Existence and Uniqueness of Mittag-Leffler-Ulam Stable Solution for Fractional Integrodifferential Equations with Nonlocal Initial Conditions

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Abstract. In this paper, the existence and uniqueness of mild solution for fractional integrodifferential equations with nonlocal initial conditions are investigated by using Hölder's inequality, $p-$mean continuity and Schauder's fixed point theorem in Banach spaces. The Mittag-Leffler-Ulam stability results are also obtained by using generalized singular Gronwall's inequality.

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1. Introduction

During the past decades, fractional differential equations have attracted many authors (see for instance [10, 11, 14] and [15]). This is mostly because they efficiently describe many phenomena arising in engineering, physics, economy and science. There has been a significant development in nonlocal problems for fractional differential equations or inclusions (see for instance [1, 7, 12] and [22]).

As we all know, the main difficulty to study the fractional evolution equations is how to obtain the suitable fractional resolvent family generated by the infinitesimal generator $A$ in Banach space. In order to solve this problem, some authors introduced an $\alpha$-resolvent family under the Riemann-Liouville fractional derivative and some constraints, (see, for example, [2, 6]), and the others introduced suitable operator families with the Caputo fractional derivative in terms of some probability density functions and operator semigroup (see, for example, [17, 22] and [23]). For the latter, a pioneering work has been reported by El-Borai [4, 5].

On the other hand, in the theory of functional equations there are some special kind of data dependence: Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin and Aoki-Rassias (see [3,
8] and [9]). Recently, J. Wang et al. [18, 19] discussed four type Mittag-Leffler-Ulam stability of fractional differential equations and obtained some new and interesting stability results. Motivated by the above work, we consider the nonlocal Cauchy problem of the following form

\[
\begin{aligned}
\frac{C^\alpha}{t}x(t) &= Ax(t) + f(t, x(t), I^\beta x(t)), \quad t \in J := [0, b] \\
x(0) + g(x) &= x_0,
\end{aligned}
\]

where \(C^\alpha/t\) is the Caputo fractional derivative of order \(0 < \alpha < 1\), \(I^\beta\) is the Riemann-Liouville fractional integration of order \(0 < \beta < 1\), \(b > 0\), \(A\) is the infinitesimal generator of a \(C_0\) semigroup \(\{Q(t)\}_{t \geq 0}\) of operators on \(E\), \(f : J \times E \times E \to E\), \(g : C(J, E) \to E\) are given functions satisfying some assumptions and \(x_0\) is an element of the Banach space \(E\).

For any strongly continuous semigroup (i.e. \(C_0\) semigroup) \(\{Q(t)\}_{t \geq 0}\) on \(E\), we define the generator

\[ Au = \lim_{t \to 0^+} \frac{Q(t)u - u}{t} \quad \text{in} \quad E. \]

The domain \(D(A)\) of this linear operator is the set of all \(u \in E\) for which the limit above exists. Then \(D(A)\) is dense in \(E\) and \(A\) is closed, meaning that for \(u_n \in D(A)\), if \(u_n \to u\) and \(Au_n \to v\) in \(E\), then \(u \in D(A)\) and \(Au = v\). For more details, we refer the reader to [13].

## 2. Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper. We assume that \(E\) is a Banach space with the norm \(|\cdot|\). Let \(C(J, E)\) be the Banach space of continuous functions from \(J\) into \(E\) with the norm \(\|x\| = \sup_{t \in J} |x(t)|\), where \(x \in C(J, E)\).

Let \(B(E)\) be the space of all bounded linear operators from \(E\) to \(E\) with the norm \(\|Q\|_{B(E)} = \sup_{u \neq 0} \frac{|Q(u)|}{|u|} : |u| = 1\), where \(Q(u) \in B(E)\) and \(u \in E\). Throughout this paper, let \(A\) be the infinitesimal generator of a \(C_0\) semigroup \(\{Q(t)\}_{t \geq 0}\) of operators on \(E\). Clearly

\[ M := \sup_{t \in [0, b]} \|Q\|_{B(E)} < \infty. \]

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details, see I. Podlubny [15].

**Definition 1.** The fractional integral of order \(\alpha\) with the lower limit zero for a function \(h \in AC([0, \infty))\) is defined as

\[ I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \quad 0 < \alpha < 1, \]

provided the right side is point-wise defined on \([0, \infty)\), where \(\Gamma(\cdot)\) is the gamma function.

**Definition 2.** Riemann-Liouville derivative of order \(\alpha\) with the lower limit zero for a function \(h \in AC([0, \infty))\) can be written as

\[ D^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{h(s)}{(t-s)^{\alpha}} ds, \quad t > 0, \quad 0 < \alpha < 1. \]
Remark 1. 
- If $h(t) \in C^1[0, \infty)$, then
  \[ C D^\alpha h(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{h'(s)}{(t-s)^\alpha} ds = t^{1-\alpha} h'(t), \quad t > 0, \quad 0 < \alpha < 1. \]

- The Caputo derivative of a constant is equal to zero.
- If $h$ is an abstract function with values in $E$, then integrals which appear in Definitions 1-3 are taken in Bochner's sense.

For measurable functions $m : J \to \mathbb{R}$, define the norm
\[
\|m\|_{L^p(J)} = \begin{cases} \left( \int_J |m(t)|^p dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \inf_{\mu(J)} \sup_{t \in J} |m(t)|, & p = \infty, \end{cases}
\]
where $\mu(J)$ is the Lebesgue measure on $J$. Let $L^p(J, \mathbb{R})$ be the Banach space of all Lebesgue measurable functions $m : J \to \mathbb{R}$ with $\|m\|_{L^p(J)} < \infty$.

Lemma 1 (Hölder's inequality). Assume that $q, p \geq 1$ and $\frac{1}{q} + \frac{1}{p} = 1$. If $l \in L^q(J, \mathbb{R})$ and $m \in L^p(J, \mathbb{R})$, then for $1 \leq p \leq \infty$, $l^m \in L^1(J, \mathbb{R})$ and $\|l^m\|_{L^1(J)} \leq \|l\|_{L^q(J)} \cdot \|m\|_{L^p(J)}$.

Lemma 2 ([21] $p$-mean continuity). For each $\psi \in L^p(J, E)$ with $1 \leq p < +\infty$, we have $\lim_{r \to 0} \int_0^t \|\psi(t + r) - \psi(t)\|^p dt = 0$, where $\psi(s) = 0$ for $s$ not in $J$.

Lemma 3 (Bochner's Theorem). A measurable function $H : [0, b] \to \mathbb{R}$ is Bochner's integrable if $|H|$ is Lebesgue integrable.

Lemma 4 (Schauder Fixed Point Theorem). If $B$ is a closed bounded and convex subset of a Banach space $E$ and $F : B \to B$ is completely continuous, then $F$ has a fixed point in $B$.

We end this section with an important singular type Gronwall inequality.

Theorem 1 ([16] Theorem 1.4). For any $t \in [0, b]$, if
\[
u(t) \leq a(t) + \sum_{i=1}^n b_i(t) \int_0^t (t-s)^{\beta_i-1} u(s) ds,
\]
where all the functions are not negative and continuous. The constants $\beta_i > 0$. $b_i$ $(i = 1, 2, \ldots, n)$ are the bounded and monotonic increasing functions on $[0, b)$, then
\[
u(t) \leq a(t) + \sum_{k=1}^\infty \left( \sum_{i', i'' \in [1, \ldots, n]} \frac{\prod_{i=1}^k b_i(t) \Gamma(\beta_i)}{\Gamma(\sum_{i=1}^k \beta_i)} \int_0^t (t-s)^{\sum_{i=1}^k \beta_i-1} a(s) ds \right).
\]
Remark 2. For \( n = 2 \), if the constants \( b_1, b_2 \geq 0, \beta_1, \beta_2 > 0 \), \( a(t) \) is nonnegative and locally integrable on \( 0 \leq t < b \) and \( u(t) \) is nonnegative and locally integrable on \( 0 \leq t < b \) with
\[
u(t) \leq a(t) + b_1 \int_0^t (t-s)^{\beta_1-1}u(s)ds + b_2 \int_0^t (t-s)^{\beta_2-1}u(s)ds,
\]
then
\[
u(t) \leq a(t) + \sum_{k=1}^{\infty} \left[ \frac{(b_1 \Gamma(\beta_1))^k}{\Gamma(kb_1)} \int_0^t (t-s)^{kb_1-1}a(s)ds + \frac{(b_2 \Gamma(\beta_2))^k}{\Gamma(kb_2)} \int_0^t (t-s)^{kb_2-1}a(s)ds \right].
\]

Remark 3. Under the hypotheses of Remark 2, let \( a(t) \) is a nondecreasing function on \( 0 \leq t < b \). Then we have
\[
u(t) \leq a(t) \left( E_{\beta_1} \left[ b_1 \Gamma(\beta_1) t^{\beta_1} \right] + E_{\beta_2} \left[ b_2 \Gamma(\beta_2) t^{\beta_2} \right] \right),
\]
where \( E_{\alpha} \) is the Mittag-Leffler function \([15]\) defined by \( E_{\alpha}[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, z \in \mathbb{C} \).

3. Existence and Uniqueness of Mild Solutions

We recall the following definition of a mild solution for the nonlocal Cauchy problem (1). For more details, one can see \([20, 24]\).

Definition 4. By the mild solution of the nonlocal Cauchy problem (1), we mean that the function \( x \in C(J, \mathbb{R}) \) which satisfies
\[
x(t) = S(t)(x_0 - g(x)) + \int_0^t (t-s)^{a-1}T(t-s)f(s, x(s), I^{\theta} x(s))ds, \quad t \in [0, b],
\]
where
\[
S(t) = \int_0^{\infty} \xi_a(\theta)Q(t^{a}\theta)d\theta,
\]
\[
T(t) = \alpha^{1-\frac{1}{\alpha}} \rho_a(\theta^{\frac{1}{\alpha}}),
\]
\[
\xi_a(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \rho_a(\theta^{-\frac{1}{\alpha}}),
\]
\[
\rho_a(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-a-n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty),
\]
where \( \xi_a \) is the probability density function defined on \( (0, \infty) \), which has properties \( \xi_a(\theta) \geq 0 \) for all \( \theta \in (0, \infty) \) and
\[
\int_0^{\infty} \xi_a(\theta)d\theta = 1, \quad \int_0^{\infty} \theta \xi_a(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}.
\]

(3)
Before stating and proving the main results, we introduce the following hypotheses.

(H1) \( Q(t) \) is a compact operator for every \( t > 0 \).

(H2) The function \( f : J \times X \times X \to E \) satisfies that \( f(.,x,y) : J \to E \) is measurable for all \( x,y \in C(J,E) \) and \( f(.,.,.) : E \times E \to E \) is continuous for all \( t \in J \) and there exists a positive function \( \mu(\cdot) \in L^p(J,\mathbb{R}^+) \) for some \( p \) with \( 1 < p < \infty \) such that
\[
|f(t,x,y)| \leq \mu(t)(\|x\| + \|y\|), \quad x, y \in C(J,E), \quad t \in J.
\]

(H3) The function \( g : C(J,E) \to E \) is completely continuous and there exist constants \( L > 0, L' > 0 \) such that:
\[
|g(x)| \leq L\|x\| + L', \quad x \in C(J,E).
\]

For each positive constant \( k \), let \( B_k = \{ x \in C(J,E) : \|x\| \leq k \} \). Then \( B_k \) is clearly a bounded closed and convex subset in \( C(J,E) \).

The following existence result for nonlocal Cauchy problem (1) is based on Schauder fixed point theorem.

**Theorem 2.** If assumptions (H1)–(H3) are satisfied, then the nonlocal Cauchy problem (1) has a mild solution provided that
\[
M \left[ L - \frac{Mk^{\alpha-\frac{1}{p}}}{\Gamma(1+\alpha)} \alpha_{p-1} \left( \frac{p-1}{p+\alpha(p-1)p-1} \right)^{p-1} \left( \|\mu\|_{L^p(J,\mathbb{R}^+)} + \frac{b^\beta + e^{\beta}}{\Gamma(1+\beta)} \|\mu\|_{L^p(J,\mathbb{R}^+)} \right) \right] < 1.
\]

**Proof.** For any positive constant \( k \) and \( x \in B_k \), according to (2) and (H3), it follows that
\[
\left| \int_0^\infty \xi_a(\theta)Q(t^a\theta)(x_0 - g(x))d\theta \right| \leq M(\|x_0\| + Lk + L').
\]
Therefore, the function \( \int_0^\infty \xi_a(\theta)Q(t^a\theta)(x_0 - g(x))d\theta \) exists.

In view of (2), (3) and (H2), we get
\[
\int_0^t \left| \int_0^\infty \theta(t-s)^{\alpha-1} \xi_a(\theta)Q((t-s)^a\theta)f(s,x(s),I^\beta x(s))d\theta \right| ds
\]
\[
\leq M \int_0^t \int_0^\infty \theta \xi_a(\theta)(t-s)^{\alpha-1} |f(s,x(s),I^\beta x(s))| d\theta ds
\]
\[
\leq \frac{M}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \mu(s)(\|x\| + \|I^\beta x\|) ds
\]
\[
\leq \frac{M}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \mu(s)(k + \frac{k}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} d\tau) ds
\]
\[
\begin{align}
&= \frac{M k}{\Gamma(1+\alpha)} \left( \Gamma(1+\alpha) \right) \int_{0}^{t} (t-s)^{\alpha-1} \mu(s) \left( 1 + \frac{s^{\beta}}{\beta \Gamma(\beta)} \right) ds \\
&= \frac{M k}{\Gamma(1+\alpha)} \left( \int_{0}^{t} (t-s)^{\alpha-1} \mu(s) ds + \frac{1}{\Gamma(1+\beta)} \int_{0}^{t} (t-s)^{\alpha-1} s^{\beta} \mu(s) ds \right) \\
&\leq \frac{M k}{\Gamma(1+\alpha)} \left( \int_{0}^{t} (t-s)^{\alpha-1} \left( \frac{s^{\beta}}{\beta \Gamma(\beta)} \right) \right)^{\frac{1}{\beta}} \left( \int_{0}^{t} (\mu(s))^{p} ds \right)^{\frac{1}{p}} \\
&+ \frac{1}{\Gamma(1+\beta)} \left( \int_{0}^{t} (t-s)^{\alpha-1} s^{\beta} \mu(s) ds \right)^{\frac{1}{p}} \left( \int_{0}^{t} (\mu(s))^{p} ds \right)^{\frac{1}{p}} \\
&\leq \frac{M k}{\Gamma(1+\alpha)} \left( \frac{p-1}{p + (\alpha - 1)p - 1} \right)^{\frac{1}{p}} \left( \int_{0}^{t} (\mu(s))^{p} ds \right)^{\frac{1}{p}} \\
&+ \frac{1}{\Gamma(1+\beta)} \left( \frac{p-1}{p + (\alpha - 1)p - 1} \right)^{\frac{1}{p}} \left( \int_{0}^{t} (\mu(s))^{p} ds \right)^{\frac{1}{p}} \\
&\leq \frac{M k}{\Gamma(1+\alpha)} \left( \frac{p-1}{p + (\alpha - 1)p - 1} \right)^{\frac{1}{p}} \left( \int_{0}^{t} (\mu(s))^{p} ds \right)^{\frac{1}{p}} \\
&\times \frac{b^{\beta} \Gamma(\alpha - 1)}{\Gamma(1+\beta)} \left( \mu(s) \right)^{\frac{\alpha - 1}{\beta}} \left( \int_{0}^{t} (\mu(s))^{p} ds \right)^{\frac{1}{p}} \\
&\leq \frac{M k b^{\alpha-1}}{\Gamma(1+\alpha)} \left( \frac{p-1}{p + (\alpha - 1)p - 1} \right)^{\frac{1}{p}} \left( \int_{0}^{t} (\mu(s))^{p} ds \right)^{\frac{1}{p}} \\
&\times \frac{\beta \Gamma(\alpha - 1)}{\Gamma(1+\beta)} \left( \mu(s) \right)^{\frac{\alpha - 1}{\beta}} \left( \int_{0}^{t} (\mu(s))^{p} ds \right)^{\frac{1}{p}}, \quad (5)
\end{align}
\]
for all \( t \in J \). Thus, \( \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \xi_{\alpha}(\theta) Q((t-s)^{\alpha}) f(s,x(s),I^{\beta} x(s)) d\theta \) is Lebesgue integrable with respect to \( s \in [0,t] \) for all \( t \in [0,b] \). From Lemma 3 (Bochner's theorem), it follows that \( \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \xi_{\alpha}(\theta) Q((t-s)^{\alpha}) f(s,x(s),I^{\beta} x(s)) d\theta \) is Bochner integrable with respect to \( s \in [0,t] \) for all \( t \in J \).

For each positive constant \( k \), define an operator \( F \) on \( B_{k} \) by the formula

\[
(Fx)(t) = S(t)(x_{0} - g(x)) + \alpha \int_{0}^{t} (t-s)^{\alpha-1} T(t-s) f(s,x(s),I^{\beta} x(s)) ds, \quad t \in [0,b], \quad (6)
\]
where \( x \in B_k \). Let

\[
k = \frac{M \left( |x_0| + L' \right)}{1 - M \left[ L - \frac{M b^{\alpha-\frac{1}{\beta}}}{\Gamma(1+\alpha)} \left( \frac{p-1}{p+(\alpha-1)p-1} \right) \mu_{L^p(J,\mathbb{R}^+)} + \frac{b^{\alpha-\frac{1}{\beta}}}{\Gamma(1+\alpha)} \mu_{L^p(J,\mathbb{R}^+)} \right]}.
\]

(7)

In the following, we will prove that \( F \) has a fixed point on \( B_k \). Our proof will be divided into two steps.

**Step I.** \( \|Fx\| \leq k \) whenever \( x \in B_k \).

For each \( x \in B_k \) and \( t \in J \), by using the similar method as we did in (4) and (5), we have

\[
|\langle Fx \rangle(t)| = \left| S(t)(x_0 - g(x)) + \alpha \int_0^t (t-s)^{\alpha-1} T(t-s)f(s,x(s), I^\beta x(s))ds \right|
\leq \left| \int_0^\infty \xi_a(\theta)Q(t^\alpha \theta)(x_0 - g(x))d\theta \right|
+ \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \xi_a(\theta)Q((t-s)^\alpha \theta)f(s,x(s), I^\beta x(s))d\theta ds
\leq M \left( |x_0| + Lk + L' \right)
+ \frac{M b^{\alpha-\frac{1}{\beta}}}{\Gamma(1+\alpha)} \left( \frac{p-1}{p+(\alpha-1)p-1} \right) \mu_{L^p(J,\mathbb{R}^+)} + \frac{b^{\alpha-\frac{1}{\beta}}}{\Gamma(1+\alpha)} \mu_{L^p(J,\mathbb{R}^+)}
= k.
\]

(8)

Hence \( \|Fx\| \leq k \) for each \( x \in B_k \).

**Step II.** \( F \) is a completely continuous operator.

Firstly, we will prove that \( F \) is continuous on \( B_k \). For any \( x_n, x \in B_k, n = 1, 2, \ldots \) with \( \lim_{n \to \infty} |x_n - x| = 0 \), we get

\[
\lim_{n \to \infty} x_n(t) = x(t), \text{ for } t \in J.
\]

Thus by condition \((H2)\), we have

\[
\lim_{n \to \infty} f(t, x_n(t), I^\beta x_n(t)) = f(t, x(t), I^\beta x(t)),
\]

for \( t \in J \).

So, we can conclude that

\[
\sup_{s \in [0,b]} |f(s, x_n(s), I^\beta x_n(s)) - f(t, x(s), I^\beta x(s))| \to 0, \text{ as } n \to \infty.
\]

On the other hand, for \( t \in J \)

\[
|Fx_n(t) - Fx(t)|
\]
\[
\begin{align*}
&= S(t)(g(x_n) - g(x)) + \alpha \int_0^t (t-s)^{a-1} T(t-s) \left( f(s,x_n(s),x_n(s)) - f(s,x(s),x(s)) \right) ds \\
&\leq \int_0^\infty \xi_a(\theta)Q(t^a \theta)(g(x_n) - g(x)) d\theta \\
&\quad + \alpha \int_0^t \int_0^\infty \theta(t-s)^{a-1} \xi_a(\theta)Q((t-s)^a \theta)(f(s,x_n(s),x_n(s)) - f(s,x(s),x(s))) d\theta ds \\
&\leq M\|g(x_n) - g(x)\| + \frac{\alpha M}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{a-1} |f(s,x_n(s),x_n(s)) - f(s,x(s),x(s))| ds \\
&\leq M\|g(x_n) - g(x)\| + \frac{Mb^a}{\Gamma(\alpha + 1)} \sup_{s\in[0,b]} |f(s,x_n(s),x_n(s)) - f(t,x(s),x(s))|,
\end{align*}
\]

which implies

\[
\|F x_n - F x\| \leq M\|g(x_n) - g(x)\| + \frac{Mb^a}{\Gamma(\alpha + 1)} \sup_{s\in[0,b]} |f(s,x_n(s),x_n(s)) - f(t,x(s),x(s))|.
\]

Hence, by condition (H3) we get \(\|F x_n - F x\| \to 0\), as \(n \to \infty\). This means that \(F\) is continuous.

Next, we will show that \(\{F x, x \in B_k\}\) is equicontinuous. For any \(x \in B_k\) and \(0 \leq t_1 \leq t_2 \leq b\), we get

\[
\begin{align*}
|F x(t_2) - F x(t_1)| &= |S(t_2) - S(t_1)|(x_0 - g(x)) \\
&\quad + \alpha \int_0^{t_1} (t_2 - s)^{a-1} T(t_2 - s) f(s,x(s),x(s)) ds \\
&\quad - \alpha \int_0^{t_1} (t_1 - s)^{a-1} T(t_1 - s) f(s,x(s),x(s)) ds \\
&\leq \int_0^\infty \xi_a(\theta)[Q(t_2^a \theta) - Q(t_1^a \theta)](x_0 - g(x)) d\theta \\
&\quad + \alpha \int_0^{t_2} \int_0^\infty \theta(t_2 - s)^{a-1} \xi_a(\theta)Q((t_2 - s)^a \theta)f(s,x(s),x(s)) d\theta ds \\
&\quad - \alpha \int_0^{t_1} \int_0^\infty \theta(t_1 - s)^{a-1} \xi_a(\theta)Q((t_1 - s)^a \theta)f(s,x(s),x(s)) d\theta ds \\
&\leq \int_0^\infty \xi_a(\theta)[Q(t_2^a \theta) - Q(t_1^a \theta)](x_0 - g(x)) d\theta \\
&\quad + \alpha \int_0^{t_2} \int_0^\infty \theta(t_2 - s)^{a-1} \xi_a(\theta)Q((t_2 - s)^a \theta)f(s,x(s),x(s)) d\theta ds \\
&\quad - \alpha \int_0^{t_1} \int_0^\infty \theta(t_1 - s)^{a-1} \xi_a(\theta)Q((t_1 - s)^a \theta)f(s,x(s),x(s)) d\theta ds.
\end{align*}
\]
where

\[ I_1 = \alpha \int_0^{t_1} \int_0^\infty \theta((t_2 - s)^{a-1} - (t_1 - s)^{a-1}) \xi_a(\theta) Q((t_2 - s)^{a\theta}) f(s,x(s),I^\beta x(s)) d\theta ds, \]

\[ I_2 = \alpha \int_0^{t_1} \int_0^\infty \theta((t_2 - s)^{a-1} - (t_1 - s)^{a-1}) \xi_a(\theta) Q((t_2 - s)^{a\theta}) f(s,x(s),I^\beta x(s)) d\theta ds, \]

\[ I_3 = \alpha \int_0^{t_1} \int_0^\infty \theta((t_2 - s)^{a-1} - (t_1 - s)^{a-1}) \xi_a(\theta) Q((t_2 - s)^{a\theta}) f(s,x(s),I^\beta x(s)) d\theta ds. \]

By using analogous argument performed in (4) and (5), we can conclude that

\[ I_1 \leq \frac{\alpha Mk}{\Gamma(a + 1)} \left( \int_0^{t_1} (t_2 - s)^{a-1} \mu(s)(1 + \frac{1}{\Gamma(\beta + 1)} s^\beta) ds \right) \]

\[ \leq \frac{M k}{\Gamma(1 + \alpha)} \left( \int_0^{t_1} (t_2 - s)^{a-1} \left( \int_0^{t_1} (\mu(s))^{p} ds \right)^{\frac{1}{p}} \right) \]

\[ + \frac{1}{\Gamma(1 + \beta)} \left( \int_0^{t_1} (t_2 - s)^{a-1} \left( \int_0^{t_1} (s^\beta \mu(s))^{p} ds \right)^{\frac{1}{p}} \right) \]

\[ \leq \frac{M k}{\Gamma(1 + \alpha)} \left( \frac{p - 1}{p + (\alpha - 1)p - 1} \right) \left( t_2 - t_1 \right)^{\frac{(p-1)p^{-1}}{p}} \]

\[ \times \left( \|\mu\|_{L^p(J,R^+)} + \frac{1}{\Gamma(1 + \beta)} \left( \int_0^{t_2} \frac{\beta^p}{s^{p+1}} ds \left( \int_0^{t_2} (\mu(s))^{p^2} ds \right)^{\frac{1}{p^2}} \right) \right) \]

\[ \leq \frac{M k}{\Gamma(1 + \alpha)} \left( \frac{p - 1}{p + (\alpha - 1)p - 1} \right) \left( t_2 - t_1 \right)^{\frac{(p-1)p^{-1}}{p}} \]

\[ \times \left( \|\mu\|_{L^p(J,R^+)} + \frac{1}{\Gamma(1 + \beta)} \left( \frac{1}{1 + \frac{\beta^p}{p-1}} \left( t_2^{\frac{p^2}{p-1} + 1} - t_1^{\frac{p^2}{p-1} + 1} \right)^{\frac{1}{p^2}} \right) \right). \]
One can deduce that \( \lim_{t_1 \to t_2} I_1 = 0 \). Also, note that

\[
I_2 \leq \frac{a M k}{\Gamma(a + 1)} \left( \int_0^{t_1} \left[ (t_2 - s)^{a-1} - (t_1 - s)^{a-1} \right] \mu(s) (1 + \frac{1}{\Gamma(\beta + 1)} s^\beta) ds \right)
\]

\[
\leq \frac{a M k}{\Gamma(a + 1)} \left( \int_0^{t_1} \left[ (t_2 - s)^{a-1} - (t_1 - s)^{a-1} \right] ds \right) \left( \int_0^{t_1} (\mu(s))^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}}
\]

\[
+ \frac{1}{\Gamma(\beta + 1)} \left( \int_0^{t_1} \left[ (t_2 - s)^{a-1} - (t_1 - s)^{a-1} \right] ds \right) \left( \int_0^{t_1} (s^\beta \mu(s))^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}}
\]

\[
\leq \frac{a M k}{\Gamma(a + 1)} \left( \int_0^{t_1} \left[ (t_2 - s)^{a-1} - (t_1 - s)^{a-1} \right] ds \right) \left\| \mu \right\|_{L_\frac{p}{p-1} (J, \mathbb{R}^+)}^{\frac{p}{p-1}}
\]

\[
+ \frac{1}{\Gamma(\beta + 1)} \left( \int_0^{t_1} \left[ (t_2 - s)^{a-1} - (t_1 - s)^{a-1} \right] ds \right) \left( \int_0^{t_1} (s^\beta \mu(s))^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}}
\]

\[
\times \left( \int_0^{t_1} (\mu(s))^{\frac{p}{p-1}} ds \right)
\]

\[
\leq \frac{a M k}{\Gamma(a + 1)} \left( \left\| \mu \right\|_{L_\frac{p}{p-1} (J, \mathbb{R}^+)}^{\frac{p}{p-1}} + \frac{b^{\beta + \frac{(p-1)^2}{p^2}}}{\Gamma(\beta + 1)} \left( \frac{1}{1 + \frac{\beta}{p-1}} \right) \left( \int_0^{t_1} (s^\beta \mu(s))^{\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}} \right)
\]

\[
\times \left( \int_0^{t_1} \left[ (t_2 - s)^{a-1} - (t_1 - s)^{a-1} \right] ds \right)^{\frac{1}{p}}
\]

Using Lagrange mean value theorem, one can obtain \( (t_2 - s)^{a-1} - (t_1 - s)^{a-1} \to 0 \) as \( t_1 \to t_2 \), for \( s \in J \). By Lemma 2, we can deduce that \( \int_0^b \left[ (t_2 - s)^{a-1} - (t_1 - s)^{a-1} \right] ds \to 0 \) as \( t_1 \to t_2 \). Thus we deduce that \( \lim_{t_1 \to t_2} I_2 = 0 \).

For \( t_1 = 0 \), \( 0 < t_2 \leq b \), it is easy to see that \( I_3 = 0 \). For \( t_1 > 0 \) and \( \varepsilon > 0 \) be enough small, we have

\[
I_3 \leq a \int_0^{t_1 - \varepsilon} \int_0^\infty \theta(t_1 - s)^{a-1} \xi_a(\theta) ||Q((t_2 - s)^{a} \theta) - Q((t_1 - s)^{a} \theta)||_{B(E)} ||f(s, x(s), I^a \theta x(s))|| d\theta ds
\]

\[
+ a \int_{t_1 - \varepsilon}^{t_1} \int_0^\infty \theta(t_1 - s)^{a-1} \xi_a(\theta) ||Q((t_2 - s)^{a} \theta) - Q((t_1 - s)^{a} \theta)||_{B(E)} ||f(s, x(s), I^a \theta x(s))|| d\theta ds
\]

\[
\leq \frac{ak}{\Gamma(a + 1)} \int_0^{t_1 - \varepsilon} (t_1 - s)^{a-1} \mu(s) (1 + \frac{s^\beta}{\Gamma(\beta + 1)}) \sup_{s \in [0, t_1 - \varepsilon]} ||Q((t_2 - s)^{a} \theta) - Q((t_1 - s)^{a} \theta)||_{B(E)} ds
\]
\[
\frac{2aMk}{\Gamma(\alpha + 1)} \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{\alpha-1} \mu(s)(1 + \frac{s^\beta}{\Gamma(\beta + 1)}) ds
\]
\[
\leq \frac{ak}{\Gamma(\alpha + 1)} \left( \int_0^{t_1-\varepsilon} (t_1-s)^{\frac{(\alpha-1)p}{p-1}} ds \right)^{\frac{p-1}{p}} \left( \int_0^{t_1-\varepsilon} (\mu(s))^\frac{p}{p-1} ds \right)^{\frac{p}{p-1}}
\]
\[
+ \frac{1}{\Gamma(1 + \beta)} \left( \frac{1}{\Gamma(1 + \beta)} \int_0^{t_1-\varepsilon} (t_1-s)^{\frac{(\alpha-1)p}{p-1}} ds \right)^{\frac{p-1}{p}} \left( \int_0^{t_1-\varepsilon} (s^\beta \mu(s))^\frac{p}{p-1} ds \right)^{\frac{1}{p}} \times \sup_{s \in [0, t_1-\varepsilon]} ||Q((t_2-s)^\alpha \theta) - Q((t_1-s)^\alpha \theta)||_{B(E)}
\]
\[
+ \frac{2aMk}{\Gamma(\alpha + 1)} \left( \int_1^{t_1} (t_1-s)^{\frac{(\alpha-1)p}{p-1}} ds \right)^{\frac{p-1}{p}} \left( \int_1^{t_1} (\mu(s))^\frac{p}{p-1} ds \right)^{\frac{p}{p-1}} + \frac{1}{\Gamma(1 + \beta)}
\]
\[
\times \left( \int_1^{t_1} (t_1-s)^{\frac{(\alpha-1)p}{p-1}} ds \right)^{\frac{p-1}{p}} \left( \int_1^{t_1} (s^\beta \mu(s))^\frac{p}{p-1} ds \right)^{\frac{1}{p}} \times \left( \int_1^{t_1} (\mu(s))^\frac{p}{p-1} ds \right)^{\frac{p}{p-1}}
\]
\[
\leq \frac{ak}{\Gamma(\alpha + 1)} \left( \frac{p-1}{p + (\alpha-1)p-1} \right)^{\frac{p-1}{p}} \left( \int_1^{t_1} (t_1-s)^{\frac{(\alpha-1)p-1}{p-1}} ds \right)^{\frac{p}{p-1}} \left( \int_1^{t_1} (\mu(s))^\frac{p}{p-1} ds \right)^{\frac{p}{p-1}} \left( \int_1^{t_1} \beta \mu(s) ds \right)^{\frac{p}{p-1}} \times ||Q((t_2-s)^\alpha \theta) - Q((t_1-s)^\alpha \theta)||_{B(E)}
\]
\[
+ \frac{2aMk}{\Gamma(\alpha + 1)} \left( \frac{p-1}{p + (\alpha-1)p-1} \right)^{\frac{p-1}{p}} \left( \int_1^{t_1} (t_1-s)^{\frac{(\alpha-1)p-1}{p-1}} ds \right)^{\frac{p}{p-1}} \left( \int_1^{t_1} (\mu(s))^\frac{p}{p-1} ds \right)^{\frac{p}{p-1}} \times \left( \int_1^{t_1} (\mu(s))^\frac{p}{p-1} ds \right)^{\frac{p}{p-1}}
\]
\[
\leq \frac{ak}{\Gamma(\alpha + 1)} \left( \frac{p-1}{p + (\alpha-1)p-1} \right)^{\frac{p-1}{p}} \left( \int_1^{t_1} (t_1-s)^{\frac{p-1}{p}} ds \right)^{\frac{p}{p-1}} \left( \int_1^{t_1} (\mu(s))^\frac{p}{p-1} ds \right)^{\frac{p}{p-1}} \left( \int_1^{t_1} \beta \mu(s) ds \right)^{\frac{p}{p-1}} \times ||Q((t_2-s)^\alpha \theta) - Q((t_1-s)^\alpha \theta)||_{B(E)}
\]
\[
+ \frac{2aMk}{\Gamma(\alpha + 1)} \left( \frac{p-1}{p + (\alpha-1)p-1} \right)^{\frac{p-1}{p}} \left( \int_1^{t_1} (t_1-s)^{\frac{p-1}{p}} ds \right)^{\frac{p}{p-1}} \left( \int_1^{t_1} (\mu(s))^\frac{p}{p-1} ds \right)^{\frac{p}{p-1}} \times \left( \int_1^{t_1} (\mu(s))^\frac{p}{p-1} ds \right)^{\frac{p}{p-1}}
\]
\[ \times \left( \|\mu\|_{L^p(J,\mathbb{R}^+)} + \frac{1}{\Gamma(1+\beta)} \left( \frac{1}{1 + \frac{\beta p^2}{p-1}} \right)^{\frac{p-1}{p^2}} e^{\frac{\beta p^2 - 1}{p^2} \|\mu\|_{L^p(J,\mathbb{R}^+)}} \right) \]

Since (H1) implies the continuity of \( \{Q(t)\}_{t \geq 0} \) in \( t \) in the uniform operator topology, it is easy to see that \( I_3 \) tends to zero independently of \( x \in B_k \) as \( t_2 - t_1 \rightarrow 0, \varepsilon \rightarrow 0 \). Thus \((Fx)(t_2) - (Fx)(t_1)\) tends to zero independently of \( x \in B_k \) as \( t_2 - t_1 \rightarrow 0 \), which means that \( \{Fx, x \in B_k\} \) is equicontinuous.

It remains to prove that for \( t \in [0, b] \), the set \( V(t) = \{(Fx)(t), x \in B_k\} \) is relatively compact in \( \mathbb{E} \). Obviously, \( V(0) \) is relatively compact in \( \mathbb{E} \). Let \( 0 < t \leq b \) be fixed. For \( \forall \varepsilon \in (0, t) \) and \( \forall \delta > 0 \), define an operator \( F_{\varepsilon, \delta} \) on \( B_k \) by the formula

\[
(F_{\varepsilon, \delta}x)(t) = \int_{\delta}^{\infty} \xi_\alpha(\theta)Q(t^\alpha \theta)(x_0 - g(x))d\theta \\
+ \alpha \int_{0}^{t-\varepsilon} \int_{0}^{\infty} \theta(t-s)^{a-1}\xi_\alpha(\theta)Q((t-s)^{\alpha} \theta)f(s, x(s), t^\beta x(s))d\theta ds \\
= Q(\varepsilon^\alpha \delta) \left( \int_{\delta}^{\infty} \theta(t-s)^{a-1}\xi_\alpha(\theta)Q((t-s)^{\alpha} \theta - \varepsilon^\alpha \delta)(x_0 - g(x))d\theta \right) \\
+ \int_{0}^{t-\varepsilon} \int_{\delta}^{\infty} \theta(t-s)^{a-1}\xi_\alpha(\theta)Q((t-s)^{\alpha} \theta - \varepsilon^\alpha \delta)f(s, x(s), t^\beta x(s))d\theta ds ,
\]

where \( x \in B_k \). Then from the compactness of \( Q(\varepsilon^\alpha \delta)(\varepsilon^\alpha \delta > 0) \), we obtain that the set \( V_{\varepsilon, \delta}(t) = \{(F_{\varepsilon, \delta}x)(t), x \in B_k\} \) is relatively compact in \( \mathbb{E} \). Obviously, \( V(0) \) is relatively compact in \( \mathbb{E} \) for \( \forall \varepsilon \in (0, t) \) and \( \forall \delta > 0 \). Moreover, for every \( x \in B_k \), we have

\[
| (Fx)(t) - (F_{\varepsilon, \delta}x)(t) | \leq \left| \int_{0}^{\delta} \xi_\alpha(\theta)Q(t^\alpha \theta)(x_0 - g(x))d\theta \right| \\
+ \alpha \left| \int_{0}^{t} \int_{0}^{\delta} \theta(t-s)^{a-1}\xi_\alpha(\theta)Q((t-s)^{\alpha} \theta)f(s, x(s), t^\beta x(s))d\theta ds \right| \\
+ \left| \int_{t}^{\infty} \int_{0}^{\delta} \theta(t-s)^{a-1}\xi_\alpha(\theta)Q((t-s)^{\alpha} \theta)f(s, x(s), t^\beta x(s))d\theta ds \right| \\
- \left| \int_{0}^{t-\varepsilon} \int_{\delta}^{\infty} \theta(t-s)^{a-1}\xi_\alpha(\theta)Q((t-s)^{\alpha} \theta)f(s, x(s), t^\beta x(s))d\theta ds \right| \\
\leq M(x_0 + Lk + L') \int_{0}^{\delta} \xi_\alpha(\theta) d\theta \\
+ \alpha Mk \int_{0}^{t} \int_{0}^{\delta} \theta(t-s)^{a-1}\xi_\alpha(\theta)\mu(s)(1 + \frac{s^\beta}{\Gamma(\beta + 1)})d\theta ds
\]

\[ \alpha M k \int_{t-\varepsilon}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \xi_a(\theta) \mu(s)(1 + \frac{s^\beta}{\Gamma(\beta + 1)}) d\theta ds \]

\[ \leq M(x_0 + Lk + L') \int_{0}^{\delta} \xi_a(\theta) d\theta \]

\[ + \alpha M k \left( \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} \mu(s)(1 + \frac{s^\beta}{\Gamma(\beta + 1)}) ds \right) \int_{0}^{\delta} \theta \xi_a(\theta) d\theta \]

\[ + \alpha M k \left( \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} \mu(s)(1 + \frac{s^\beta}{\Gamma(\beta + 1)}) ds \right) \int_{0}^{\infty} \theta \xi_a(\theta) d\theta ds \]

\[ \leq M(x_0 + Lk + L') \int_{0}^{\delta} \xi_a(\theta) d\theta \]

\[ + \alpha M k \left( \int_{0}^{t} (t-s)^{\alpha-1} \mu(s)(1 + \frac{s^\beta}{\Gamma(\beta + 1)}) ds \right) \int_{0}^{\delta} \theta \xi_a(\theta) d\theta \]

\[ + \alpha M k \left( \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} \mu(s)(1 + \frac{s^\beta}{\Gamma(\beta + 1)}) ds \right) \]

Using again the same analogous performed in (5), we have

\[ \int_{0}^{t} (t-s)^{\alpha-1} \mu(s)(1 + \frac{s^\beta}{\Gamma(\beta + 1)}) ds \leq \left( \frac{p-1}{p + (\alpha - 1)p - 1} \right)^{\frac{p-1}{p}} t^{\frac{p + (\alpha - 1)p - 1}{p}} \]

\[ \times \left( \|\mu\|_{L^p(J, \mathbb{R}^n)} + \frac{t}{1 + \beta} \|\mu\|_{L^2(J, \mathbb{R}^n)} \right), \]

and

\[ \int_{t-\varepsilon}^{t} (t-s)^{\alpha-1} \mu(s)(1 + \frac{s^\beta}{\Gamma(\beta + 1)}) ds \leq \left( \frac{p-1}{p + (\alpha - 1)p - 1} \right)^{\frac{p-1}{p}} \varepsilon^{\frac{p + (\alpha - 1)p - 1}{p}} \]

\[ \times \left( \|\mu\|_{L^p(J, \mathbb{R}^n)} + \frac{\varepsilon^{\frac{p + (\alpha - 1)p - 1}{p}}}{1 + \beta} \|\mu\|_{L^2(J, \mathbb{R}^n)} \right), \]

we obtain

\[ |(Fx(t) - (F_{\varepsilon,\delta}x)(t))| \leq M(x_0 + Lk + L') \int_{0}^{\delta} \xi_a(\theta) d\theta \]

\[ + \alpha M k \left( \frac{p-1}{p + (\alpha - 1)p - 1} \right)^{\frac{p-1}{p}} t^{\frac{p + (\alpha - 1)p - 1}{p}} \left( \|\mu\|_{L^p(J, \mathbb{R}^n)} + \frac{\varepsilon^{\frac{p + (\alpha - 1)p - 1}{p}}}{1 + \beta} \|\mu\|_{L^2(J, \mathbb{R}^n)} \right) \]
There exists a positive constant \( L \).

**Theorem 3.** If assumptions (H2)–(H4) are satisfied, then the nonlocal Cauchy problem (1) has a unique mild solution provided that

\[
\times \int_0^\delta \theta \xi_a(\theta)d\theta + aMk \left( \frac{p-1}{p+(\alpha-1)p-1} \right)^{\frac{p-1}{p}} \times \epsilon \left( \frac{\rho^2+\gamma-1}{\gamma^2} \left( \frac{\mu}{\Gamma(1+\beta)} \right) \right).
\]

Therefore, there are relatively compact sets \( \{(F_{\epsilon, t}x)(t), x \in B_k\} \) arbitrarily close to the set \( \{(Ft)x(t), x \in B_k\} \) for \( t \in (0, b) \). Hence, \( \{(Ft)x(t), x \in B_k\} \) is relatively compact in \( E \). Moreover, \( \{(Ft)x(t), x \in B_k\} \) is uniformly bounded by (8). Therefore, \( \{(Ft)x(t), x \in B_k\} \) is relatively compact by Ascoli-Arzelà Theorem.

Also, since \( F \) is continuous on \( B_k \). Then \( F \) is a completely continuous operator. Obviously \( F \) maps \( B_k \) into itself. Hence, Schauder fixed point theorem shows that \( F \) has a fixed point \( x \in B_k \), which means that the nonlocal Cauchy problem (1) has at least one mild solution on \( J \). The proof is complete.

The following existence and uniqueness result for the nonlocal Cauchy problem (1) is based on Banach contraction principle. We will need the following assumption.

(H4) There exists a positive constant \( L_f \) such that for any \( x, x^*, y, y^* \in C(J, B_k) \), we have

\[
|f(t, x, x^*) - f(t, y, y^*)| \leq L_f (|x - y| + |x^* - y^*|),
\]

for \( t \in J \), where \( k \) is defined as in (7).

**Theorem 3.** If assumptions (H2)–(H4) are satisfied, then the nonlocal Cauchy problem (1) has a unique mild solution provided that

\[
ML + \frac{\alpha Mf}{(\alpha+1)} \left( \frac{b^\alpha}{\alpha} + \frac{\Gamma(\alpha)b^{\alpha+\beta}}{\Gamma(\alpha+\beta)} \right) < 1.
\]

**Proof.** It is easy to see that \( \int_0^\infty \xi_a(\theta)Q(t^\alpha \theta)(x_0 - g(x))d\theta \) exists and

\[
\int_0^t \int_0^\infty \theta(t-s)^{a-1} \xi_a(\theta)Q((t-s)^\alpha \theta)f(s, x(s), t^\beta x(s))d\theta ds
\]

Bochner’s integrable with respect to \( s \in [0, t] \) for all \( t \in J \). For \( x \in B_k \), Consider the operator \( F \) on \( B_k \), which is given by (6).

Obviously, it is sufficient to proof that \( F \) has a unique fixed point on \( B_k \).

According to (8), we know that \( F \) is an operator from \( B_k \) into itself. For any \( x, y \in B_k \) and \( t \in J \), according to (H3), (H4) and (2), we have

\[
|(Fx)(t) - (Fy)(t)| \leq \int_0^\infty \xi_a(\theta)Q(t^\alpha \theta)(g(y) - g(x))d\theta
\]
\[
\begin{align*}
+ \alpha \left| \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \xi_a(\theta)[f(s,x(s),I^a y(s)) - f(s,y(s),I^a y(s))] d\theta ds \right| \\
\leq M L \|x - y\| \\
+ \frac{\alpha M L_f}{\Gamma(\alpha + 1)} \left( \int_0^t (t-s)^{\alpha-1} |x(s) - y(s)| ds + \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\beta-1} |x(\tau) - y(\tau)| d\tau ds \right).
\end{align*}
\]

By changing the order of the second integral, we get
\[
\int_0^t \int_0^s (t-s)^{\alpha-1} |x(s) - y(s)| ds + \int_0^t (t-s)^{\alpha-1} \int_0^s (s-\tau)^{\beta-1} |x(\tau) - y(\tau)| d\tau ds = \int_0^t \int_\tau^t (t-s)^{\alpha-1} |x(s) - y(s)| ds d\tau \\
= \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} |x(s) - y(s)| ds d\tau \\
= \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} |x(s) - y(s)| ds.
\]
Therefore, we get
\[
\begin{align*}
|\langle F x \rangle(t) - \langle F y \rangle(t) | & \leq M L \|x - y\| \\
+ \frac{\alpha M L_f}{\Gamma(\alpha + 1)} \left( \int_0^t (t-s)^{\alpha-1} ds + \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha+\beta-1} ds \right) \|x - y\| \\
\leq M L \|x - y\| + \frac{\alpha M L_f}{\Gamma(\alpha + 1)} \left( \frac{b^a}{\alpha} + \frac{\Gamma(\alpha) b^{a+\beta}}{\Gamma(\alpha + \beta)} \right) \|x - y\|.
\end{align*}
\]
Thus
\[
\|F x - F y\| \leq \left( M L + \frac{\alpha M L_f}{\Gamma(\alpha + 1)} \left( \frac{b^a}{\alpha} + \frac{\Gamma(\alpha) b^{a+\beta}}{\Gamma(\alpha + \beta)} \right) \right) \|x - y\|.
\]
which means that \( F \) is a contraction according to (9). By applying Banach contraction principle, we know that \( F \) has a unique fixed point on \( B_k \). The proof is complete. \( \square \)

## 4. Mittag-Leffler-Ulam Stabilities

In this section, we consider the Mittag-Leffler-Ulam stability of the nonlocal Cauchy problem (1). Let \( \varepsilon \) be a positive real number and \( \varphi : J \to \mathbb{R}^+ \) be a continuous function. We consider the following inequalities
\[
\begin{align*}
|C D^a y(t) - Ay(t) - f(t, y(t), I^b y(t))| & \leq \varepsilon, \ t \in J \\
|C D^a y(t) - Ay(t) - f(t, y(t), I^b y(t))| & \leq \varphi(t), \ t \in J \\
|C D^a y(t) - Ay(t) - f(t, y(t), I^b y(t))| & \leq \varepsilon \varphi(t), \ t \in J
\end{align*}
\]
**Definition 5.** Eq. (1) is Mittag-Leffler-Ulam-Hyers stable, with respect to $E_a$ if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C^1(J, E)$ of the inequality (10), there exists a mild solution $x \in C(J,E)$ of Eq. (1) with

$$|y(t) - x(t)| \leq c\epsilon E_a[t], \quad t \in J.$$ 

**Definition 6.** Eq. (1) is generalized Mittag-Leffler-Ulam-Hyers stable, with respect to $E_a$ if there exists $\theta \in C(\mathbb{R}^+, \mathbb{R}^+), \theta(0) = 0$ such that for each solution $y \in C^1(J, E)$ of the inequality (10), there exists a mild solution $x \in C(J, E)$ of Eq. (1) with

$$|y(t) - x(t)| \leq \theta(\epsilon) E_a[t], \quad t \in J.$$ 

**Definition 7.** Eq. (1) is Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to $\varphi E_a$ if there exists $C_{\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in C^1(J, E)$ of the inequality (12), there exists a mild solution $x \in C(J,E)$ of Eq. (1) with

$$|y(t) - x(t)| \leq C_{\varphi} \epsilon \varphi(t) E_a[t], \quad t \in J.$$ 

**Definition 8.** Eq. (1) is generalized Mittag-Leffler-Ulam-Hyers-Rassias stable with respect to $\varphi E_a$ if there exists $C_{\varphi} > 0$ such that for each solution $y \in C^1(J, E)$ of the inequality (11), there exists a mild solution $x \in C(J,E)$ of Eq. (1) with

$$|y(t) - x(t)| \leq C_{\varphi} \varphi(t) E_a[t], \quad t \in J.$$ 

**Remark 4.** It is clear that: (i) Definition 5 $\implies$ Definition 6; (ii) Definition 7 $\implies$ Definition 8.

**Remark 5.** A function $y \in C^1(J,E)$ is a solution of the inequality (10) if and only if there exist a function $h \in C(J,E)$ (which depend on $y$) such that

(i) $|h(t)| \leq \epsilon, \quad t \in J,$

(ii) $C^\alpha D^\alpha y(t) = Ay(t) + f(t, y(t), I^\beta y(t)) + h(t), \quad t \in J.$

One can have similar remarks for the inequalities (11) and (12).

**Remark 6.** If $y \in C^1(J,E)$ i a solution of the inequality (10), then $y$ is a solution of the following integral inequality

$$|y(t) - S(t)(y_0 - g(y)) - \int_0^t (t - s)^{\alpha - 1} T(t - s)f(s, y(s), I^\beta y(s))ds| \leq \frac{Mb^\alpha}{\Gamma(\alpha + 1)} \epsilon.$$

We have similar remarks for the solutions of inequalities (11) and (12).

**Theorem 4.** If assumptions (H3) and (H4) are satisfied, then the nonlocal Cauchy problem (1) is Mittag-Leffler-Ulam-Hyers stable.
Then we get
\[
\begin{align*}
\text{From these relations, we have}
\end{align*}
\]
\[
\frac{C}{D^\alpha x(t)} = Ax(t) + f(t, x(t), I^\beta x(t)), \quad t \in J
\]
x(0) = y(0).
\] (13)

We have
\[
x(t) = S(t)(y_0 - g(y)) + \int_0^t (t-s)^{\alpha-1} T(t-s)f(s, y(s), I^\beta y(s)) ds.
\]
Then we get
\[
\begin{align*}
|y(t) - x(t)| &= |y(t) - S(t)(y_0 - g(y)) - \int_0^t (t-s)^{\alpha-1} T(t-s)f(s, x(s), I^\beta x(s)) ds | \\
&\leq |y(t) - S(t)(y_0 - g(y)) - \int_0^t (t-s)^{\alpha-1} T(t-s)f(s, y(s), I^\beta y(s)) ds | \\
&\quad + \int_0^t (t-s)^{\alpha-1} T(t-s)[f(s, y(s), I^\beta y(s)) - f(s, x(s), I^\beta x(s))] ds \\
&\leq \frac{M b^\alpha}{\Gamma(\alpha + 1)} \epsilon + \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \xi_\alpha(\theta) Q((t-s)^{\alpha-1} \theta) |f(s, y(s), I^\beta y(s)) - f(s, x(s), I^\beta x(s))| d\theta ds \\
&\leq \frac{M b^\alpha}{\Gamma(\alpha + 1)} \epsilon + \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \xi_\alpha(\theta) Q((t-s)^{\alpha-1} \theta) |f(s, y(s), I^\beta y(s)) - f(s, x(s), I^\beta x(s))| d\theta ds
\end{align*}
\]

From these relations, we have
We have
\[\frac{Mb^\alpha}{\Gamma(\alpha + 1)} \epsilon + \frac{aML_f}{\Gamma(\alpha + 1)} \left( \int_0^t (t-s)^{a-1}|x(s) - y(s)|ds \right) + \frac{1}{\Gamma(\beta)} \int_0^t \int_0^\tau (t-s)^{\beta-1}(s-\tau)^{\beta-1}|x(\tau) - y(\tau)|d\tau ds \] (13).

Using a similar manner of the second integral as in Proof of Theorem 3, we get
\[|y(t) - x(t)| \leq \frac{Mb^\alpha}{\Gamma(\alpha + 1)} \epsilon + \frac{aML_f}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{a-1}|x(s) - y(s)|ds \]
\[+ \frac{ML_f}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{a+\beta-1}|x(s) - y(s)|ds. \]

An application of Remark 2 and Remark 3 (with \(b_1 = \frac{aML_f}{\Gamma(\alpha + 1)}\), \(b_2 = \frac{ML_f}{\Gamma(\alpha + 1)}\), \(\beta_1 = \alpha\) and \(\beta_2 = \alpha + \beta\)) to the last inequality yields the desired estimation
\[|y(t) - x(t)| \leq \frac{M b^\alpha}{\Gamma(\alpha + 1)} \left( E_{\alpha} [ML_f t^\alpha] + E_{\alpha+\beta} [ML_f t^{a+\beta}] \right) \epsilon. \]

Thus, the conclusion of our theorem hold. \(\square\)

The following theorem gives generalized Mittag-Leffler-Ulam-Hyers stability.

**Theorem 5.** If assumptions (H3) and (H4) are satisfied. Suppose there exist \(\lambda > 0\) such that
\[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{a-1}\varphi(s)ds \leq \lambda \varphi(t), \]
for all \(t \in J\), where \(\varphi \in C(J, \mathbb{R}^+)\) is nondecreasing. Then the nonlocal Cauchy problem (1) is generalized Mittag-Leffler-Ulam-Hyers stable with respect to \(\varphi E_{\alpha} \).

**Proof.** Let \(y \in C^1(J, E)\) is a solution of the inequality (11). By Remark 5, we have for \(t \in J\) that \(y\) is a solution of the following integral inequality
\[\left| y(t) - S(t)(y_0 - g(y)) - \int_0^t (t-s)^{a-1}T(t-s)f(s, y(s), I^\beta y(s))ds \right| \leq M \lambda \varphi(t). \]

Let us denote by \(x \in C(J, E)\) the unique mild solution of the nonlocal Cauchy problem
\[\begin{cases} CD^a x(t) = Ax(t) + f(t, x(t), I^\beta x(t)), & t \in J \\ x(0) = y(0). \end{cases} \quad (14)\]

We have
\[x(t) = S(t)(y_0 - g(y)) + \int_0^t (t-s)^{a-1}T(t-s)f(s, y(s), I^\beta y(s))ds. \]
Then we get
\[
\left| y(t) - S(t)(y_0 - g(y)) - \int_0^t (t-s)^{a-1} T(t-s)f(s, y(s), I^\beta y(s))ds \right|
\leq \int_0^t (t-s)^{a-1} T(t-s)\varphi(s)ds
\leq a \int_0^t \int_0^\infty \theta(t-s)^{a-1} \xi_a(\theta)Q((t-s)^{a-1}\theta)\varphi(s)d\theta ds
\leq \frac{aM}{\Gamma(\alpha+1)} \int_0^t (t-s)^{a-1}\varphi(s)ds
\leq M\lambda \varphi(t).
\]

Again, from these relations, we have
\[
|y(t) - x(t)| = \left| y(t) - S(t)(y_0 - g(y)) - \int_0^t (t-s)^{a-1} T(t-s)f(s, x(s), I^\beta x(s))ds \right|
\leq \left| \int_0^t (t-s)^{a-1} T(t-s)f(s, y(s), I^\beta y(s))ds - f(s, x(s), I^\beta x(s))ds \right|
\leq M\lambda \varphi(t) + a \left| \int_0^t \int_0^\infty \theta(t-s)^{a-1} \xi_a(\theta)Q((t-s)^{a-1}\theta)\right| x[f(s, y(s), I^\beta y(s)) - f(s, x(s), I^\beta x(s))]d\theta ds
\leq M\lambda \varphi(t) + \frac{aM}{\Gamma(\alpha+1)} \int_0^t (t-s)^{a-1}|f(s, y(s), I^\beta y(s)) - f(s, x(s), I^\beta x(s))|ds
\leq M\lambda \varphi(t) + \frac{aM\ell_f}{\Gamma(\alpha+1)} \left( \int_0^t (t-s)^{a-1}|x(s) - y(s)|ds \right.
\left. + \frac{1}{\Gamma(\beta)} \int_0^t \int_0^s (t-s)^{a-1}(s-\tau)^{\beta-1}|x(\tau) - y(\tau)|d\tau ds \right).
\]

Using a similar manner of the second integral as in Proof of Theorem 3, we get
\[
|y(t) - x(t)| \leq \frac{Mb^a}{\Gamma(\alpha+1)} \epsilon + \frac{aM\ell_f}{\Gamma(\alpha+1)} \int_0^t (t-s)^{a-1}|x(s) - y(s)|ds
\left. + \frac{ML_f}{\Gamma(\alpha+1)} \int_0^t (t-s)^{a+\beta-1}|x(s) - y(s)|ds. \right.
\]
According to Remark 2 and Remark 3, we obtain our required assertion

$$|y(t) - x(t)| \leq M\lambda \left( E_\alpha \left[ M L f t^\alpha \right] + E_{\alpha+\beta} \left[ M L f t^{\alpha+\beta} \right] \right) \phi(t).$$

The conclusion of our theorem hold.

References


