The Exact Order of Approximation for Bivariate Complex Bernstein-Schurer Polynomials

Nurhayat İspir, Şule Yüksel Güngör

Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey

Abstract. In this paper we study the approximation properties of the tensor product kind bivariate complex Bernstein-Schurer polynomials. We obtain the order of simultaneous approximation and Voronovskaja-type results with quantitative estimate for bivariate complex Bernstein-Schurer polynomials attached to analytic functions on compact polydisks.

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1. Introduction

The Bernstein polynomial attached to \( f : [0, 1] \to \mathbb{R} \) was introduced by Bernstein to give a proof of the Weierstrass Theorem. If \( f \) is continuous on \([0, 1]\) then \( \lim_{n} B_{n}(f)(x) = f(x) \) where \( B_{n}(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^{k}(1-x)^{n-k} f\left(\frac{k}{n}\right), x \in [0, 1] \).

If \( f : G \to \mathbb{C} \) is an analytic function in the open set \( G \subset \mathbb{C} \), with \( D_{1} \subset G \) (where \( D_{1} = \{ z \in \mathbb{C} : |z| < 1 \} \)), then S. N. Bernstein [6] proved that the complex Bernstein polynomials defined by

\[
B_{n}(f)(z) = \sum_{k=0}^{n} \binom{n}{k} z^{k}(1-z)^{n-k} f\left(\frac{k}{n}\right)
\]

uniformly converge to \( f \) in \( D_{1} \). But Bernstein obtained this convergence result without any quantitative estimate. Recently, Voronovskaja- type results with quantitative estimates for the complex Bernstein, complex q-Bernstein, complex Bernstein- Kantorovich, complex Kantorovich- Stancu polynomials attached to analytic functions on compact disks and the exact order of simultaneous approximation by these complex operators were obtained by S. G. Gal [5].

*Corresponding author.

Email addresses: nispir@gazi.edu.tr (N. İspir), sulegungor@gazi.edu.tr (Ş. Güngör)

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The complex Bernstein-Schurer polynomials (introduced and studied in the case of real variable in [7]) are defined for any fixed $p = 0, 1, 2, \ldots$ by

$$B_{n,p}(f)(z) = \sum_{k=0}^{n+p} \binom{n+p}{k} z^k (1-z)^{n+p-k} f(k/n), z \in \mathbb{C}.$$  

The approximation properties of these polynomials are investigated and the exact order of approximation with quantitative estimates were gave in [1]. It is clear that for $p = 0$ these polynomials become the classical complex Bernstein polynomials studied in [5]. In real case the approximation properties of bivariate Bernstein-Schurer polynomials is studied by D. Barbasu [2-4].

In this note we would like to extend the approximation results from the univariate case, obtained for the complex Bernstein-Schurer polynomials, to the bivariate case.

First we present a few concepts in the bivariate case which are natural extensions of the usual concepts in the univariate case. Let $D_{R_j} := \{z_j \in \mathbb{C} : |z_j| < R_j, j = 1, 2\}$ and $P(0;R) = D_{R_1} \times D_{R_2}$ denotes an open polydisk (of center $0$ and radius $R$) where $R = (R_1, R_2)$ and $|z_1| \leq r_1, |z_2| \leq r_2, r_1 < R_1$ with $r_2 < R_2$. Let also

$$\overline{P}_R := \overline{P}(0,R) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_j| \leq R_j, \ j = 1, 2\}$$

denotes the closed polydisk. For $f(z_1, z_2)$ is an analytic function of two complex variables $(z_1, z_2)$ in the polydisk $P(0; R)$ we can define the tensor product kind Bernstein-Schurer polynomials as follows

$$B_{n,m,p,q}(f)(z_1, z_2) = \sum_{k=0}^{n+p} \sum_{j=0}^{m+q} b_{n,k} z_1^k (1-z_1)^{n+p-k} b_{m,j} z_2^j (1-z_2)^{m+q-j} f(k/n, j/m)$$  \hspace{1cm} (1)

where $b_{n,k} (z_1) = \binom{n+p}{k} z_1^k (1-z_1)^{n+p-k}$, $b_{m,j} (z_2) = \binom{m+q}{j} z_2^j (1-z_2)^{m+q-j}$, $n, m \in \mathbb{N}$ and $p, q \in \mathbb{N} \cup \{0\}$.

The goal of this paper is to obtain the exact order of approximation for the polynomials given by (1) on compact polydisks. First we give the order of approximation and the Voronovskaja-type theorems with quantitative estimate for the polynomials $B_{n,m}(f)(z)$ defined by (1). These results allow us to obtain the exact order in approximation by the polynomials $B_{n,m}(f)(z)$.

2. The Convergence Results with Quantitative Estimates

**Theorem 1.** For fixed $p, q \in \mathbb{N} \cup \{0\}$ and $R_1 > p + 1, R_2 > q + 1$ suppose that $f : P(0; R) \to \mathbb{C}$ is analytic in $P(0; R) = D_{R_1} \times D_{R_2}$, that is $f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} z_1^k z_2^j$ for all $(z_1, z_2) \in P(0; R)$, $R = (R_1; R_2)$. Then we have

(i) For all $|z_1| \leq r_1, |z_2| \leq r_2$ with $1 < r_1$, $(p+1)r_1 < R_1$, $1 < r_2$, $(q+1)r_2 < R_2$ and $n, m \in \mathbb{N}$

$$|B_{n,m,p,q}(f)(z_1, z_2) - f(z_1, z_2)| \leq M_{r_1, r_2, p, q}^{n,m} (f)$$

where

\[
M_{r_1, r_2, n, m}^{p, q}(f) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |c_{k, j}| r_j^k \left[ \frac{3k(k-1)}{(n+p)} \left( (p+1) r_1 \right)^k + \frac{1}{n} \left( (p+1) r_1 \right)^k - \frac{r_1^k}{n} \right] \\
+ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |c_{k, j}| r_1^k \left[ \frac{3j(j-1)}{(m+q)} \left( (q+1) r_2 \right)^j + \frac{1}{m} \left( (q+1) r_2 \right)^j - \frac{r_2^j}{m} \right]
\]

and \(M_{r_1, r_2, n, m}^{p, q}(f) < \infty\).

(ii) Let \(k_1, k_2 \in \mathbb{N}\) be with \(k_1 + k_2 \geq 1\), \(1 \leq r_1 < r_1^* \leq (1+p)r_1 < R_1\), \(1 \leq r_2 < r_2^* \leq (1+q)r_2 < R_2\). Then for all \(|z_1| \leq r_1\), \(|z_2| \leq r_2\) and \(n, m \in \mathbb{N}, p, q \in \mathbb{N} \cup \{0\}\) we have

\[
\left| \frac{\partial^{|k_1+k_2|} B_{n, m, p, q}(f)}{\partial z_1^{k_1} \partial z_2^{k_2}}(z_1, z_2) \right| \leq \left| \frac{\partial^{|k_1+k_2|} B_{r_1, r_2, n, m}(f)}{\partial z_1^{k_1} \partial z_2^{k_2}}(z_1, z_2) \right| \leq M_{r_1, r_2, n, m}^{p, q}(f) \cdot \frac{k_1!}{(r_1^*-r_1)^{k_1+1}} \frac{k_2!}{(r_2^*-r_2)^{k_2+1}}
\]

where \(M_{r_1, r_2, n, m}^{p, q}(f)\) is given as at the above point (i).

Proof. (i) Denote \(e_{k, i}(z_1, z_2) = e_k(z_1) e_i(z_2)\) where \(e_k(u) = u^k\). Since \(f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k, j} e_{k, j}(z_1, z_2)\) and by the definition of the operator (1) we get

\[
|B_{n, m, p, q}(f)(z_1, z_2) - f(z_1, z_2)| \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} |c_{k, j}| |B_{n, m, p, q}(e_{k, j})(z_1, z_2) - e_{k, j}(z_1, z_2)|.
\]

By the simple calculation we can write

\[
|B_{n, m, p, q}(e_{k, i})(z_1, z_2) - e_{k, i}(z_1, z_2)| = |B_{n, p}(e_k)(z_1), B_{m, q}(e_i)(z_2) - z_1^k z_2^i| \\
\leq |z_1^k| |B_{n, p}(e_k)(z_1) - z_1^k| + |B_{p, q}(e_i)(z_2) - z_2^i| \\
\leq r_1^k |A| + |B_{n, p}(e_k)(z_1)| B,
\]

say. By the Stirling numbers of second kind \(S(k, j)\), we can write

\[
B_{n, p}(e_k)(z_1) = \sum_{j=1}^{k} S(k, j) \left( \frac{(n+p) \ldots (n+p-(j-1))}{(n+p)^j} \right) z_1^j
\]

(for complex Bernstein polynomials in one variable see [5, page 27]). Since \(S(k, j) \geq 0\), for all \(n, k \in \mathbb{N}, p \in \mathbb{N} \cup \{0\}\) and \(\sum_{j=1}^{k} S(k, j) \left( \frac{(n+p) \ldots (n+p-(j-1))}{(n+p)^j} \right) r_j \leq r_k\) it follows

\[
|B_{n, p}(e_k)(z_1)| \leq \sum_{j=1}^{k} S(k, j) \left( \frac{(n+p) \ldots (n+p-(j-1))}{(n+p)^k} \right) r_1^j \leq r_k
\]
for all $|z_1| \leq r_1$, with $1 < r_1$, $(p+1)r_1 < R_1$. To estimate $A$ and $B$ for fixed $n, m \in \mathbb{N}$, we should consider two possible cases:

1. $0 \leq k \leq n + p$, $0 \leq j \leq m + q$ and
2. $k > n + p$, $j > m + q$.

We start with case (1). If $k = 0$, $j = 0$ then obviously we get $|B_{n,p}(e_k)(z_1) - (e_k)(z_1)| = 0$ and $|B_{m,q}(e_j)(z_2) - (e_j)(z_2)| = 0$. Therefore, let us suppose that $1 \leq k \leq n + p$, $1 \leq j \leq m + q$. Denoting by $\Delta^k$ the finite difference of order $k$, as in the case of the classical Bernstein polynomials we easily can write the representation formulas

$$B_{n,p}(f)(z_1) = \sum_{v=0}^{n+p} \Delta^v_{1/n} f(0) e_v(z_1), \quad B_{m,q}(f)(z_2) = \sum_{w=0}^{m+q} \Delta^w_{1/m} f(0) e_w(z_2).$$

For simplicity, we use the following notations

$$C_{n,v,k}^p = \binom{n+p}{v} \Delta^v_{1/n} e_k(0) = \binom{n+p}{v} \left[ \frac{1}{n^v} \right]_{0, \ldots, \frac{j}{n}} e_k v! / n^v,$$

$$C_{n,w,j}^q = \binom{m+q}{w} \Delta^w_{1/m} e_j(0) = \binom{m+q}{w} \left[ \frac{1}{m^w} \right]_{0, \ldots, \frac{j}{m}} e_j w! / m^w.$$

Since $e_k, e_j$ are convex of any order, it follows that all $C_{n,v,k}^p \geq 0$, $C_{n,w,j}^q \geq 0$ and taking into account that $B_{n,p}(f)(1) = f((n + p)/p)$, $B_{m,q}(f)(1) = f((m + q)/m)$ we get

$$\sum_{v=0}^{n+p} C_{n,v,k}^p = B_{n,p}(e_1)(1) = \left( \frac{n+p}{n} \right)^k, \quad \sum_{w=0}^{m+q} C_{n,w,j}^q = B_{n,p}(e_2)(1) = \left( \frac{m+q}{m} \right)^j. \quad (3)$$

Using the result in the proof of Theorem 2.1 in [1] for any $|z_1| \leq r_1$, $|z_2| \leq r_2$ with $1 \leq r_1 \leq (p + 1)r_1 < R_1$, $1 \leq r_2 \leq (q + 1)r_2 < R_2$, directly we can write

$$A = \left| B_{m,q}(e_j)(z_2) - z_2^j \right| \leq \frac{j(j-1)}{m+q} \left( (q+1)r_2 \right)^j + \frac{1}{m} \left[ (q+1)r_2 \right]^j - \frac{r_2^j}{m},$$

and

$$B = \left| B_{n,p}(e_k)(z_1) - e_k(z_1) \right| \leq \frac{\binom{k}{n+p}}{n} \left( (p+1)r_1 \right)^k + \frac{1}{n} \left[ (p+1)r_1 \right]^k - \frac{r_1^k}{n}.$$
Combining all of the results obtained for $A$ and $B$, we have the desired inequality.

(ii) Now we give the rate of convergence in simultaneous approximation. Let

$$1 \leq r_1 < r_2^*, \quad 1 \leq r_2 < r_2^* \quad \text{and} \quad \gamma_1 = |u_1 - z_1| = r_1^*, \quad \gamma_2 = |u_2 - z_2| = r_2^*.$$ By the Cauchy’s formula

$$\frac{\partial^{k_1+k_2} B_{n,m,p,q}(f)}{\partial z_1^{k_1} \partial z_2^{k_2}}(z_1, z_2) - \frac{\partial^{k_1+k_2} f}{\partial z_1^{k_1} \partial z_2^{k_2}}(z_1, z_2) = k_1! k_2! \int \int \frac{[(B_{n,m,p,q}(u_1, u_2) - f(u_1, u_2)) du_1 du_2]}{(u_1 - z_1)^{k_1+1} (u_2 - z_2)^{k_2+1}}$$

passing to absolute value with $|z_1| \leq r_1$, $|z_2| \leq r_2$ and taking into account that $|u_1 - z_1| \geq r_1^* - r_1$, $|u_2 - z_2| \geq r_2^* - r_2$, by applying the estimate in (i) we easily obtain

$$\left| \frac{\partial^{k_1+k_2} B_{n,m,p,q}(f)}{\partial z_1^{k_1} \partial z_2^{k_2}}(z_1, z_2) - \frac{\partial^{k_1+k_2} f}{\partial z_1^{k_1} \partial z_2^{k_2}}(z_1, z_2) \right| \leq M_{n,m,p,q}^{r_1^*, r_2^*, m,n}(f) \frac{k_1!}{(r_1^* - r_1)^{k_1+1}} \frac{k_2!}{(r_2^* - r_2)^{k_2+1}}$$

which proves the theorem. \(\square\)

In what follows a Voronovskaja’s result for $B_{n,m,p,q}(f)$ is presented. It will be the product of the parametric extensions generated by the Voronovskaja’s formula in univariate case in Theorem 2.2 in [1]. Indeed, for $f(z_1, z_2)$ defining the parametric extensions of the Voronovskaja’s formula by

$$z_1 L_n(f)(z_1, z_2) := B_{n,p}(f(., z_2))(z_1) - f(z_1, z_2) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}(z_1, z_2) - \frac{z_1 (1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2),$$

$$z_2 L_m(f)(z_1, z_2) := B_{m,q}(f(z_1, .))(z_2) - f(z_1, z_2) - \frac{q}{m} z_2 \frac{\partial f}{\partial z_2}(z_1, z_2) - \frac{z_2 (1 - z_2)}{2m} \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2),$$

their product (composition) gives

$$z_1 L_m(f)(z_1, z_2) o z_2 L_n(f)(z_1, z_2)

= B_{m,q} \left[ B_{n,p}(f(., .))(z_1) - f(z_1, .) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}(z_1, .) - \frac{z_1 (1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, .) \right](z_2)

- B_{n,p}(f(., z_2))(z_1) - f(z_1, z_2) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}(z_1, z_2) - \frac{z_1 (1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2)

- \frac{q}{m} z_2 \left[ B_{n,p}(\frac{\partial f}{\partial z_2}(., z_2))(z_1) - \frac{\partial f}{\partial z_2}(z_1, z_2) \right]

- \frac{p}{n} z_1 \left[ \frac{\partial f}{\partial z_1}(z_1, z_2) - \frac{z_1 (1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2) \right]

- \frac{z_2 (1 - z_2)}{2m} \left[ B_{n,p}(\frac{\partial^2 f}{\partial z_2^2}(., z_2))(z_1) - \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2) \right]

- \frac{p}{n} z_1 \left[ \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2) - \frac{z_1 (1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2) \right]\]
Now we can give the Voronovskaja-type theorem.

After simple calculation, we can write

\[
\begin{align*}
z_2 L_m(f)(z_1, z_2) o/z_1 L_n(f)(z_1, z_2) &= B_n,m,p,q \left( f \right) (z_1, z_2) - B_{m,q} \left( f \left( z_1, \ldots \right) \right) (z_2) \\
- \frac{p}{n} z_1 B_{m,q} \left( \frac{\partial f}{\partial z_1} \right) (z_2) - z_2 \left( \frac{1 - z_1}{2n} B_{m,q} \left( \frac{\partial^2 f}{\partial z_1^2} \right) (z_2) \right) \\
- B_{n,p} \left( f \left( ., z_2 \right) \right) (z_1) - f \left( z_1, z_2 \right) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1} (z_1, z_2) - \frac{z_1 (1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2} (z_1, z_2) \\
- \frac{q}{m} z_2 B_{n,p} \left( \frac{\partial f}{\partial z_2} \right) (z_1) + \frac{q}{m} z_2 \frac{\partial f}{\partial z_2} (z_1, z_2) + \frac{p}{n} \frac{q}{m} z_2 \frac{\partial^2 f}{\partial z_1 \partial z_2} (z_1, z_2) \\
+ \frac{q}{m} z_2 \frac{z_1 (1 - z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2} \left( \frac{\partial^2 f}{\partial z_2^2} \right) (z_1, z_2) \\
- \frac{z_2 (1 - z_2)}{2m} B_{n,p} \left( \frac{\partial^2 f}{\partial z_2^2} \right) (z_1) + \frac{z_2 (1 - z_2)}{2m} \frac{\partial^2 f}{\partial z_2^2} (z_1, z_2) \\
+ \frac{z_2 (1 - z_2)}{2m} p z_1 \frac{\partial f}{\partial z_1} \left( \frac{\partial^2 f}{\partial z_2^2} \right) (z_1, z_2) + \frac{z_1 (1 - z_1)}{2n} \frac{z_2 (1 - z_2)}{2m} \frac{\partial^2 f}{\partial z_1^2 \partial z_2^2} (z_1, z_2)
\end{align*}
\]

from which immediately can be derived the commutativity property

\[
z_2 L_m(f)(z_1, z_2) o/z_1 L_n(f)(z_1, z_2) = z_2 L_m(f)(z_1, z_2) o/z_2 L_n(f)(z_1, z_2).
\]

Now we can give the Voronovskaja-type theorem.

**Theorem 2.** For fixed \( p, q \in \mathbb{N} \cup \{0\} \) and \( R_1 > p + 1, R_2 > q + 1 \) suppose that \( f : P(0; R) \to \mathbb{C} \) is analytic in \( P(0; R) = D_{R_1} \times D_{R_2} \) that is \( f(z_1; z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} z_1^k z_2^j \) for all \( (z_1, z_2) \in P(0; R), R = (R_1, R_2) \). For all \( |z_1| \leq r_1, |z_2| \leq r_2 \) with \( 1 < r_1, (p + 1)r_1 < R_1, 1 < r_2, (q + 1)r_2 < R_2 \) and \( n, m \in \mathbb{N} \) we have

\[
\left| z_2 L_m(f)(z_1, z_2) o/z_1 L_n(f)(z_1, z_2) \right| \leq M_{r_1 r_2}^{p,q} \left( f \right) \left[ \frac{1}{n^2} + \frac{1}{m^2} \right],
\]

where

\[
M_{r_1 r_2}^{p,q} \left( f \right) = \max \left\{ \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} \left[ \left( q + 1 \right) r_2 \right]^{j} D_{k,p,r_1}, \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left| c_{k,j} \right| j \left[ \left( q + 1 \right) r_2 \right]^{j-1} D_{k,p,r_1}, \right. \\
\left. \sum_{k=2}^{\infty} \sum_{j=2}^{\infty} c_{k,j} j \left( j - 1 \right) \left[ \left( q + 1 \right) r_2 \right]^{j-2} D_{k,p,r_1} \right\},
\]

\[
D_{k,p,r_1} = (k - 1) A_k \left[ \left( p + 1 \right) r_1 \right]^{k-1} + C_{k,p} \left[ \left( p + 1 \right) r_1 \right]^{k-2}, A_k = (k - 1) \left[ 4 \left( k - 1 \right) \left( k - 2 \right) + 2 \right]
\]

and

\[
C_{k,p} = (k - 1) \left[ p \left( 5k - 4 \right) + p^2 + k \left( 4k - 7 \right) \right].
\]
Proof. Since \( f(z_1; z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} z_1^k z_2^j \) for all \((z_1, z_2) \in P(0; R)\), we can write
\[
\frac{\partial f}{\partial z_1}(z_1, z_2) = \sum_{k=0}^{\infty} f_k(z_2) z_1^k \text{ with } f_k(z_2) = \sum_{j=0}^{\infty} c_{k,j} z_2^j. \]
It follows \( \frac{\partial f}{\partial z_1}(z_1, z_2) = \sum_{k=1}^{\infty} f_k(z_2) k z_1^{k-1} \),
\( \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2) = \sum_{k=2}^{\infty} f_k(z_2) k (k-1) z_1^{k-2} \) and \( \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} j (j-1) z_2^{j-2} \).
Hence \( B_{n,p} (f(., z_2))(z_1) = \sum_{k=0}^{\infty} f_k(z_2) B_{n,p} (e_1^k)(z_1) \)
and
\[
B_{n,p} (f(., z_2))(z_1) = \frac{p}{n} z_1 \left( \frac{1}{n} - \frac{k z_1^k}{n} \right) - \frac{p}{n} z_1 \left( \frac{1}{n} - \frac{k z_1^k}{n} \right) - \frac{k z_1^{k-1} (1 - z_1) k (k-1)}{2n}.
\]
Applying \( B_{m,q} \) to the last expression with respect to \( z_2 \), we obtain
\[
E_1 = \sum_{k=2}^{\infty} B_{m,q} (f_k)(z_2) \left[ B_{n,p} (e_1^k)(z_1) - (e_1^k)(z_1) - \frac{p}{n} k z_1^k - \frac{p}{n} k z_1^k - \frac{k z_1^{k-1} (1 - z_1) k (k-1)}{2n} \right].
\]
Passing now to absolute value with \( |z_1| \leq r_1, |z_2| \leq r_2 \) and considering the estimates in the proof of Theorem 2.1 and Theorem 2.2 in [1], we can write
\[
|E_1| \leq \sum_{k=2}^{\infty} \sum_{j=0}^{\infty} |c_{k,j}| \left[ (1 + q) r_2 \right] \left[ (1 - 1) A_k \left[ \frac{p + 1}{n} r_1 \right] - C_k, p \left[ \frac{p + 1}{n} r_1 \right] \right]^{-k-2}
\]
where \( A_k = (k-1) \left[ 4 (k-1) (k-2) + 2 \right] \) and \( C_k, p = (k-1) \left[ p (5k-4) + p^2 + k (4k-7) \right] \).
Similarly,
\[
|E_2| \leq \sum_{k=2}^{\infty} |f_k(z_2)| \left[ (1 - 1) A_k \left[ \frac{p + 1}{n} r_1 \right] - C_k, p \left[ \frac{p + 1}{n} r_1 \right] \right]^{-k-2}
\]
\[
\leq \sum_{k=2}^{\infty} \sum_{j=0}^{\infty} |c_{k,j}| \left[ (1 + q) r_2 \right] \left[ (1 - 1) A_k \left[ \frac{p + 1}{n} r_1 \right] - C_k, p \left[ \frac{p + 1}{n} r_1 \right] \right]^{-k-2}.
\]
Then
\[
B_{n,p} \left( \frac{\partial f}{\partial z_2}(., z_2) \right)(z_1) = \sum_{k=0}^{\infty} \frac{\partial f_k}{\partial z_2}(z_2) B_{n,p} (e_1^k)(z_1)
\]
\[
= \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} c_{k,j} z_2^{j-1} B_{n,p} (e_1^k)(z_1)
\]
and
\[
\left[ B_{n,p} \left( \frac{\partial f}{\partial z_2} \right)(z_1, z_2) \right](z_1) = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} c_{k,j} j z_2^{-1} \left[ B_{n,p} \left( e_1^k \right)(z_1) - (e_1^k)(z_1) \right] - \frac{p}{n} k z_1^{k-1} \frac{z_1(1-z_1)k(k-1)}{2n},
\]
in the same way, we get
\[
|E_3| \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |c_{k,j}| j \left[(q + 1) r_2 \right]^{j-1} \left[ (k-1)A_k \left[ (p+1) r_1 \right] + C_{k,p} \left[ (p+1) r_1 \right] \right].
\]

Similarly
\[
B_{n,p} \left( \frac{\partial^2 f}{\partial z_2^2} \right)(z_1) = \sum_{k=0}^{\infty} \frac{\partial^2 f_k}{\partial z_2^2} \left[ B_{n,p} \left( e_1^k \right)(z_1) \right] = \sum_{k=0}^{\infty} \sum_{j=2}^{\infty} c_{k,j} j \left[(q + 1) r_2 \right]^{j-2} B_{n,p} \left( e_1^k \right)(z_1)
\]
and
\[
B_{n,p} \left( \frac{\partial^2 f}{\partial z_2^2} \right)(z_1) = \sum_{k=2}^{\infty} \sum_{j=2}^{\infty} c_{k,j} j \left[(q + 1) r_2 \right]^{j-2} B_{n,p} \left( e_1^k \right)(z_1) - \frac{p}{n} k z_1^{k-1} \frac{z_1(1-z_1)k(k-1)}{2n},
\]
hence by similar opinion we have
\[
|E_4| \leq \sum_{k=2}^{\infty} \sum_{j=2}^{\infty} |c_{k,j}| j \left[(q + 1) r_2 \right]^{j-2} \left[ (k-1)A_k \left[ (p+1) r_1 \right] + C_{k,p} \left[ (p+1) r_1 \right] \right].
\]

Interchanging above the places of \( n \) and \( m \) we obtain a similar order of approximation for
\[|z_1 L_m(f)(z_1, z_2) o_{z_1} L_n(f)(z_1, z_2)|\]
therefore
\[
|z_1 L_m(f)(z_1, z_2) o_{z_1} L_n(f)(z_1, z_2)| \leq |E_1| + |E_2| + |E_3| + |E_4| \leq M_{r_1, r_2}^{p, f} \left[ \frac{1}{n^2} + \frac{1}{m^2} \right]
\]
with \( M_{r_1, r_2}^{p, f} \) given by the statement.
\[\square\]

The Voronovskaja- type theorem will be used to find the exact order in approximation by \( B_{n,p,p}(f) \). We present the following Theorem.
Theorem 3. For fixed \(p, q \in \mathbb{N} \cup \{0\}\) and \(R_1 > p + 1, R_2 > q + 1\) suppose that \(f : \mathbb{P}(0; R) \to \mathbb{C}\) is analytic in \(\mathbb{P}(0; R) = D_{R_1} \times D_{R_2}\), that is \(f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} z_1^{k} z_2^{j}\) for all \((z_1, z_2) \in \mathbb{P}(0; R), R = (R_1; R_2)\). Denoting \(\|f\|_{r_1, r_2} = \sup \{|f(z_1, z_2)| : |z_1| \leq r_1, |z_2| \leq r_2\}\), if \(f\) is not a solution of the complex partial differential equation

\[
pz_1 \frac{\partial f}{\partial z_1} + \frac{z_1(1-z_1)}{2} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2) + qz_2 \frac{\partial f}{\partial z_2} + \frac{z_2(1-z_2)}{2} \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2) = 0,
\]

for any \((z_1, z_2) \in \mathbb{P}(0; R)\), then we have

\[
\|B_{n,n,p,p} - f\|_{r_1, r_2} \geq \frac{K_{r_1,r_2}f}{n}, \text{ for all } n \in \mathbb{N}
\]

where \(K_{r_1,r_2}f\) is independent on \(n\).

Proof. We can write

\[
B_{n,n,p,p}(f)(z_1, z_2) - f(z_1, z_2)
\]

\[
= \frac{n}{2} \left( B_{n,p}(f(z_1, \cdot))(z_2) - f(z_1, z_2) - \frac{p}{n} z_2 \frac{\partial f}{\partial z_2}(z_1, z_2) - \frac{z_2(1-z_2)}{2n} \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2) \right)
\]

\[
+ \frac{n}{2} \left( B_{n,p}(f(\cdot, z_2))(z_1) - f(z_1, z_2) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}(z_1, z_2) - \frac{z_1(1-z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2) \right)
\]

\[
+ \frac{n}{2} \left( \frac{z_1}{2n} \frac{\partial f}{\partial z_1}(z_1, z_2) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}(z_1, z_2) \right)
\]

\[
+ \frac{z_2}{4} \left( B_{n,p}(f(\cdot, \cdot))(z_1, z_2) \right) - \frac{p}{n} z_2 \frac{\partial f}{\partial z_2}(z_1, z_2) - \frac{z_2(1-z_2)}{2n} \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2)
\]

\[
+ \frac{z_1}{4} \left( B_{n,p}(f(\cdot, \cdot))(z_1, z_2) \right) - \frac{p}{n} z_1 \frac{\partial f}{\partial z_1}(z_1, z_2) - \frac{z_1(1-z_1)}{2n} \frac{\partial^2 f}{\partial z_1^2}(z_1, z_2)
\]

\[
+ \frac{z_2(1-z_2)}{8n} \frac{\partial^2 f}{\partial z_2^2}(z_1, z_2)
\]

From Theorem 2.2 in [1] and by the reasonings in the above Theorem 2, it is immediate that \(\|R_n(f)\|_{r_1, r_2} \to 0\) as \(n \to \infty\). Also, by Theorem 2 we obtain

\[
\frac{n^2}{4} \|z_n L_m(f) \alpha_n L_n(f)\|_{r_1, r_2} \leq \frac{M_{r_1,r_2}^p(f)}{2}
\]
which implies
\[
\left\| \frac{2}{n} \left[ \frac{n^2}{4} \left( z_n L_m(f) o_{z_1} L_n(f) \right) \right] + R_n(f) \right\|_{r_1, r_2} \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

Denoting
\[
H(z_1, z_2) = \frac{p}{2} \frac{\partial f}{\partial z_1} + \frac{z_1(1-z_1)}{4} \frac{\partial^2 f}{\partial z_1^2} (z_1, z_2) + \frac{q}{2} \frac{\partial f}{\partial z_2} + \frac{z_2(1-z_2)}{4} \frac{\partial^2 f}{\partial z_2^2} (z_1, z_2)
\]
and from the inequalities
\[
\| F + G \|_{r_1, r_2} \geq \| F \|_{r_1, r_2} - \| G \|_{r_1, r_2} \geq \| F \|_{r_1, r_2} - \| G \|_{r_1, r_2}
\]
it follows
\[
\left\| B_{n,n,p,p} - f \right\|_{r_1, r_2} \geq \frac{2}{n} \left\{ \| H \|_{r_1, r_2} - \left\| \frac{2}{n} \left[ \frac{n^2}{4} \left( z_n L_m(f) o_{z_1} L_n(f) \right) \right] + R_n(f) \right\|_{r_1, r_2} \right\}
\]
\[
\geq \frac{1}{n} \| H \|_{r_1, r_2} = \frac{1}{n} \| H \|_{r_1, r_2},
\]
for all \( n \geq n_0 \), with \( n_0 \) depending only on \( r_1, r_2 \). We used here that by hypothesis we have \( \| H \|_{r_1, r_2} > 0 \).

For \( n \in \{1, 2, \ldots, n_0 - 1\} \) it is easily seen that \( \| B_{n,n,p,p} - f \|_{r_1, r_2} \geq \frac{A_{r_1,r_2,n,p}(f)}{n} \) with
\[
A_{r_1,r_2,n,p} (f) = n \left\| B_{n,n,p,p} - f \right\|_{r_1, r_2}
\]
which finally implies (5) where
\[
K_{r_1, r_2, f} = \max \left\{ A_{r_1,r_2,1,p} (f), \ldots, A_{r_1,r_2,n_0-1,p} (f), \frac{1}{2} \| H \|_{r_1, r_2} \right\}.
\]

This completes the proof. \( \square \)

Combining now Theorem 1 with Theorem 3 we immediately obtain the following exact order.

**Corollary 1.** For fixed \( p, q \in \mathbb{N} \cup \{0\} \) and \( R_1 > p + 1, R_2 > q + 1 \) suppose that \( f : P(0; R) \rightarrow \mathbb{C} \) is analytic in \( P(0; R) = D_{R_1} \times D_{R_2} \) that is \( f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} c_{k,j} z_1^k z_2^j \) for all \( (z_1, z_2) \in P(0; R) \), \( R = (R_1; R_2) \). Assume that \( 1 < r_1, (p+1)r_1 < R_1, 1 < r_2, (q+1)r_2 < R_2 \). If \( f \) is not a solution of the equation (4) then we have
\[
\left\| B_{n,n,p,p} - f \right\|_{r_1, r_2} \sim \frac{1}{n}
\]
for all \( n \in \mathbb{N} \).
References


