On Degree Sum Energy of a Graph

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Abstract. The degree sum energy of a graph $G$ is defined as the sum of the absolute values of the eigenvalues of the degree sum matrix of $G$. In this paper, we obtain some lower bounds for the degree sum energy of a graph $G$.

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1. Introduction

We consider finite, undirected and simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. Let $G = (V, E)$ be a graph. The number of vertices of $G$ we denote by $n$ and the number of edges we denote by $m$, thus $|V(G)| = n$ and $|E(G)| = m$. The degree of a vertex $v$, denoted by $d_i$. Specially, $\Delta = \Delta(G)$ and $\delta = \delta(G)$ are called the maximum and minimum degree of vertices of $G$ respectively. $G$ is said to be $r$-regular if $\delta(G) = \Delta(G) = r$ for some positive integer $r$. For any integer $x, \lfloor x \rfloor$ is the positive integer less than or equal to $x$. For undefined terminologies we refer the reader to [5].

The energy $E(G)$ of a graph $G$ is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of $G$. This quantity, introduced almost 30 years ago [6] and having a clear connection to chemical problems, has in newer times attracted much attention of mathematicians and mathematical chemists [3, 7–9, 13–15].

Motivated by work on maximum degree energy [1], Ramane et al. [12] introduced the concept of degree sum energy, which is defined as follow:

**Definition 1.** Let $G$ be a simple graph with $n$ vertices $v_1, v_2, \ldots, v_n$ and let $d_i$ be the degree of $v_i, i = 1, 2, \ldots, n$. Then $DS(G) = [d_{ij}]$ is called the degree sum matrix of a graph $G$, where

\[ d_{ij} = \begin{cases} d_i + d_j & \text{if } i \neq j; \\ 0 & \text{otherwise}. \end{cases} \]

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The characteristic polynomial of $DS(G)$ is denoted by $f_n(G, \lambda) := \det(\lambda I - DS(G))$. Since $DS(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The maximum degree energy of $G$ is then defined as

$$E_{DS}(G) = \sum_{i=1}^{n} |\lambda_i|.$$ 

In this paper, we are interested in to obtain some new lower bounds for the degree sum energy of a graph $G$.

2. Results

For the sake of completeness, we mention below some results which are important throughout the paper.

**Lemma 1** ([12]). Since $\text{trace}(DS(G)) = 0$, the eigenvalues of $DS(G)$ satisfied the following relations

1. $\sum_{i=1}^{n} \lambda_i = 0$

2. $\sum_{i=1}^{n} \lambda_i^2 = 2R$, where $R = \sum_{1 \leq i < j \leq n} (d_i + d_j)^2$

**Lemma 2** ([12]). If $G$ is any graph with $n$ vertices, then $\sqrt{2R} \leq E_{DS}(G)$.

**Theorem 1** ([11]). Suppose $a_i$ and $b_i$, $1 \leq i \leq n$ are positive real numbers, then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \left( \sum_{i=1}^{n} a_i b_i \right)^2$$

where $M_1 = \max_{1 \leq i \leq n} (a_i)$; $M_2 = \max_{1 \leq i \leq n} (b_i)$; $m_1 = \min_{1 \leq i \leq n} (a_i)$ and $m_2 = \min_{1 \leq i \leq n} (b_i)$.

**Theorem 2** ([10]). Let $a_i$ and $b_i$, $1 \leq i \leq n$ are nonnegative real numbers, then

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \frac{n^2}{4} (M_1 M_2 - m_1 m_2)^2$$

where $M_i$ and $m_i$ are defined similarly to Theorem 1.

**Theorem 3** ([2]). Suppose $a_i$ and $b_i$, $1 \leq i \leq n$ are positive real numbers, then

$$|n \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i| \leq \alpha(n)(A-a)(B-b)$$

where $a, b, A$ and $B$ are real constants, that for each $i$, $1 \leq i \leq n$, $a \leq a_i \leq A$ and $b \leq b_i \leq B$. Further, $\alpha(n) = n \left( \frac{n}{2} \right) \left( 1 - \frac{n}{2} \right)$.
Theorem 4 ([4]). Let \(a_i\) and \(b_i\), \(1 \leq i \leq n\) are nonnegative real numbers, then

\[
\sum_{i=1}^{n} b_i^2 + rR \sum_{i=1}^{n} a_i^2 \leq (r + R)(\sum_{i=1}^{n} a_i b_i)
\]

(4)

where \(r\) and \(R\) are real constants, so that for each \(i\), \(1 \leq i \leq n\), holds, \(ra_i \leq b_i \leq Ra_i\).

3. Bounds for the Degree Sum Energy of Graphs

Theorem 5. Let \(G\) be a graph of order \(n\) and size \(m\), then

\[
E_{DS}(G) \geq \sqrt{2R n - \frac{n^2}{4} (\lambda_1 - \lambda_n)^2}
\]

(5)

where \(\lambda_1\) and \(\lambda_n\) are maximum and minimum of the absolute value of \(\lambda_i's\).

Proof. Suppose \(\lambda_1, \lambda_2, \ldots, \lambda_n\) are the eigenvalues of \(DS(G)\). We assume that \(a_i = 1\) and \(b_i = |\lambda_i|\), which by Theorem 2 implies

\[
\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\lambda_i|^2 - (\sum_{i=1}^{n} |\lambda_i|)^2 \leq \frac{n^2}{4} (\lambda_1 - \lambda_n)^2
\]

\[
2R n - (E_{DS}(G))^2 \leq \frac{n^2}{4} (\lambda_1 - \lambda_n)^2
\]

\[
E_{DS}(G) \geq \sqrt{2R n - \frac{n^2}{4} (\lambda_1 - \lambda_n)^2},
\]

as asserted. \(\Box\)

Theorem 6. Suppose zero is not an eigenvalue of \(DS(G)\). Then

\[
E_{DS}(G) \geq \frac{2 \sqrt{\lambda_1 \lambda_n} \sqrt{2R n}}{\lambda_1 + \lambda_n}
\]

(6)

where \(\lambda_1\) and \(\lambda_n\) are minimum and maximum of the absolute value of \(\lambda_i's\).

Proof. Suppose \(\lambda_1, \lambda_2, \ldots, \lambda_n\) are the eigenvalues of \(DS(G)\). We assume that \(a_i = |\lambda_i|\) and \(b_i = 1\), which by Theorem 1 implies

\[
\sum_{i=1}^{n} |\lambda_i|^2 \sum_{i=1}^{n} 1^2 \leq \frac{1}{4} (\sqrt{\frac{\lambda_n}{\lambda_1}} + \sqrt{\frac{\lambda_1}{\lambda_n}})^2 (\sum_{i=1}^{n} |\lambda_i|)^2
\]

\[
2R n \leq \frac{1}{4} (\frac{\lambda_1 + \lambda_n}{\lambda_1 \lambda_n})(E_{DS}(G))^2
\]

\[
E_{DS}(G) \geq \frac{2 \sqrt{\lambda_1 \lambda_n} \sqrt{2R n}}{\lambda_1 + \lambda_n},
\]

as desired. \(\Box\)
Theorem 7. Let $G$ be a graph of order $n$ and size $m$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $DS(G)$. Then

$$E_{DS}(G) \geq \sqrt{2Rn - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$$

where $\alpha(n) = n\left(\frac{2}{n^2}\right)(1 - \frac{1}{n^2})$.

Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $DS(G)$. We assume that $a_i = |\lambda_i| = b_i$, $a = |\lambda_n| = b$ and $A = |\lambda_1| = b$, which by Theorem 3 implies

$$|n \sum_{i=1}^n |\lambda_i|^2 - \left( \sum_{i=1}^n |\lambda_i| \right)^2 | \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$$

(8)

Since, $E_{DS}(G) = \sum_{i=1}^n |\lambda_i|$, $\sum_{i=1}^n |\lambda_i|^2 = 2R$, the above inequality becomes

$$2Rn - E_{DS}(G)^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$$

and a simple calculation gives us the required result. \qed

Corollary 1. Since $\alpha(n) \leq \frac{n^2}{4}$, then according to (7), we have

$$E_{DS}(G) \geq \sqrt{2Rn - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$$

$$\geq \sqrt{2Rn - \frac{n^2}{4}(|\lambda_1| - |\lambda_n|)^2}.$$ 

This means that inequality (7) is stronger of inequality (5).

Theorem 8. Let $G$ be a graph of order $n$ and size $m$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be a non-increasing arrangement of eigenvalues of $DS(G)$. Then

$$E_{DS}(G) \geq \frac{\lambda_1||\lambda_n|n + 2R}{|\lambda_1| + |\lambda_n|}$$

(9)

where $\lambda_1$ and $\lambda_n$ are minimum and maximum of the absolute value of $\lambda_i$'s.

Proof. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $DS(G)$. We assume that $b_i = |\lambda_i|$, $a_i = 1, r = |\lambda_n|$ and $R = |\lambda_1|$, which by Theorem 4 implies

$$\sum_{i=1}^n |\lambda_i|^2 + |\lambda_1||\lambda_n|\sum_{i=1}^n 1 \leq (|\lambda_1| + |\lambda_n|)\sum_{i=1}^n |\lambda_i|.$$ 

(10)

Since, $E_{DS}(G) = \sum_{i=1}^n |\lambda_i|$, $\sum_{i=1}^n |\lambda_i|^2 = 2R$, from (10), inequality (9) directly follows from Theorem 4. \qed
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