Weak Separation Axioms via $e$-$\mathcal{I}$-Sets in Ideal Topological Spaces

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Abstract. In this paper, we use the notion of $e$-$\mathcal{I}$-open sets to introduce and define some new weak separation axioms. Also we study some of their basic properties. Additionally, we investigate the relationship and implications of these axioms among themselves and with other known axioms.

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1. Introduction

The notion of $R_0$ topological spaces is introduced by Shanin [15] in 1943. Later, Davis [4] rediscovered it and studied some properties of this weak separation axiom. Several topologists (e.g. [6, 10, 13]) further investigated properties of $R_0$ topological spaces and many interesting results have been obtained in various contexts. In the same paper, Davis also introduced the notion of $R_1$ topological spaces which are independent of both $T_0$ and $T_1$ but strictly weaker than $T_2$. A subset $A$ of a space $(X, \tau)$ is said to be regular open (resp. regular closed) [16] if $A = \text{Int}(\text{Cl}(A))$ (resp. $A = \text{Cl}(\text{Int}(A))$). $A$ is said to be $\delta$-open [18] if for each $x \in A$, there exists a regular open set $G$ such that $x \in G \subset A$. The complement of a $\delta$-open set is said to be $\delta$-closed. A point $x \in X$ is called a $\delta$-cluster point of $A$ if $\text{Int}(\text{Cl}(U)) \cap A \neq \emptyset$ for each open set $U$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $\text{Cl}_\delta(A)$ [18]. The set $\delta$-interior of $A$ [18] is the union of all regular open sets of $X$ contained in $A$ and is denoted by $\text{Int}_\delta(A)$. $A$ is $\delta$-open if $\text{Int}_\delta(A) = A$. The collection of all $\delta$-open sets of $(X, \tau)$ is denoted by $\delta\text{O}(X)$ and forms a topology $\tau^\delta$.

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An ideal $\mathcal{I}$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies the following conditions: $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$; $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Applications to various fields were further investigated by Jankovic and Hamlett [11]; Dontchev [5]; Mukherjee et al. [12]; Arenas et al. [3]; Nasef and Mahmoud [14], etc. Given a topological space $(X, \mathcal{I})$ with an ideal $\mathcal{I}$ on $X$ and if $\varphi(X)$ is the set of all subsets of $X$, a set operator $(\cdot)^\ast : \varphi(X) \rightarrow \varphi(X)$, called a local function [11, 17] of $A$ with respect to $\tau$ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$,

$$A^\ast(\mathcal{I}, \tau) = \{ x \in X \mid U \cap A \notin \mathcal{I} \text{ for every } U \in \tau(x) \}$$

where $\tau(x) = \{ U \in \tau \mid x \in U \}$. Furthermore $C(l)^\ast(A) = A \cup A^\ast(\mathcal{I}, \tau)$ defines a Kuratowski closure operator for the topology $\tau^\ast$. When there is no chance for confusion, we will simply write $A^\ast$ for $A^\ast(\mathcal{I}, \tau)$. $X^\ast$ is often a proper subset of $X$. By a space, we always mean a topological space $(X, \tau)$ with no separation properties assumed. If $A \subseteq X$, $CL(A)$ and $Int(A)$ will denote the closure and interior of $A$ in $(X, \tau)$, respectively. A subset $A$ of a topological space $(X, \tau)$ is said to be $e$-open [9] if $A \subseteq Int(\delta CL(A)) \cup Int(\delta Int(A))$. The notion of $e$-open sets has been studied extensively in recent years by many topologists. In this paper, we use the notion of $e^{-}\mathcal{I}$-open sets to introduce and define some new weak separation axioms. Also we study some of their basic properties. Additionally, we investigate the relationship and implications of these axioms among themselves and with other known axioms.

2. Preliminaries

A subset $A$ of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $e^{-}\mathcal{I}$-open [2] if

$$A \subseteq CL(\delta Int(A)) \cup Int(\delta CL(A)).$$

The complement of an $e^{-}\mathcal{I}$-open set is called an $e^{-}\mathcal{I}$-closed set [2]. The intersection of all $e^{-}\mathcal{I}$-closed sets containing $A$ is called the $e^{-}\mathcal{I}$-closure of $A$ and is denoted by $CL^\ast(A)$. The $e^{-}\mathcal{I}$-interior of $A$ is defined by the union of all $e^{-}\mathcal{I}$-open sets contained in $A$ and is denoted by $Int^\ast(A)$. The family of all $e^{-}\mathcal{I}$-open (resp. $e^{-}\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $E.\mathcal{I}O(X, x)$ (resp. $E.\mathcal{I}C(X, x)$). A subset $U$ of $X$ is called an $e^{-}\mathcal{I}$-neighborhood of a point $x \in X$ if there exists an $e^{-}\mathcal{I}$-open set $V$ of $(X, \tau, \mathcal{I})$ such that $x \in V \subseteq U$. A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be $e^{-}\mathcal{I}$-continuous if $f^{-1}(V) \in E.\mathcal{I}O(X)$ for every open set $V$ of $Y$.

**Definition 1.** A topological space $(X, \tau)$ is said to be:

(i) $R_0$ [4] if every open set contains the closure of each of its singletons.

(ii) $R_1$ [4] if for $x, y$ in $X$ with $CL(\{x\}) \neq CL(\{y\})$, there exist disjoint open sets $U$ and $V$ such that $CL(\{x\}) \subset U$ and $CL(\{y\}) \subset V$.

**Definition 2.** A topological space $(X, \tau)$ is said to be:

(i) $e^{-}T_1$ [7, 8] if for each pair of distinct points $x$ and $y$ in $X$, there exist $e$-open sets $U$ and $V$ containing $x$ and $y$, respectively, such that $y \notin U$ and $x \notin V$. 

...
Theorem 1. Let $X$ be an ideal topological space.

Definition 5. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be:

(i) an $\mathcal{I}$-T$_1$ space if for each pair of distinct points $x$ and $y$ in $X$, there exists $\mathcal{I}$-open sets $U$ and $V$ of $X$ such that $x \in U$ and $y \notin U$, $y \in V$ and $x \notin V$.

(ii) an $\mathcal{I}$-T$_2$ space if for each pair of distinct points $x$ and $y$ in $X$, there exist disjoint $\mathcal{I}$-open sets $U$ and $V$ in $X$ such that $x \in U$ and $y \in V$.

3. On $\mathcal{I}$-$\mathcal{R}_0$ Spaces

Definition 4. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. Then the $\mathcal{I}$-kernel of $A$, denoted by $\mathcal{I}\text{Ker}(A)$, is defined to be the set $\mathcal{I}\text{Ker}(A) = \{G \in E\mathcal{I}O(X)|A \subset G\}$.

Lemma 1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $x, y \in X$. Then, $y \in \mathcal{I}\text{Ker}(\{x\})$ if and only if $x \in Cl^\mathcal{I}_x(\{y\})$.

Proof. Suppose that $y \notin \mathcal{I}\text{Ker}(\{x\})$. Then there exists $U \in E\mathcal{I}O(X, x)$ such that $y \notin U$. Therefore, we have $x \notin Cl^\mathcal{I}_x(\{y\})$. The proof of the converse case can be done similarly.

Lemma 2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $S$ a subset of $X$. Then, $\mathcal{I}\text{Ker}(S) = \{x \in X|Cl^\mathcal{I}_x(\{x\}) \cap S \neq \emptyset\}$.

Proof. Let $x \in \mathcal{I}\text{Ker}(S)$. Suppose that $Cl^\mathcal{I}_x(\{x\}) \cap S = \emptyset$. Hence $x \notin X \setminus Cl^\mathcal{I}_x(\{x\})$ which is an $\mathcal{I}$-open set containing $S$. Since $x \notin \mathcal{I}\text{Ker}(S)$, this is a contradiction. Hence $Cl^\mathcal{I}_x(\{x\}) \cap S \neq \emptyset$. Conversely, suppose that $Cl^\mathcal{I}_x(\{x\}) \cap S \neq \emptyset$. Next, let $x \in X$ such that $Cl^\mathcal{I}_x(\{x\}) \cap S \neq \emptyset$ and suppose that $x \notin \mathcal{I}\text{Ker}(S)$. Then, there exists an $\mathcal{I}$-open set $U$ containing $S$ and $x \notin U$. Let $y \in Cl^\mathcal{I}_x(\{x\}) \cap S$. Hence, $U$ is an $\mathcal{I}$-neighborhood of $y$ which does not contains $x$. By this contradiction $x \in \mathcal{I}\text{Ker}(S)$ and hence the claim.

Definition 5. An ideal topological space $(X, \tau, \mathcal{I})$ is called an $\mathcal{I}$-$\mathcal{R}_0$ space if every $\mathcal{I}$-open set contains the $\mathcal{I}$-closure of each of its singletons.

Definition 6. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-$T_0$ if for each pair of distinct points $x$ and $y$ in $X$, there exists an $\mathcal{I}$-open set $U$ such that $x \in U$ and $y \notin U$, or there exists an $\mathcal{I}$-open set $V$ such that $y \in V$ and $x \notin V$.

Theorem 1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $X$ is $\mathcal{I}$-$T_1$ if and only if it is $\mathcal{I}$-$T_0$ and $\mathcal{I}$-$R_0$.

Proof. Let $X$ be an $\mathcal{I}$-$T_1$ space. By the definition of an $\mathcal{I}$-$T_1$ space, it is an $\mathcal{I}$-$T_0$ and $\mathcal{I}$-$R_0$ space.

Conversely, let $X$ be an $\mathcal{I}$-$T_0$ and $\mathcal{I}$-$R_0$ space. Let $x, y$ be any two distinct points of $X$. Since $X$ is $\mathcal{I}$-$T_0$, then there exists an $\mathcal{I}$-open set $U$ such that $x \in U$ and $y \notin U$ or
there exists an e-\mathcal{I}_-open set \( V \) such that \( y \in V \) and \( x \notin V \). Let \( x \in U \) and \( y \notin U \). Since \( X \) is e-\mathcal{I}_{R_0}, then \( \text{Cl}^*_e(x) \subseteq U \). We have \( y \notin U \) and then \( y \notin \text{Cl}^*_e(x) \). We obtain \( y \in X \setminus \text{Cl}^*_e(x) \). Take \( S = X \setminus \text{Cl}^*_e(x) \). Thus, \( U \) and \( S \) are e-\mathcal{I}-open sets containing \( x \) and \( y \), respectively, such that \( y \notin U \) and \( x \notin S \). Hence, \( X \) is e-\mathcal{I}-T_1. \(\Box\)

**Remark 1.** Since an ideal topological space \((X, \tau, \mathcal{I})\) is e-\mathcal{I}-T_1 if and only if the singletons are e-\mathcal{I}-closed, it is clear that every e-\mathcal{I}-T_1 space e-\mathcal{I}_{R_0}. But the converse is not true in general.

**Example 1.** Let \( X = \{a, b, c\} \) with a topology \( \tau = \{\emptyset, X, \{a\}\} \) and \( \mathcal{I} = \{\emptyset, \{c\}, \{b\}, \{b, c\}\} \). Since e-\mathcal{I}-open = \{\emptyset, X, \{a\}\} and \( \mathcal{I} = \{\emptyset, \{c\}, \{b\}, \{b, c\}\} \). It is clear that every e-\mathcal{I}-open set contains the e-\mathcal{I}-closure of each of its singletons so the ideal topological space is e-\mathcal{I}_{R_0}, but none of e-\mathcal{I}-T_0 and e-\mathcal{I}_-T_1.

**Remark 2.** The following example and Example 1 show that the notions e-\mathcal{I}_-T_0-ness and e-\mathcal{I}_{R_0}-ness are independent.

**Example 2.** Let \( X = \{a, b, c\} \) with a topology \( \tau = \{\emptyset, X, \{a\}\} \) and \( \mathcal{I} = \{\emptyset, \{a\}\} \). Now, we determine e-\mathcal{I}_-open = \{\emptyset, X, \{a\}\} \) and \( \mathcal{I} = \{\emptyset, \{a\}\} \). Then \((X, \tau, \mathcal{I})\) is e-\mathcal{I}_-T_0 but it is not e-\mathcal{I}_{R_0}.

**Lemma 3.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Then for any points \( x \) and \( y \) in \( X \), the following statements are equivalent:

(i) \( \mathcal{I}_e\text{Ker}(\{x\}) \neq \mathcal{I}_e\text{Ker}(\{y\}) \).

(ii) \( \text{Cl}^*_e(\{x\}) \neq \text{Cl}^*_e(\{y\}) \).

**Proof.** (i) \(\Rightarrow\) (ii): Let \( \mathcal{I}_e\text{Ker}(\{x\}) \neq \mathcal{I}_e\text{Ker}(\{y\}) \), then there exists a point \( k \) in \( X \) such that \( k \in \mathcal{I}_e\text{Ker}(\{x\}) \) and \( k \notin \mathcal{I}_e\text{Ker}(\{y\}) \). By Lemma 1, \( x \in \text{Cl}^*_e(\{x\}) \) and \( y \notin \text{Cl}^*_e(\{x\}) \). Therefore, \( \text{Cl}^*_e(\{x\}) \subseteq \text{Cl}^*_e(\{k\}) = \text{Cl}^*_e(\{k\}) \) and hence \( y \notin \text{Cl}^*_e(\{x\}) \). Hence \( \text{Cl}^*_e(\{x\}) \neq \text{Cl}^*_e(\{y\}) \). By using \( \mathcal{I}_e\text{Ker}(\{x\}) \neq \mathcal{I}_e\text{Ker}(\{y\}) \), we obtain \( \text{Cl}^*_e(\{x\}) \neq \text{Cl}^*_e(\{y\}) \).

(ii) \(\Rightarrow\) (i): Let \( \text{Cl}^*_e(\{x\}) \neq \text{Cl}^*_e(\{y\}) \), then there exists a point \( k \) in \( X \) such that \( k \in \text{Cl}^*_e(\{x\}) \) and \( k \notin \text{Cl}^*_e(\{y\}) \) and then there exists an e-\mathcal{I}_-open set containing \( k \) and therefore \( x \) but not \( y \), namely, \( y \notin \mathcal{I}_e\text{Ker}(\{x\}) \) and thus \( \mathcal{I}_e\text{Ker}(\{x\}) \neq \mathcal{I}_e\text{Ker}(\{y\}) \). \(\Box\)

**Proposition 1.** For an ideal topological space \((X, \tau, \mathcal{I})\), the following properties are equivalent:

(i) \((X, \tau, \mathcal{I})\) is an e-\mathcal{I}_{R_0} space,

(ii) For any \( K \in E\mathcal{I}C(X) \), \( x \notin K \) implies \( K \subseteq U \) and \( x \notin U \) for some \( U \in E\mathcal{I}O(X) \),

(iii) For any \( K \in E\mathcal{I}C(X) \), \( x \notin K \) implies \( K \cap \text{Cl}^*_e(\{x\}) = \emptyset \),

(iv) For any distinct points \( x \) and \( y \) of \( X \), either \( \text{Cl}^*_e(\{x\}) = \text{Cl}^*_e(\{y\}) \) or \( \text{Cl}^*_e(\{x\}) \cap \text{Cl}^*_e(\{y\}) = \emptyset \).

**Proof.** (i) \(\Rightarrow\) (ii): Let \( K \in E\mathcal{I}C(X) \) and \( x \notin K \). Then by (i), \( \text{Cl}^*_e(\{x\}) \subseteq X \setminus K \). Set \( U = X \setminus \text{Cl}^*_e(\{x\}) \), then \( U \in E\mathcal{I}O(X) \), \( K \subseteq U \) and \( x \notin U \).

(ii) \(\Rightarrow\) (iii): Let \( K \in E\mathcal{I}C(X) \) and \( x \notin K \). There exists \( U \in E\mathcal{I}O(X) \) such that \( K \subseteq U \) and \( x \notin U \).
For an ideal topological space \( X, \tau, \mathcal{I} \)

Let

\[
\text{Theorem 2.} \quad \text{An ideal topological space } (X, \tau, \mathcal{I}) \text{ is } e\mathcal{I}-R_0 \text{ space if and only if for any } x \text{ and } y \text{ in } X, \text{ } Cl^*_e(x) \neq Cl^*_e(y) \implies Cl^*_e(x) \cap Cl^*_e(y) = \emptyset.
\]

**Proof.** Let \( (X, \tau, \mathcal{I}) \) is \( e\mathcal{I}-R_0 \). By Proposition 1, we obtain the assertion. Conversely, let \( V \in E\mathcal{I}O(X; x) \). We will show that \( Cl^*_e(x) \subset V \). Let \( y \in X \setminus V \). Then \( x \neq y \) and \( x \notin Cl^*_e(y) \). This shows that \( Cl^*_e(x) \neq Cl^*_e(y) \). By assumption, \( Cl^*_e(x) \cap Cl^*_e(y) = \emptyset \). Hence \( y \notin Cl^*_e(x) \) and therefore \( Cl^*_e(x) \subset V \).

**Theorem 3.** Let \( (X, \tau, \mathcal{I}) \) be an ideal topological space. Then the following properties are equivalent:

(i) \( (X, \tau, \mathcal{I}) \) is an \( e\mathcal{I}-R_0 \) space,

(ii) \( x \in Cl^*_e(y) \) if and only if \( y \in Cl^*_e(x) \) for any points \( x \) and \( y \) in \( X \).

**Proof.** (i) \( \Rightarrow \) (ii): Assume that \( (X, \tau, \mathcal{I}) \) is \( e\mathcal{I}-R_0 \). Let \( x \in Cl^*_e(y) \) and \( A \in E\mathcal{I}O(X, y) \). Now by hypothesis, \( x \in Cl^*_e(y) \subset A \) and \( x \in A \). Therefore, every \( e\mathcal{I} \)-open set containing \( y \) contains \( x \). Hence \( y \in Cl^*_e(x) \).

(iii) \( \Rightarrow \) (i): Let \( U \in E\mathcal{I}O(X, x) \). If \( y \notin U \), then \( x \notin Cl^*_e(y) \) and hence \( y \notin Cl^*_e(x) \). This implies that \( Cl^*_e(x) \subset U \). Hence \( (X, \tau, \mathcal{I}) \) is \( e\mathcal{I}-R_0 \).

**Theorem 4.** For an ideal topological space \( (X, \tau, \mathcal{I}) \), the following properties are equivalent:

(i) \( (X, \tau, \mathcal{I}) \) is an \( e\mathcal{I}-R_0 \) space;

(ii) For any nonempty set \( S \) of \( X \) and any \( G \in E\mathcal{I}O(X) \) such that \( S \cap G \neq \emptyset \), there exists \( K \in E\mathcal{I}C(X) \) such that \( S \cap K \neq \emptyset \) and \( K \subset G \);

(iii) For any \( G \in E\mathcal{I}O(X) \), \( G = \bigcup \{ K \in E\mathcal{I}C(X) | K \subset G \} \);

(iv) For any \( K \in E\mathcal{I}C(X) \), \( K = \bigcap \{ G \in E\mathcal{I}O(X) | K \subset G \} \);

(v) For any \( x \in X \), \( Cl^*_e(x) \subset \mathcal{I}Ker(\{ x \}) \).
Proof. (i) ⇒ (ii): Let $S$ be a nonempty set of $X$ and $G \in E\mathcal{O}(X)$ such that $S \cap G \neq \emptyset$. There exists $x \in S \cap G$. Since $x \in G \in E\mathcal{O}(X)$, it follows that $\text{Cl}_e^*(\{x\}) \subset G$. Take $K = \text{Cl}_e^*(\{x\})$, then $K \in E\mathcal{O}(X)$, $K \subset G$ and $S \cap K \neq \emptyset$.

(ii) ⇒ (iii): Let $G \in E\mathcal{O}(X)$. We have $G \supset \cup \{K \in E\mathcal{O}(X) | K \subset G\}$. Let $x$ be any point of $G$.

By (ii) there exists $K \in E\mathcal{O}(X)$ such that $x \in K$ and $K \subset G$. Thus, we have $x \in K \subset \cup \{K \in E\mathcal{O}(X) | K \subset G\}$ and hence $G = \cup \{K \in E\mathcal{O}(X) | K \subset G\}$.

(iii) ⇒ (iv): This is obvious.

(iv) ⇒ (v): Let $x$ be any point of $X$ and $y \notin J_e\text{Ker}(\{x\})$. There exists $V \in E\mathcal{O}(X)$ such that $x \in V$ and $y \notin V$; hence $\text{Cl}_e^*(\{y\}) \cap V = \emptyset$. By (iv), $[\cap \{G \in E\mathcal{O}(X) | \text{Cl}_e^*(\{y\}) \subset G\}] \cap V = \emptyset$ and there exists $G \in E\mathcal{O}(X)$ such that $x \notin G$ and $\text{Cl}_e^*(\{y\}) \subset G$. Hence, $\text{Cl}_e^*(\{x\}) \cap G = \emptyset$ and $y \notin \text{Cl}_e^*(\{x\})$. Thus, $\text{Cl}_e^*(\{x\}) \subset J_e\text{Ker}(\{x\})$.

(v) ⇒ (i): Let $G \in E\mathcal{O}(X)$ and $x \in G$. Let $y \in J_e\text{Ker}(\{x\})$. We have $x \in \text{Cl}_e^*(\{y\})$ and $y \in G$. It follows that $J_e\text{Ker}(\{x\}) \subset G$. Thus, we obtain $x \in \text{Cl}_e^*(\{x\}) \subset J_e\text{Ker}(\{x\}) \subset G$. This shows that $(X, \tau, J)$ is an $e-J-R_0$ space.

\[\square\]

**Theorem 5.** An ideal topological space $(X, \tau, J)$ is $e-J-R_0$ if and only if for any pair of points $x$ and $y$ in $X$, $J_e\text{Ker}(\{x\}) \neq J_e\text{Ker}(\{y\})$ implies $J_e\text{Ker}(\{x\}) \cap J_e\text{Ker}(\{y\}) = \emptyset$.

**Proof.** Suppose that $(X, \tau, J)$ is an $e-J-R_0$ space. Thus by Lemma 3, for any points $x$ and $y$ in $X$ if $J_e\text{Ker}(\{x\}) \neq J_e\text{Ker}(\{y\})$, then $\text{Cl}_e^*(\{x\}) \neq \text{Cl}_e^*(\{y\})$. Now we prove that $J_e\text{Ker}(\{x\}) \cap J_e\text{Ker}(\{y\}) = \emptyset$. Assume that $z \in J_e\text{Ker}(\{x\}) \cap J_e\text{Ker}(\{y\})$. By $z \in J_e\text{Ker}(\{x\})$ and Lemma 1, it follows that $x \in \text{Cl}_e^*(\{z\})$. Since $x \in \text{Cl}_e^*(\{x\})$, by Theorem 2, $\text{Cl}_e^*(\{x\}) = \text{Cl}_e^*(\{z\})$. Similarly, we have $\text{Cl}_e^*(\{x\}) = \text{Cl}_e^*(\{z\}) = \text{Cl}_e^*(\{y\})$. This is a contradiction. Therefore, we have $J_e\text{Ker}(\{x\}) \cap J_e\text{Ker}(\{y\}) = \emptyset$. Conversely, let $(X, \tau, J)$ be an ideal topological space such that for any points $x$ and $y$ in $X$, $J_e\text{Ker}(\{x\}) \neq J_e\text{Ker}(\{y\})$ implies $J_e\text{Ker}(\{x\}) \cap J_e\text{Ker}(\{y\}) = \emptyset$. If $\text{Cl}_e^*(\{x\}) \neq \text{Cl}_e^*(\{y\})$, then by Lemma 3, $J_e\text{Ker}(\{x\}) \neq J_e\text{Ker}(\{y\})$. Hence, $J_e\text{Ker}(\{x\}) \cap J_e\text{Ker}(\{y\}) = \emptyset$ which implies $\text{Cl}_e^*(\{x\}) \cap \text{Cl}_e^*(\{y\}) = \emptyset$. Because $z \in \text{Cl}_e^*(\{x\})$ implies that $x \in J_e\text{Ker}(\{z\})$ and therefore $J_e\text{Ker}(\{x\}) \cap J_e\text{Ker}(\{z\}) \neq \emptyset$. By hypothesis, we have $J_e\text{Ker}(\{x\}) = J_e\text{Ker}(\{z\})$. Then $z \in \text{Cl}_e^*(\{x\}) \cap \text{Cl}_e^*(\{y\})$ implies that $J_e\text{Ker}(\{x\}) = J_e\text{Ker}(\{z\}) = J_e\text{Ker}(\{y\})$. This is a contradiction. Therefore, $\text{Cl}_e^*(\{x\}) \cap \text{Cl}_e^*(\{y\}) = \emptyset$ and by Theorem 2 $(X, \tau, J)$ is an $e-J-R_0$ space.

\[\square\]

**Theorem 6.** For an ideal topological space $(X, \tau, J)$, the following properties are equivalent:

(i) $(X, \tau, J)$ is an $e-J-R_0$ space,

(ii) If $F$ is an $e-J$-closed subset of $X$, then $F = J_e\text{Ker}(F)$,

(iii) If $F$ is an $e-J$-closed subset of $X$ and $x \in F$, then $J_e\text{Ker}(\{x\}) \subset F$,

(iv) If $x \in X$, then $J_e\text{Ker}(\{x\}) \subset \text{Cl}_e^*(\{x\})$.

**Proof.** (i) ⇒ (ii): Let $F$ be an $e-J$-closed subset of $X$ and $x \notin F$. Thus $X \setminus F \in E\mathcal{O}(X \setminus F)$. Since $(X, \tau, J)$ is an $e-J-R_0$, $\text{Cl}_e^*(\{x\}) \subset X \setminus F$. Since $F \subset X \setminus \text{Cl}_e^*(\{x\})$, $J_e\text{Ker}(F) \subset X \setminus \text{Cl}_e^*(\{x\})$
Theorem 7. For an ideal topological space $X$. Therefore, $\mathcal{I} \text{Ker}(F) = F$.

(ii) $\Rightarrow$ (iii): In general, $A \subseteq B$ implies $\mathcal{I} \text{Ker}(A) \subseteq \mathcal{I} \text{Ker}(B)$. Therefore, it follows from (ii) that $\mathcal{I} \text{Ker}(\{x\}) \subseteq \mathcal{I} \text{Ker}(F) = F$.

(iii) $\Rightarrow$ (iv): Since $x \in C_l^*\{x\}$ and $C_l^*\{x\}$ is $\mathcal{I}$-closed, (iii) $\mathcal{I} \text{Ker}(\{x\}) \subseteq C_l^*\{x\}$.

(iv) $\Rightarrow$ (i): We show the implication by using Theorem 3. Let $x \in C_l^*(\{y\})$. Then by Lemma 1 $y \in \mathcal{I} \text{Ker}(\{x\})$. By (iv), we obtain $y \in \mathcal{I} \text{Ker}(\{x\}) \subseteq C_l^*(\{x\})$. Therefore, $x \in C_l^*(\{y\})$ implies $y \in C_l^*(\{x\})$. The converse is obvious and $(X, \tau, \mathcal{I})$ is an e-$\mathcal{I}$-R$_0$ space.

Corollary 1. For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:

(i) $(X, \tau, \mathcal{I})$ is an e-$\mathcal{I}$-R$_0$ space,

(ii) $C_l^*(\{x\}) = \mathcal{I} \text{Ker}(\{x\})$ for all $x \in X$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $(X, \tau, \mathcal{I})$ is an e-$\mathcal{I}$-R$_0$ space. By Theorem 4, $C_l^*(\{x\}) \subseteq \mathcal{I} \text{Ker}(\{x\})$ for each $x \in X$. By Theorem 6, $\mathcal{I} \text{Ker}(\{x\}) \subseteq C_l^*(\{x\})$. This shows that $C_l^*(\{x\}) = \mathcal{I} \text{Ker}(\{x\})$.

(ii) $\Rightarrow$ (i): This is obvious by Theorem 6. 

Corollary 2. Let $(X, \tau, \mathcal{I})$ be an e-$\mathcal{I}$-R$_0$ and $x \in X$. If $C_l^*(\{x\}) \cap \mathcal{I} \text{Ker}(\{x\}) = \{x\}$, then $\mathcal{I} \text{Ker}(\{x\}) = \{x\}$.

Proof. The proof follows from Theorem 6 (iv).

Definition 7. A net $\{x_\lambda\}_{\lambda \in \Lambda}$ is said to be e-$\mathcal{I}$-convergent to a point $x$ in $X$, if for any $U \in E \mathcal{I}O(X, x)$, there exists $\lambda_0 \in \wedge$ such that $x_\lambda \in U$ for any $\lambda \in \wedge$ such that $\lambda \geq \lambda_0$.

Lemma 4. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $x$ and $y$ be any two points in $X$ such that every net in $X$ e-$\mathcal{I}$-converging to $y$ e-$\mathcal{I}$-converges to $x$. Then $x \in C_l^*(\{y\})$.

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n \in N}$ is a net in $C_l^*(\{y\})$. Since $\{x_n\}_{n \in N}$ e-$\mathcal{I}$-converges to $y$, then $\{x_n\}_{n \in N}$ e-$\mathcal{I}$-converges to $x$ and this implies that $x \in C_l^*(\{y\})$.

Theorem 7. For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:

(i) $(X, \tau, \mathcal{I})$ is an e-$\mathcal{I}$-R$_0$ space,

(ii) If $x, y \in X$, then $y \in C_l^*(\{x\})$ if and only if every net in $X$ e-$\mathcal{I}$-converging to $x$ e-$\mathcal{I}$-converges to $x$.

Proof. (i) $\Rightarrow$ (ii): Let $x, y \in X$ such that $y \in C_l^*(\{x\})$. Suppose that $\{x_\alpha\}_{\alpha \in N}$ be a net in $X$ such that $\{x_\alpha\}_{\alpha \in N}$ e-$\mathcal{I}$-converges to $y$. Since $y \in C_l^*(\{x\})$, by Theorem 2 we have $C_l^*(\{x\}) = C_l^*(\{y\})$. Therefore $x \in C_l^*(\{y\})$. This means that $\{x_\alpha\}_{\alpha \in N}$ e-$\mathcal{I}$-converges to $x$. Conversely, let $x, y \in X$ such that every net in $X$ e-$\mathcal{I}$-converging to $y$ e-$\mathcal{I}$-converges to $x$. Then $x \in C_l^*(\{y\})$ by Lemma 4. By Theorem 2, we have $C_l^*(\{x\}) = C_l^*(\{y\})$. Therefore
y ∈ Cl_e^*(\{x\}).

(iii) ⇒ (i): Assume that x and y are any two points of X such that Cl_e^*(\{x\}) ∩ Cl_e^*(\{y\}) ≠ ∅. Let z ∈ Cl_e^*(\{x\}) ∩ Cl_e^*(\{y\}). So there exists a net \{x_a\}_{a∈N} in Cl_e^*(\{x\}) such that \{x_a\}_{a∈N} e.-\mathcal{I}-converges to z. Since z ∈ Cl_e^*(\{y\}), then \{x_a\}_{a∈N} e.-\mathcal{I}-converges to y. It follows that y ∈ Cl_e^*(\{x\}). By the same token we obtain x ∈ Cl_e^*(\{y\}). Therefore Cl_e^*(\{x\}) = Cl_e^*(\{y\}) and by Theorem 2 \((X, τ, \mathcal{I})\) is an e.-\mathcal{I}-R_0 space.

4. On e.-\mathcal{I}-R_1 Spaces

Definition 8. An ideal topological space \((X, τ, \mathcal{I})\) is said to be e.-\mathcal{I}-R_1 if for x, y in X with Cl_e^*(\{x\}) ≠ Cl_e^*(\{y\}), there exist disjoint e.-\mathcal{I}-open sets U and V such that Cl_e^*(\{x\}) is a subset of U and Cl_e^*(\{y\}) is a subset of V.

Proposition 2. If \((X, τ, \mathcal{I})\) is e.-\mathcal{I}-R_1, then it is e.-\mathcal{I}-R_0.

Proof. Let U ∈ E.\mathcal{I}O(X, x). If y ∉ U, then x \notin Cl_e^*(\{y\}), we have Cl_e^*(\{x\}) ≠ Cl_e^*(\{y\}). So, there exists an e.-\mathcal{I}-open set V_y such that Cl_e^*(\{y\}) ⊂ V_y and x \notin V_y, which implies y \notin Cl_e^*(\{x\}). Thus Cl_e^*(\{x\}) ⊂ U. Therefore \((X, τ, \mathcal{I})\) is e.-\mathcal{I}-R_0.

Theorem 8. An ideal topological space \((X, τ, \mathcal{I})\) is e.-\mathcal{I}-R_1 if and only if for x, y in X, \mathcal{I}_R((\{x\}) ≠ \mathcal{I}_R((\{y\}), there exist disjoint e.-\mathcal{I}-open sets U and V such that Cl_e^*(\{x\}) ⊂ U and Cl_e^*(\{y\}) ⊂ V.

Proof. It follows from Lemma 3.

Remark 3. In the following diagram we denote by arrows the implications between the separation axioms which we have introduced and discussed in this paper and examples show that no other implications hold between them:

![Diagram](https://via.placeholder.com/150)

Example 3. Let X = \{a, b, c\}, τ = \{φ, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\} and \mathcal{I} = \{φ, \{b\}\}. E.\mathcal{I}O = \{φ, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}. Then \((X, τ, \mathcal{I})\) is e.-\mathcal{I}-R_0 but not R_0 and e.-\mathcal{I}-R_1.

Example 4. Let X = \{a, b, c\} with a topology τ = \{φ, X, \{a\}, \{b\}, \{a, b\}\} and \mathcal{I} = \{φ, \{a\}\}. Since E.\mathcal{I}O = \{φ, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}. Then \((X, τ, \mathcal{I})\) is e.-\mathcal{I}-R_0 but not R_0.

Example 5. Let X = \{a, b, c\} with a topology τ = \{φ, X, \{a\}, \{b\}\} and \mathcal{I} = \{φ, \{c\}\}. Since E.\mathcal{I}O = \{φ, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}. Then \((X, τ, \mathcal{I})\) is e.-\mathcal{I}-R_1 but not R_1.
Example 6. Let $X = \{a, b, c\}$ with a topology $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$. Since $E \Phi O = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $e$-open sets is $\{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Then $(X, \tau, \mathcal{I})$ is $e$-$R_0$ but not $e$-$\mathcal{I}$-$R_0$.

Theorem 9. The following properties are equivalent:

(i) $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$R_1$,

(ii) for each $x$, $y \in X$ one of the following holds:

- If $U$ is $e$-$\mathcal{I}$-open, then $x \in U$ if and only if $y \in U$,
- there exist disjoint $e$-$\mathcal{I}$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

(iii) If $x$, $y \in X$ such that $CL^*(\{x\}) \neq CL^*(\{y\})$, then there exist $e$-$\mathcal{I}$-closed sets $F_1$ and $F_2$ such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, and $X = F_1 \cup F_2$.

Proof: (i) $\Rightarrow$ (ii): Let $x, y \in X$. Then $CL^*(\{x\}) = CL^*(\{y\})$ or $CL^*(\{x\}) \neq CL^*(\{y\})$. If $CL^*(\{x\}) = CL^*(\{y\})$ and $U$ is $e$-$\mathcal{I}$-open, then $x \in U$ implies $y \in CL^*(\{x\}) \subset U$ and $y \in U$ implies $x \in CL^*(\{y\}) \subset U$. Thus consider the case that $CL^*(\{x\}) \neq CL^*(\{y\})$. Then there exist disjoint $e$-$\mathcal{I}$-open sets $U$ and $V$ such that $x \in CL^*(\{x\}) \setminus U$ and $y \in CL^*(\{y\}) \setminus V$.

(ii) $\Rightarrow$ (iii): Let $x, y \in X$ such that $CL^*(\{x\}) \neq CL^*(\{y\})$. Then $x \notin CL^*(\{y\})$ or $y \notin CL^*(\{x\})$, say $x \notin CL^*(\{y\})$. Then there exists an $e$-$\mathcal{I}$-open set $A$ such that $x \in A$ and $y \notin A$, which implies there exist disjoint $e$-$\mathcal{I}$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$. Then $F_1 = X \setminus V$ and $F_2 = X \setminus U$ are $e$-$\mathcal{I}$-closed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, and $X = F_1 \cup F_2$.

(iii) $\Rightarrow$ (i): First, we show that $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$R_0$. Let $U$ be $e$-$\mathcal{I}$-open and let $x \in U$. Suppose that $CL^*(\{x\}) \notin U$. Let $y \in CL^*(\{x\}) \cap (X \setminus U)$. Then $CL^*(\{x\}) \neq CL^*(\{y\})$ and there exist $F_1, F_2 \in E \Phi C(X)$ such that $x \in F_1$, $y \in F_2$, $x \notin F_1$, $y \notin F_2$, and $X = F_1 \cup F_2$. Then $y \in F_2 \setminus F_1 = X \setminus F_1$, which is $e$-$\mathcal{I}$-open, and $x \notin X \setminus F_1$, which is a contradiction. Hence, $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$R_0$. To show $X$ to be $e$-$\mathcal{I}$-$R_1$ assume that $a, b \in X$ such that $CL^*(\{a\}) \neq CL^*(\{b\})$. Then there exist $P_1, P_2 \in E \Phi C(X)$ such that $a \in P_1$, $b \notin P_1$, $a \notin P_2$, $b \in P_2$ and $X = P_1 \cup P_2$. Thus $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$, which are $e$-$\mathcal{I}$-open. This implies $CL^*(\{a\}) \subset P_1 \setminus P_2 = X \setminus P_2 \in E \Phi O(X)$ and $CL^*(\{b\}) \subset P_2 \setminus P_1$. Thus, $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$R_1$. □

Theorem 10. The following properties are equivalent:

(i) $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$T_2$,

(ii) $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$R_1$ and $e$-$\mathcal{I}$-$T_1$,

(iii) $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$R_1$ and $e$-$\mathcal{I}$-$T_0$.

Proof: (i) $\Rightarrow$ (ii): Since $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$T_1$, then it is $e$-$\mathcal{I}$-$T_1$. If $x, y \in X$ such that $CL^*(\{x\}) \neq CL^*(\{y\})$, then $x \neq y$ and there exist disjoint $e$-$\mathcal{I}$-open sets $U$ and $V$ such that $x \in U$ and $y \in V$. Therefore, $CL^*(\{x\}) = \{x\} \subset U$ and $CL^*(\{y\}) = \{y\} \subset V$. Hence $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$R_1$.

(ii) $\Rightarrow$ (iii): Since $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$T_1$, then $(X, \tau, \mathcal{I})$ is $e$-$\mathcal{I}$-$T_0$. 


(iii) \(\Rightarrow\) (i): Since \((X, \tau, \mathcal{I})\) is \(e_{\mathcal{I}}\)-\(R_1\), then \((X, \tau, \mathcal{I})\) is \(e_{\mathcal{I}}\)-\(R_0\) and \(e_{\mathcal{I}}\)-\(T_0\) and hence by Theorem 1 \((X, \tau, \mathcal{I})\) is \(e_{\mathcal{I}}\)-\(T_1\). Let \(x, y \in X\) such that \(x \neq y\). Since 
\[\text{Cl}_{e_{\mathcal{I}}}^*(\{x\}) = \{x\} \neq \{y\} = \text{Cl}_{e_{\mathcal{I}}}^*(\{y\}),\]
then there exist disjoint \(e_{\mathcal{I}}\)-open sets \(U\) and \(V\) such that 
\(x \in U\) and \(y \in V\). Hence, \((X, \tau, \mathcal{I})\) is \(e_{\mathcal{I}}\)-\(T_2\).

In view of Definition 3, it follows that

**Theorem 11.** An ideal topological space \((X, \tau, \mathcal{I})\) is \(e_{\mathcal{I}}\)-\(T_2\) if and only if for \(x, y \in X\) such that 
\(x \neq y\), there exist \(e_{\mathcal{I}}\)-closed sets \(F_1\) and \(F_2\) such that \(x \in F_1\), \(y \notin F_1\), \(y \in F_2\), \(x \notin F_2\), and 
\(X = F_1 \cup F_2\).

**Remark 4.** Let \(\{x_\lambda\}_{\lambda \in \Lambda}\) be a net in \((X, \tau, \mathcal{I})\) and \(e_{\mathcal{I}}\text{lim}(\{x_\lambda\}_{\lambda \in \Lambda})\) denote \(\{x \in X: \mathcal{I} - \text{converges to } x}\). 

**Theorem 12.** The following properties are equivalent:

(i) \((X, \tau, \mathcal{I})\) is \(e_{\mathcal{I}}\)-\(R_1\),

(ii) for \(x, y \in X\) 
\[\text{Cl}_{e_{\mathcal{I}}}^*(\{x\}) = \text{Cl}_{e_{\mathcal{I}}}^*(\{y\}),\]
whenever there exists a net \(\{x_\lambda\}_{\lambda \in \Lambda}\) such that 
\(x, y \in e_{\mathcal{I}}\text{lim}(\{x_\lambda\}_{\lambda \in \Lambda})\),

(iii) \((X, \tau, \mathcal{I})\) is \(e_{\mathcal{I}}\)-\(R_0\), and for every \(e_{\mathcal{I}}\)-convergent net \(\{x_\lambda\}_{\lambda \in \Lambda}\) in \(X\), 
\[e_{\mathcal{I}}\text{lim}(\{x_\lambda\}_{\lambda \in \Lambda}) = \text{Cl}_{e_{\mathcal{I}}}^*(\{x\})\] for some \(x \in X\).

**Proof.** (i) \(\Rightarrow\) (ii): Let \(x, y \in X\) such that there exists a net \(\{x_\lambda\}_{\lambda \in \Lambda}\) in \(X\) such that 
\(x, y \in e_{\mathcal{I}}\text{lim}(\{x_\lambda\}_{\lambda \in \Lambda})\). Then, by Theorem 9, (a) if \(U\) is \(e_{\mathcal{I}}\)-open, then \(x \in U\) if and only if 
\(y \in U\) or (b) there exist disjoint \(e_{\mathcal{I}}\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\). Since 
\(x, y \in e_{\mathcal{I}}\text{lim}(\{x_\lambda\}_{\lambda \in \Lambda})\), then (a) is satisfied, and we obtain 
\[\text{Cl}_{e_{\mathcal{I}}}^*(\{x\}) = \text{Cl}_{e_{\mathcal{I}}}^*(\{y\}).\]

(ii) \(\Rightarrow\) (iii): Let \(U \in e_{\mathcal{I}}\text{O}(X, x)\). Let \(y \notin U\). For each \(n \in N\) let \(x_n = x\). Then \(\{x_n\}_{n \in N}\) \(e_{\mathcal{I}}\)-converges to \(x\) and since 
\[\text{Cl}_{e_{\mathcal{I}}}^*(\{x\}) \neq \text{Cl}_{e_{\mathcal{I}}}^*(\{y\})\], by (ii) \(\{x_n\}\) does not \(e_{\mathcal{I}}\)-converge to \(y\) and there exists \(A \in e_{\mathcal{I}}\text{O}(X)\) such that \(y \in A\) and \(x \notin A\). Thus, \(y \notin \text{Cl}_{e_{\mathcal{I}}}^*(\{x\})\) and \(\text{Cl}_{e_{\mathcal{I}}}^*(\{x\}) \subset U\). Hence \((X, \tau, \mathcal{I})\) is \(e_{\mathcal{I}}\)-\(R_0\).

Let \(\{x_\lambda\}_{\lambda \in \Lambda}\) be an \(e_{\mathcal{I}}\)-convergent net in \(X\). Let \(x \in X\) such that 
\(\{x_\lambda\}_{\lambda \in \Lambda}\) \(e_{\mathcal{I}}\)-converges to \(x\). If \(y \in \text{Cl}_{e_{\mathcal{I}}}^*(\{x\})\), then \(\{x_\lambda\}_{\lambda \in \Lambda}\) \(e_{\mathcal{I}}\)-converges to \(y\), which implies 
\[\text{Cl}_{e_{\mathcal{I}}}^*(\{x\}) \subset e_{\mathcal{I}}\text{lim}(\{x_\lambda\}_{\lambda \in \Lambda})\]. Let \(y \in e_{\mathcal{I}}\text{lim}(\{x_\lambda\}_{\lambda \in \Lambda})\), then \(x, y \in e_{\mathcal{I}}\text{lim}(\{x_\lambda\}_{\lambda \in \Lambda})\), which implies 
\(y \in \text{Cl}_{e_{\mathcal{I}}}^*(\{y\}) = \text{Cl}_{e_{\mathcal{I}}}^*(\{x\})\). Hence 
\[e_{\mathcal{I}}\text{lim}(\{x_\lambda\}_{\lambda \in \Lambda}) = \text{Cl}_{e_{\mathcal{I}}}^*(\{x\})\].

(iii) \(\Rightarrow\) (i): Assume that \((X, \tau, \mathcal{I})\) is not \(e_{\mathcal{I}}\)-\(R_1\). Then there exist \(x, y \in X\) such that 
\[\text{Cl}_{e_{\mathcal{I}}}^*(\{x\}) \neq \text{Cl}_{e_{\mathcal{I}}}^*(\{y\})\] and every \(e_{\mathcal{I}}\)-open set containing \(\text{Cl}_{e_{\mathcal{I}}}^*(\{x\})\) intersects every \(e_{\mathcal{I}}\)-open set containing \(\text{Cl}_{e_{\mathcal{I}}}^*(\{y\})\). Since \((X, \tau, \mathcal{I})\) is \(e_{\mathcal{I}}\)-\(R_0\), then every \(e_{\mathcal{I}}\)-open set containing \(x\) contains 
\(\text{Cl}_{e_{\mathcal{I}}}^*(\{x\})\) and every \(e_{\mathcal{I}}\)-open set containing \(y\) contains \(\text{Cl}_{e_{\mathcal{I}}}^*(\{y\})\), which implies that every \(e_{\mathcal{I}}\)-open set containing \(x\) intersects every \(e_{\mathcal{I}}\)-open set containing \(y\). Let 
\(D_x = \{U \subset X | U \in e_{\mathcal{I}}\text{O}(X, x)\}\). Let \(\geq_x\) be the binary relation on \(D_x\) defined by \(U_1 \geq_x U_2\) if and only if 
\(U_1 \subset U_2\). Then, clearly \((D_x, \geq_x)\) is a directed set. Let 
\(D_y = \{U \subset X | U \in e_{\mathcal{I}}\text{O}(X, y)\}\) and let \(\geq_y\) be the binary relation on \(D\) defined by \(U_1 \geq_y U_2\) if and only if 
\(U_1 \geq_x U_2\). Then, \((D, \geq)\) is a directed set. For each \((U_1, U_2) \in D\), let 
\(x_{(U_1,U_2)} \in (U_1, U_2)\).
Then \( \{ x_{(U_1, U_2)} \}_{U_1, U_2} \in D \) is a net in \( X \) that \( e\mathcal{I}\)-converges to both \( x \) and \( y \). Thus, there exists \( z \in X \) such that \( e\mathcal{I}\lim(\{ x_{(U_1, U_2)} \}_{U_1, U_2} \in D) = CL^*_e(\{z\}) \), which implies \( x, y \in CL^*_e(\{z\}) \). Since \( \{ CL^*_e(\{w\}); w \in X \} \) is a decomposition of \( X \), then \( CL^*_e(\{x\}) = CL^*_e(\{z\}) = CL^*_e(\{y\}) \), which is a contradiction. Hence \( (X, \tau, \mathcal{I}) \) is \( e\mathcal{I}-R_1 \).

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**References**


