right]-\left[\frac{m}{p}\right]} \frac{\Gamma_p(n+m)h_p(n+m)}{\Gamma_p(n)h_p(n)\Gamma_p(m)h_p(m)}$$
$$= (-1)^{1+\left[\frac{n+m}{p}\right]-\left[\frac{n}{p}\right]-\left[\frac{m}{p}\right]} \frac{h_p(n+m)}{h_p(n)h_p(m)} \frac{1}{B_p(n,m)}.$$

□

Theorem 12. *If $n, m \in \mathbb{N}$, then*

$$B_p(-n,m) = \begin{cases} (-1)^{m-\left[\frac{n}{p}\right]+\left[\frac{-m+n}{p}\right]} \frac{h_p(n-m)}{h_p(n)} B_p(n-m,m) & \text{if } m < n \\ (-1)^{n+1-\left[\frac{n}{p}\right]} \frac{1}{h_p(n)} (B_p(m-n,n))^{-1} & \text{if } n \leq m \end{cases}$$

Proof. We know that

$$B_p(-n,m) = \frac{\Gamma_p(-n)\Gamma_p(m)}{\Gamma_p(-n+m)}.$$

Assume that $m < n$. Then, by Proposition 2 we can write

$$\begin{aligned} B_p(-n,m) =& \frac{(-1)^{n+1-\left[\frac{n}{p}\right]}(\Gamma_p(n+1))^{-1}\Gamma_p(m)}{(-1)^{-m+n+1-\left[\frac{-m+n}{p}\right]}(\Gamma_p(-m+n+1))^{-1}} \\ =& (-1)^{n+1-\left[\frac{n}{p}\right]+m-n-1+\left[\frac{-m+n}{p}\right]}\frac{\Gamma_p(n-m+1)\Gamma_p(m)}{\Gamma_p(n+1)}. \end{aligned}$$

Using Proposition 1 we obtain

$$\begin{aligned} B_p(-n,m) =& (-1)^{m-\left[\frac{n}{p}\right]+\left[\frac{-m+n}{p}\right]}\frac{\Gamma_p(n-m)h_p(n-m)\Gamma_p(m)}{\Gamma_p(n)h_p(n)} \\ =& (-1)^{m-\left[\frac{n}{p}\right]+\left[\frac{-m+n}{p}\right]}\frac{h_p(n-m)}{h_p(n)}B_p(n-m,m). \end{aligned}$$

Assume that $n \leq m$. By Proposition 2 we get

$$\begin{aligned} B_p(-n,m) =& \frac{(-1)^{n+1-\left[\frac{n}{p}\right]}(\Gamma_p(n+1))^{-1}\Gamma_p(m)}{\Gamma_p(m-n)} \\ =& (-1)^{n+1-\left[\frac{n}{p}\right]}\frac{\Gamma_p(m)}{\Gamma_p(m-n)\Gamma_p(n+1)}, \end{aligned}$$

and by Proposition 1 we have

$$\begin{aligned} B_p(-n,m) =& (-1)^{n+1-\left[\frac{n}{p}\right]}\frac{\Gamma_p(m)}{\Gamma_p(m-n)\Gamma_p(n)h_p(n)} \\ =& \frac{(-1)^{n+1-\left[\frac{n}{p}\right]}}{h_p(n)}(B_p(m-n,n))^{-1} \end{aligned}$$

□

Theorem 13. *If* $n, m \in \mathbb{N}$ *then*

$$B_p(n,-m) = \begin{cases} \frac{(-1)^{m+1-\left[\frac{m}{p}\right]}}{h_p(m)}B_p(n-m,m)^{-1} & \text{if } m \leq n \\ \frac{(-1)^{n-\left[\frac{m}{p}\right]+\left[\frac{m-n}{p}\right]}h_p(m-n)}{h_p(m)}B_p(m-n,n) & \text{if } n < m \end{cases}$$

Proof. From Definition 2 we write

$$B_p(n,-m) = \frac{\Gamma_p(n)\Gamma_p(-m)}{\Gamma_p(n-m)}.$$

If $m \leq n$, using Proposition 2 we get

$$\begin{aligned} B_p(n,-m) &= \frac{\Gamma_p(n)(-1)^{m+1-\left[\frac{m}{p}\right]}\Gamma_p(m+1)^{-1}}{\Gamma_p(n-m)} \\ &= (-1)^{m+1-\left[\frac{m}{p}\right]}\frac{\Gamma_p(n)}{\Gamma_p(m+1)\Gamma_p(n-m)}. \end{aligned}$$

According to Proposition 1 and Definition 2 we have

$$\begin{aligned} B_p(n,-m) &= (-1)^{m+1-\left[\frac{m}{p}\right]}\frac{\Gamma_p(n)}{\Gamma_p(m)h_p(m)\Gamma_p(n-m)} \\ B_p(n,-m) &= \frac{(-1)^{m+1-\left[\frac{m}{p}\right]}}{h_p(m)}B_p(n-m,m)^{-1}. \end{aligned}$$

If $n < m$, then by Proposition 2 we have

$$\begin{aligned} B_p(n,-m) &= \frac{\Gamma_p(n)(-1)^{m+1-\left[\frac{m}{p}\right]}\Gamma_p(m+1)^{-1}}{(-1)^{(m-n)+1-\left[\frac{m-n}{p}\right]}\Gamma_p(m-n+1)^{-1}} \\ &= (-1)^{m+1-\left[\frac{m}{p}\right]-(m-n)-1+\left[\frac{m-n}{p}\right]}\frac{\Gamma_p(n)\Gamma_p(m-n+1)}{\Gamma_p(m+1)}. \end{aligned}$$

Using Proposition 1 and Definition 2 we obtain

$$\begin{aligned} B_p(n,-m) &= (-1)^{n-\left[\frac{m}{p}\right]+\left[\frac{m-n}{p}\right]}\frac{\Gamma_p(n)\Gamma_p(m-n)h_p(m-n)}{\Gamma_p(m)h_p(m)} \\ B_p(n,-m) &= \frac{(-1)^{n-\left[\frac{m}{p}\right]+\left[\frac{m-n}{p}\right]}h_p(m-n)}{h_p(m)}B_p(m-n,n). \end{aligned}$$

□

3. Conclusions

In the present work we prove that the p-adic beta function $B_p : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{Q}_p$ has the following properties:

- If $x, y \in \mathbb{Z}_p$, then

$$B_p(x,y) = B_p(y,x)$$

- If $x, y \in \mathbb{Z}_p$, then

$$B_p(x,y)B_p(x+y,1-y) = \begin{cases} \frac{(-1)^{\ell(y)}}{h_p(x)}, & p \neq 2 \\ \frac{(-1)^{\sigma(y)+1}}{h_p(x)} & p = 2 \end{cases}$$

- If $x, y \in \mathbb{Z}_p$, then

$$B_p(x+1,y) = \frac{h_p(x)}{h_p(x+y)} B_p(x,y)$$

- If $x, y \in \mathbb{Z}_p$, then

$$B_p(x,y+1) = \frac{h_p(y)}{h_p(x+y)} B_p(x,y)$$

- If $x, y \in \mathbb{Z}_p$, then

$$B_p(x+1,y) + B_p(x,y+1) = \frac{h_p(x)+h_p(y)}{h_p(x+y)} B_p(x,y)$$

- If $x, y \in \mathbb{Z}_p$, then

$$B_p(x,y+1) = \frac{h_p(y)}{h_p(x)} B_p(x+1,y)$$

- If $x, y \in \mathbb{Z}_p$, then

$$B_p(x+1,y+1) = \frac{h_p(x)h_p(y)}{h_p(x+y+1)h_p(x+y)} B_p(x,y)$$

- If $x, y, z, w \in \mathbb{Z}_p$, then

$$B_p(x,y)B_p(x+y,z)B_p(x+y+z,w) = \frac{\Gamma_p(x)\Gamma_p\left(y\right)\Gamma_p(z)\Gamma_p(w)}{\Gamma_p\left(x+y+z+w\right)}$$

- If $x \in \mathbb{Z}_p$, then

$$B_p(x,1-x) = \begin{cases} (-1)^{\ell(x)+1} & \text{if } p \neq 2 \\ (-1)^{\sigma_1(y)+2} & \text{if } p = 2 \end{cases}$$

- If $n, k \in \mathbb{N}, k \leq n$, then

$$\binom{n}{k}_p B_p(n-k+1,k+1) = \frac{-1}{h_p(n+1)}$$

- If $m,n \in \mathbb{N}$, then

$$B(n+1,m+1) = -B_p(n,m)\frac{h_p(n)h_p(m)}{h_p(m+n)(m+n+1)}\frac{\left[\frac{n}{p}\right]!\left[\frac{m}{p}\right]!}{\left[\frac{m+n}{p}\right]!}p^{\left[\frac{n}{p}\right]+\left[\frac{m}{p}\right]-\left[\frac{m+n}{p}\right]}$$

- If $m,n \in \mathbb{N}$, then

$$B(n+1,m+1) = B_p(n+1,m+1)\frac{\left[\frac{n}{p}\right]!\left[\frac{m}{p}\right]!}{\left[\frac{m+n+1}{p}\right]!}p^{\left[\frac{n}{p}\right]+\left[\frac{m}{p}\right]-\left[\frac{m+n+1}{p}\right]}$$

- If $m,n \in \mathbb{N}$, then

$$B(p^n+1,p^m+1) = B_p(p^n,p^m)\frac{(p^{n-1})!(p^{m-1})!}{(p^{n-1}+p^{m-1})!}\frac{1}{h_p(p^n+p^m)(p^n+p^m+1)}$$

- If $p \neq 2$, then

$$B_p(\frac{1}{2},\frac{1}{2}) = \begin{cases} -1 & \text{if } p \equiv 3 \pmod 4 \\ 1 & \text{if } p \equiv 1 \pmod 4 \end{cases}$$

- If $m,n \in \mathbb{N}$, then

$$B_p(-n,-m) = (-1)^{\left[1+\left[\frac{n+m}{p}\right]-\left[\frac{n}{p}\right]-\left[\frac{m}{p}\right]\right]}\frac{h_p(n+m)}{h_p(n)h_p(m)}\frac{1}{B_p(n,m)}$$

- If $m,n \in \mathbb{N}$, then

$$B_p(-n,m) = \begin{cases} (-1)^{m-\left[\frac{n}{p}\right]+\left[\frac{-m+n}{p}\right]}\frac{h_p(n-m)}{h_p(n)}B_p(n-m,m) & \text{if } m < n \\ (-1)^{n+1-\left[\frac{n}{p}\right]}\frac{1}{h_p(n)}(B_p(m-n,n))^{-1} & \text{if } n \leq m \end{cases}$$

- If $m,n \in \mathbb{N}$, then

$$B_p(n,-m) = \begin{cases} \frac{(-1)^{m+1-\left[\frac{m}{p}\right]}}{h_p(m)}B_p(n-m,m)^{-1} & \text{if } m \leq n \\ \frac{(-1)^{n-\left[\frac{m}{p}\right]+\left[\frac{m-n}{p}\right]}h_p(m-n)}{h_p(m)}B_p(m-n,n) & \text{if } n < m \end{cases}$$

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