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A Note on Primary and Weakly Primary Submodules

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Abstract. In this paper, the generalizations of primary submodules and weakly primary submodules are proposed as $P(N)$-locally primary submodules and $P(N)$-locally weakly primary submodules, respectively. The relationships of these submodules are investigated extensively.

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1. Introduction

Throughout this paper, we assume that all rings are commutative with identity $1 \neq 0$. An ideal I of R is called a proper ideal if $I \neq R$. Then the radical of a proper ideal I of R is denoted by $rad(I)$ and $rad(I) = \{x \in R \mid x^n \in I$ for some positive integer $n\}$. A proper ideal P of R is called prime (primary) if $ab \in P$ for some $a, b \in R$ implies that either $a \in P$ or $b \in P$ (either $a \in P$ or $b^n \in P$ for some positive integer n). A proper ideal P of R is said to be a weakly prime ideal if $0 \neq ab \in P$ for some $a, b \in R$ implies that either $a \in P$ or $b \in P$, and it is called a weakly primary ideal if $0 \neq ab \in P$ for some $a, b \in R$ implies that either $a \in P$ or $b^n \in P$ for some positive integer n (see [2, 3]).

Let M be an R-module. A submodule N of M is called a proper submodule if $N \neq M$. A proper submodule N of M is called a prime submodule if $rm \in N$ for some $r \in R$ and $m \in M$ implies that either $m \in N$ or $rM \subseteq N$ and it is said to be a weakly prime submodule if $0 \neq rm \in N$ for some $r \in R$ and $m \in M$ implies that either $m \in N$ or $rM \subseteq N$. A non empty subset S of R is said to be multiplicative closed set if $0 \notin S$ and whenever $a, b \in S$, then $ab \in S$. Let S be a multiplicative closed set in R. It can be easily seen that M_S is an R_S-module under the operations $\frac{a}{s} + \frac{b}{u} = \frac{ua+sb}{su}$ and $\frac{r}{v}\frac{a}{s} = \frac{ra}{vs}$ for any $\frac{r}{v} \in R_S$ and $\frac{a}{s}, \frac{b}{u} \in M_S$ [5]. A proper submodule N of M is said to be $S(N)$-locally prime ($S(N)$-weakly prime) submodule if $N_{\mathfrak{M}}$ is a prime (a weakly prime) submodule of $M_{\mathfrak{M}}$ for each maximal ideal $\mathfrak{M}$ with $S(N) \subseteq \mathfrak{M}$ [4].

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A proper submodule N of M is said to be a primary submodule if $rm \in N$ for some $r \in R$, $m \in M$ implies that either $m \in N$ or $r^n M \subseteq N$ for some positive integer n and it is said to be a weakly primary submodule if $0 \neq rm \in N$ for some $r \in R$, $m \in M$ implies that either $m \in N$ or $r^n M \subseteq N$ for some positive integer n. The ideal $\{r \in R \mid rM \subseteq N\}$ will be denoted by $(N : M)$ and $(0 : N) = \{r \in R \mid rN = 0\}$ where N is a submodule of M. Then the annihilator of M is $(0 : M)$ where $(0 : M) = \{r \in R \mid rM = 0\}$. An R-module M is called a faithful module if $(0 : M) = (0)$. Note that if N is a primary submodule of M, then $(N : M)$ is a primary ideal of R and $rad(N : M) = \{r \in R \mid r^n M \subseteq N$ for some positive integer $n\}$ is a prime ideal of R ([1, 6, 7]).

Main aim is to obtain the two generalization on primary submodules and weakly primary submodules of an R-module M.Let N be a proper submodule of M. An element $r \in R$ is said to be primary to N if $r^n m \in N$, where $m \in M$ and n is a positive integer, then $m \in N$. Then $r \in R$ is said to be not primary to N if $r^n m \in N$ for some positive integer n and for some $m \in M \setminus N$. The set of all elements of R that are not primary to N is denoted by $P(N)$. Then we get $P(N) = \{r \in R \mid r^n m \in N$ for some positive integer n, for some element $m \in M \setminus N\}$. If $N = (0)$, then $P((0)) = \{r \in R \mid r^n m = 0$ for some positive integer n, for some $0 \neq m \in M\}$. A proper submodule N of M is said to be an $\mathfrak{M}$-primal if $P(N)$ forms an ideal of R.

2. $P(N)$-Locally Primary and $P(N)$-Locally Weakly Primary Submodules

Definition 1. *Let N be a proper submodule of an R-module M. Then N is called a $P(N)$-locally primary submodule of M if $N_{\mathfrak{M}}$ is a primary submodule of $M_{\mathfrak{M}}$ for all maximal ideal $\mathfrak{M}$ where $P(N) \subseteq \mathfrak{M}$.*

Definition 2. *A proper submodule N of an R-module M is called a $P(N)$-locally weakly primary submodule of M if $N_{\mathfrak{M}}$ is a weakly primary submodule of $M_{\mathfrak{M}}$ for every maximal ideal $\mathfrak{M}$ where $P(N) \subseteq \mathfrak{M}$.*

Lemma 1. *Let N be a proper submodule of an R-module M. Then $rad(N : M) \subseteq P(N)$.*

Proof. Let $r \in rad(N : M)$. Then $r^n M \subseteq N$ for some positive integer n. There exists $m \in M \setminus N$ such that $r^n m \in N$. Then $r \in P(N)$. Thus $rad(N : M) \subseteq P(N)$. □

Every primary submodule N is proposed as $P(N)$-locally primary submodule and every weakly primary submodule N is proposed as $P(N)$-locally weakly primary submodule in the following propositions, respectively.

Proposition 1. *A primary submodule N of an R-module M is a $P(N)$-locally primary submodule.*

Proof. Let $\mathfrak{M}$ be a maximal ideal of R where $P(N) \subseteq \mathfrak{M}$. By the previous lemma, we say that $rad(N : M) \subseteq P(N) \subseteq \mathfrak{M}$. $N_{\mathfrak{M}}$ is a proper submodule of $M_{\mathfrak{M}}$. Indeed, if $N_{\mathfrak{M}} = M_{\mathfrak{M}}$, then $\frac{m}{1} \in M_{\mathfrak{M}}$ for any $m \in M$. Then $rm \in N$ for some $r \notin \mathfrak{M}$. We get $r^n \notin rad(N : M)$. Since N is a primary submodule of M, then $m \in N$. Thus $N = M$, a contradiction. Since $rad(N : M) \cap (R \setminus \mathfrak{M}) = \emptyset$, then $N_{\mathfrak{M}}$ is a primary submodule of $M_{\mathfrak{M}}$. Consequently, N is a $P(N)$-locally primary submodule. □

Proposition 2. *A weakly primary submodule N of an R-module M is a $P(N)$-locally weakly primary submodule.*

Proof. Suppose that $\mathfrak{M}$ is a maximal ideal of R where $P(N) \subseteq \mathfrak{M}$. In the same manner as in the proof of the previous proposition, we have that $N_{\mathfrak{M}}$ is a proper submodule of $M_{\mathfrak{M}}$. Let $0_{\mathfrak{M}} \neq \frac{r}{s}\frac{m}{p} \in N_{\mathfrak{M}}$ for some $\frac{r}{s} \in R_{\mathfrak{M}}$ and $\frac{m}{p} \in M_{\mathfrak{M}}$ (for some $r \in R$, $m \in M$ and $s, p \in R \setminus \mathfrak{M}$). Then there is a $q \in R \setminus \mathfrak{M}$ such that $qrm \in N$. Assume that $qrm = 0$. Then $\frac{r}{s}\frac{m}{p} = \frac{q}{q}\frac{r}{s}\frac{m}{p} = \frac{qrm}{qsp} = 0_{\mathfrak{M}}$, this is a contradiction. So $0 \neq qrm \in N$. As $rad(N : M) \subseteq P(N) \subseteq \mathfrak{M}$, then $q \notin rad(N : M)$. Thus $rm \in N$ since N is a weakly primary submodule. It is clear that $rm \neq 0$. Hence $0 \neq rm \in N$ implies that $m \in N$ or $r^n M \subseteq N$ for some positive integer n. Thus we get $\frac{m}{p} \in N_{\mathfrak{M}}$ or $\frac{r^n}{s^n} M_{\mathfrak{M}} \subseteq N_{\mathfrak{M}}$ for some positive integer n by [4, Corollary 2.9]. Then we get that $N_{\mathfrak{M}}$ is a weakly primary submodule of $M_{\mathfrak{M}}$. Consequently, N is a $P(N)$-locally weakly primary submodule. □

Corollary 1. *Let N be a proper submodule of an R-module M. If N is primary, then N is $P(N)$-locally weakly primary.*

Proof. Assume that N is a primary submodule. Then N is a weakly primary submodule. Thus, N is a $P(N)$-locally weakly primary submodule by Proposition 2. □

Note that if N is a $P(N)$-locally primary submodule of M, then N is a $P(N)$-locally weakly primary submodule of M.

We give an example to show the converse is not true.

Example 1. *Consider $R = F[X,Y,Z]$—module $M = F[X,Y,Z]/(X^2, YZ)$ and the zero submodule $N = (0)$ of M. One can easily see that $P(N) = \{0, X, Y, Z, \ldots\}$. Note that $P(N) \subseteq \mathfrak{M} = (X,Y,Z)$ which is the unique maximal ideal of R. Then $N_{\mathfrak{M}} = (0)$ is weakly primary submodule of $R_{\mathfrak{M}}$— module $M_{\mathfrak{M}}$. Thus N is $P(N)$—weakly primary submodule. But $N_{\mathfrak{M}}$ is not primary submodule since, $\frac{Y}{1} \cdot \frac{\bar{Z}}{1} = \frac{\bar{0}}{1} \in N_{\mathfrak{M}}$ but $\left(\frac{Y}{1}\right)^n \notin (N_{\mathfrak{M}} : M_{\mathfrak{M}})$, $\left(\frac{\bar{Z}}{1}\right) \notin N_{\mathfrak{M}}$ for all positive integer n. Thus N is not $P(N)$—primary.*

In the following example, it is illustrated that a submodule N can be both $P(N)$-locally primary submodule of M and $P(N)$-locally weakly primary submodule of M but it is neither primary submodule of M nor weakly primary submodule of M.

Example 2. *Let $R = \mathbb{Z}$ and consider the $R-module$ $M{=}\mathbb{Z}_{12}$. Let N be the submodule of $\mathbb{Z}_{12}$ generated by $\overline{6}$. It is easily seen that $\overline{0} \neq 2\overline{3} (= 3\overline{2}) \in N$ but $2 \notin (N : M)$ and $\overline{3} \notin N$ ($3 \notin (N : M)$ and $\overline{2} \notin N$), that is, N is not a weakly primary submodule of M, hence N is not a primary submodule of M. Assume that N is not a $P(N)$-locally primary submodule of M. Then there exists a maximal ideal $\mathfrak{M}$ of R with $P(N) \subseteq \mathfrak{M}$ where $N_{\mathfrak{M}}$ is not a primary submodule of $M_{\mathfrak{M}}$. Note that $2, 3 \in P(N)$. Thus $1 \in \mathfrak{M}$, a contradiction. Therefore, N is a $P(N)$-locally primary submodule of M. Hence N is a $P(N)$-locally weakly primary submodule of M.*

Theorem 1. *Let N be a proper submodule of an R-module M. Then the following statements hold:*

(i) *N is a primary submodule if and only if $P(N) = rad(N : M)$.*

(ii) *Let $P(0) \subseteq rad(N : M)$. Then N is a primary submodule if and only if N is a weakly primary submodule.*

Proof. *(i)* Assume that N is a primary submodule. Let $r \in P(N)$. Then $r^n m \in N$ for some positive integer n and for some $m \in M \setminus N$. Since N is a primary submodule, then $(r^n)^k M = r^{nk} M \subseteq N$ for some positive integer k, that is, $r \in rad(N : M)$. Hence $P(N) \subseteq rad(N : M)$. By Lemma 1, we get $P(N) = rad(N : M)$.

Suppose that $P(N) = rad(N : M)$. Let $rm \in N$ and $m \in M \setminus N$ where $r \in R$, $m \in M$. Then $r \in P(N)$. Thus $r \in rad(N : M)$, that is, $r^k M \subseteq N$ for some positive integer k. Consequently, N is a primary submodule.

(ii) It is clear that every primary submodule is a weakly primary submodule.

Now, assume that N is a weakly primary submodule. Let $r \in P(N)$. Then $r^n m \in N$ for some positive integer n and for some $m \in M \setminus N$. Suppose that $r^n m = 0$. Since $m \in M \setminus N$, then we get $m \neq 0$. So $r \in P(0)$. Thus $r \in rad(N : M)$, by assumption. Hence $P(N) = rad(N : M)$ by Lemma 1. Suppose that $0 \neq r^n m \in N$. Since $m \in M \setminus N$ and N is a weakly primary submodule, then $(r^n)^k M \subseteq N$ for some positive integer k, that is, $r \in rad(N : M)$ and so $P(N) = rad(N : M)$. By *(i)*, N is a primary submodule. □

Corollary 2. *Let N be a proper submodule of an R-module M with $P(N) = rad(N : M)$. Then N is a $P(N)$-locally primary submodule and $P(N)$-locally weakly primary submodule.*

Proof. We get that N is a primary submodule by Theorem 1*(i)*. Then N is a $P(N)$-locally primary submodule by Proposition 1. Since N is primary submodule, then N is weakly primary submodule. Therefore, N is $P(N)$-locally weakly primary submodule by Proposition 2. □

Note that, by [4, Lemma 2.19], if $\mathfrak{M}$ is a maximal ideal of R, then $(N : M)_{\mathfrak{M}} \subseteq (N_{\mathfrak{M}} : M_{\mathfrak{M}})$. Now, we explain that $rad((N : M)_{\mathfrak{M}}) = rad(N_{\mathfrak{M}} : M_{\mathfrak{M}})$ when $\mathfrak{M}$ is a maximal ideal of R with $P(N) \subseteq \mathfrak{M}$.

Proposition 3. *Let N be a proper submodule of an R-module M. Then $rad((N : M)_{\mathfrak{M}}) = rad(N_{\mathfrak{M}} : M_{\mathfrak{M}})$ for any maximal ideal $\mathfrak{M}$ of R with $P(N) \subseteq \mathfrak{M}$.*

Proof. Since $S(N) \subseteq P(N)$ for any proper submodule N of M, it is clear from [4, Lemma 2.19 and Lemma 2.20]. □

Lemma 2. *Let N be a proper submodule of an R-module M. Then $rad((N : M)_{\mathfrak{M}}) = (rad(N : M))_{\mathfrak{M}}$ for any maximal ideal $\mathfrak{M}$ of R with $P(N) \subseteq \mathfrak{M}$.*

Proof. Let $\frac{r}{p} \in rad((N : M)_{\mathfrak{M}})$ for some $r \in R$ and $p \in R \setminus \mathfrak{M}$. Then $(\frac{r}{p})^n = \frac{r^n}{p^n} \in (N : M)_{\mathfrak{M}}$ for some positive integer n. There exists an element $q \in R \setminus \mathfrak{M}$ such that $qr^n \in (N : M)$, that is, $qr^n m \in N$ for every $m \in M$. Then $r^n m \in N$ for every $m \in M$ since $q \notin P(N)$. Thus $r \in rad(N : M)$. Then $\frac{r}{p} \in rad((N : M)_{\mathfrak{M}})$. Conversely, assume that $\frac{r}{p} \in rad((N : M)_{\mathfrak{M}})$. There is an $u \in R \setminus \mathfrak{M}$ such that $ur \in rad(N : M)$. Then $(ur)^n = u^n r^n \in (N : M)$. Hence $\frac{u^n}{u^n} \frac{r^n}{p^n} \in (N : M)_{\mathfrak{M}}$. Consequently, $\frac{r^n}{p^n} = (\frac{r}{p})^n \in (N : M)_{\mathfrak{M}}$ and so $\frac{r}{p} \in rad((N : M)_{\mathfrak{M}})$. □

Corollary 3. *Let N be a proper submodule of an R-module. If $\mathfrak{M}$ is any maximal ideal of R with $P(N) \subseteq \mathfrak{M}$, then $rad((N:M)_{\mathfrak{M}}) = rad(N_{\mathfrak{M}} : M_{\mathfrak{M}})$.*

Proof. It is clear from Proposition 3 and Lemma 2. □

Proposition 4. *Let N be a proper submodule of an R-module M and $m \in M$. Then $rad((N:Rm)_{\mathfrak{M}}) = rad(N_{\mathfrak{M}} : (Rm)_{\mathfrak{M}})$ for any maximal ideal $\mathfrak{M}$ of R with $P(N) \subseteq \mathfrak{M}$.*

Proof. It is clear. □

If we put $N = 0$ in Proposition 4, we have the following corollary.

Corollary 4. *Let M be an R-module and $m \in M$. Then $rad((0:Rm)_{\mathfrak{M}}) = rad(0_{\mathfrak{M}} : (Rm)_{\mathfrak{M}})$ for any maximal ideal $\mathfrak{M}$ of R with $P(0) \subseteq \mathfrak{M}$.*

Proposition 5. *Let N be a proper submodule of an R-module M and $\mathfrak{M}$ be a maximal ideal of R with $P(N) \subseteq \mathfrak{M}$. Then the following statements hold:*

(i) Let $P(0) \subseteq P(N)$. Then $rad(N:M)$ is a weakly prime ideal of R if and only if $rad((N:M)_{\mathfrak{M}})$ is a weakly prime ideal of $R_{\mathfrak{M}}$.

(ii) $rad(N:M)$ is a prime ideal of R if and only if $rad((N:M)_{\mathfrak{M}})$ is a prime ideal of $R_{\mathfrak{M}}$.

Proof. *(i)* Suppose that $rad(N:M)$ is a weakly prime ideal of R. If $rad(N:M)_{\mathfrak{M}} = R_{\mathfrak{M}}$, then $\frac{1}{1} \in rad((N:M)_{\mathfrak{M}}) = (rad(N:M))_{\mathfrak{M}}$ and so $q1 = q \in rad(N:M)$ for some $q \in R \setminus \mathfrak{M}$. But by Lemma 1, $rad(N:M) \subseteq P(N) \subseteq \mathfrak{M}$, which is a contradiction. So $rad((N:M)_{\mathfrak{M})} \neq R_{\mathfrak{M}}$, that is, $rad((N:M)_{\mathfrak{M}})$ is a proper ideal of $R_{\mathfrak{M}}$. Let $0 \neq \frac{r}{p}\frac{s}{q} \in rad((N:M)_{\mathfrak{M}})$, where $r, s \in R$ and $p, q \in R \setminus \mathfrak{M}$. Then we have $\frac{r}{p}\frac{s}{q} = \frac{rs}{pq} \in (rad(N:M))_{\mathfrak{M}}$, then there exists an $u \in R \setminus \mathfrak{M}$ such that $urs \in rad(N:M)$. If $urs = 0$, then $\frac{r}{p}\frac{s}{q} = \frac{u}{u}\frac{r}{p}\frac{s}{q} = \frac{urs}{upq} = 0$, this is a contradiction. So $urs \neq 0$. Since $0 \neq urs \in rad(N:M)$ and $rad(N:M)$ is a weakly prime ideal of R, then $ur \in rad(N:M)$ or $s \in rad(N:M)$. Hence $\frac{r}{p} = \frac{u}{u}\frac{r}{p} \in (rad(N:M))_{\mathfrak{M}}$ or $\frac{s}{q} \in (rad(N:M))_{\mathfrak{M}}$, that is, $\frac{r}{p} \in rad((N:M)_{\mathfrak{M}})$ or $\frac{s}{q} \in rad((N:M)_{\mathfrak{M}})$.

Assume that $rad((N:M)_{\mathfrak{M}})$ is a weakly prime ideal of $R_{\mathfrak{M}}$. If $rad(N:M) = R$, then $rad((N:M)_{\mathfrak{M}}) = R_{\mathfrak{M}}$, a contradiction. So $rad(N:M)$ is a proper ideal of R. Let $0 \neq ab \in rad(N:M)$ for some $a, b \in R$. Then $\frac{ab}{1} = \frac{a}{1}\frac{b}{1} \in rad((N:M)_{\mathfrak{M}})$. If $\frac{a}{1}\frac{b}{1} = 0$, then $qab = 0$ for some $q \in R \setminus \mathfrak{M}$. As $0 \neq ab$, then $q \in P(0)$. Thus $q \in \mathfrak{M}$, which is a contradiction. So $0 \neq \frac{a}{1}\frac{b}{1} \in rad((N:M)_{\mathfrak{M}})$. Since $rad((N:M)_{\mathfrak{M}})$ is a weakly prime ideal of $R_{\mathfrak{M}}$, then $\frac{a}{1} \in rad((N:M)_{\mathfrak{M}})$ or $\frac{b}{1} \in rad((N:M)_{\mathfrak{M}})$. Therefore $pa \in rad(N:M)$ for some $p \in R \setminus \mathfrak{M}$ or $sb \in rad(N:M)$ for some $s \notin \mathfrak{M}$. As $p \in R \setminus \mathfrak{M}$ and $s \in R \setminus \mathfrak{M}$, then $p, s \notin P(N)$. Consequently, $a \in rad(N:M)$ or $b \in rad(N:M)$.

(ii) Assume that $rad(N:M)$ is a prime ideal of R. In a similar way, we get $rad((N:M)_{\mathfrak{M}})$ is a proper ideal of $R_{\mathfrak{M}}$. Now, let $\frac{r}{p}\frac{s}{q} \in rad((N:M)_{\mathfrak{M}})$, where $r, s \in R$ and $p, q \in R \setminus \mathfrak{M}$. Then we have $\frac{rs}{pq} \in (rad(N:M))_{\mathfrak{M}}$ and so we have $urs \in rad(N:M)$ for some $u \in R \setminus \mathfrak{M}$. Since $rad(N:M)$ is a prime ideal of R, then $ur \in rad(N:M)$ or $s \in rad(N:M)$. Consequently, $\frac{r}{p} = \frac{u}{u}\frac{r}{p} \in (rad(N:M))_{\mathfrak{M}}$ or $\frac{s}{q} \in (rad(N:M))_{\mathfrak{M}}$, that is, $\frac{r}{p} \in rad((N:M)_{\mathfrak{M}})$ or $\frac{s}{q} \in rad((N:M)_{\mathfrak{M}})$.

Suppose that $rad((N:M)_{\mathfrak{M}})$ is a prime ideal of $R_{\mathfrak{M}}$. From *(i)*, it is clear that $rad(N:M)$ is a proper ideal of R. Then $\frac{ab}{1}=\frac{a}{1}\frac{b}{1}\in rad((N:M)_{\mathfrak{M}})$ for some $a,b\in R$ and since $rad(N:M)_{\mathfrak{M}}$ is a prime ideal of $R_{\mathfrak{M}}$, then $\frac{a}{1}\in rad((N:M)_{\mathfrak{M}})$ or $\frac{b}{1}\in rad((N:M)_{\mathfrak{M}})$. Thus $pa\in rad(N:M)$ for some $p\in R\setminus\mathfrak{M}$ or $sb\in rad(N:M)$ for some $s\in R\setminus\mathfrak{M}$. As $p\in R\setminus\mathfrak{M}$ and $s\in R\setminus\mathfrak{M}$, then $p,s\notin P(N)$. Therefore, $a\in rad(N:M)$ or $b\in rad(N:M)$. □

Proposition 6. *Let M be a faithful cyclic R-module and N be a proper submodule of M with $P(0)\subseteq P(N)$. If N is a $P(N)$-locally weakly primary submodule of M, then $rad(N:M)$ is a weakly prime ideal of R.*

Proof. Let $\mathfrak{M}$ be a maximal ideal of R with $P(N)\subseteq\mathfrak{M}$. By [4, Proposition 2.18], $M_{\mathfrak{M}}$ is a faithful cyclic $R_{\mathfrak{M}}$-module. Then $N_{\mathfrak{M}}$ is a weakly primary submodule of $M_{\mathfrak{M}}$. Thus by [1, Proposition 2.3], $rad(N_{\mathfrak{M}}:M_{\mathfrak{M}})$ is a weakly prime submodule of $M_{\mathfrak{M}}$. By Proposition 3, $rad((N:M)_{\mathfrak{M}})$ is a weakly prime submodule of $M_{\mathfrak{M}}$. By Proposition 5 i), $rad(N:M)$ is a weakly prime ideal of R. □

Proposition 7. *Let M be an R-module. Suppose that N is an $\mathfrak{M}$-primal and a $P(N)$-locally weakly primary submodule of M not primary submodule of M. If $P(0)\subseteq P(N)$ and I is an ideal of R such that $I\subseteq rad(N:M)$, then $IN=0$. Particularly, $rad(N:M)N=0$.*

Proof. Suppose that $P(0)\subseteq P(N)$ and I is an ideal of R such that $I\subseteq rad(N:M)$. Since N is $\mathfrak{M}$-primal, then $P(N)$ is an ideal of R. As $1\notin P(N)$, then $P(N)$ is a proper ideal. Hence there is a maximal ideal $\mathfrak{M}$ of R such that $P(N)\subseteq\mathfrak{M}$. Then, $N_{\mathfrak{M}}$ is a weakly primary submodule of $M_{\mathfrak{M}}$ because N is a $P(N)$-locally weakly primary submodule of M. Our aim is to show that $N_{\mathfrak{M}}$ is not a primary submodule of $M_{\mathfrak{M}}$. Assume that $N_{\mathfrak{M}}$ is a primary submodule of $M_{\mathfrak{M}}$. Let $rm\in N$ for some $r\in R$, $m\in M$. Then $\frac{rm}{1}=\frac{r}{1}\frac{m}{1}\in N_{\mathfrak{M}}$. By assumption, $\frac{m}{1}\in N_{\mathfrak{M}}$ or $(\frac{r}{1})^nM_{\mathfrak{M}}\subseteq N_{\mathfrak{M}}$ for some positive integer n. By using a similar technique in the previous proofs, $m\in N$ or $r^nM\subseteq N$ for some positive integer n since $P(N)\subseteq\mathfrak{M}$, but this contradicts with N which is not a primary submodule of M. By [4, Lemma 2.19], $I_{\mathfrak{M}}\subseteq rad((N:M)_{\mathfrak{M}})\subseteq rad(N_{\mathfrak{M}}:M_{\mathfrak{M}})$. By [1, Corollary 3.4], $I_{\mathfrak{M}}N_{\mathfrak{M}}=0$. We get $\frac{r}{1}\frac{m}{1}=\frac{rm}{1}=0$ for every $r\in I$ and every $m\in N$. Therefore $qrm=0$ for some $q\in R\setminus\mathfrak{M}$. If $rm\neq 0$, then $q\in P(0)$ and so $q\in\mathfrak{M}$, which is a contradiction. Hence $rm=0$, that is, $IN=0$. Particularly, by putting $I=rad(N:M)$, we have $rad(N:M)N=0$. □

Proposition 8 ([4, Proposition 2.16]). *Let M be an R-module and $\mathfrak{M}$ be a maximal ideal of R. If $\overline{I}$ is an ideal of $R_{\mathfrak{M}}$ and $\overline{N}$ is a submodule of $M_{\mathfrak{M}}$, then*

(i) $I=\{a\in R\mid \frac{a}{1}\in\overline{I}\}$ *is an ideal of R and $\overline{I}=I_{\mathfrak{M}}$.*

(ii) $N=\{m\in M\mid \frac{m}{1}\in\overline{N}\}$ *is a submodule of M and $\overline{N}=N_{\mathfrak{M}}$.*

Theorem 2. *Let N be an $\mathfrak{M}$-primal submodule of an R-module M with $P(0)\subseteq P(N)$. Then N is a $P(N)$-locally weakly primary submodule of M if and only if $0\neq ID\subseteq N$ for some ideal I of R and some submodule D of M implies $I\subseteq rad(N:M)$ or $D\subseteq N$.*

Proof. Assume that N is a $P(N)$-locally weakly primary submodule of M. Let $0 \neq ID \subseteq N$ for some ideal I of R and some submodule D of M. Since N is $\mathfrak{M}$-primal, then $P(N)$ is an ideal of R. As $1 \notin P(N)$, then $P(N)$ is a proper ideal. So we have $P(N) \subseteq \mathfrak{M}$ for some maximal ideal $\mathfrak{M}$ of R. Thus $N_{\mathfrak{M}}$ is a weakly primary submodule of $M_{\mathfrak{M}}$. Now, $I_{\mathfrak{M}}$ is an ideal of $R_{\mathfrak{M}}$ and $D_{\mathfrak{M}}$ is a submodule of $M_{\mathfrak{M}}$ with $(ID)_{\mathfrak{M}} = I_{\mathfrak{M}}D_{\mathfrak{M}} \subseteq N_{\mathfrak{M}}$. Suppose that $I_{\mathfrak{M}}D_{\mathfrak{M}} = 0_{\mathfrak{M}}$. Then $\frac{r}{1}\frac{m}{1} = \frac{rm}{1} = 0$ for every $r \in I$ and every $m \in D$. So there exists a $q \in R \setminus \mathfrak{M}$ such that $qrm = 0$. If $rm \neq 0$, then $q \in P(0)$. Thus $q \in \mathfrak{M}$, which is a contradiction. So $rm = 0$, hence $ID = 0$, that is a contradiction. Then $0_{\mathfrak{M}} \neq I_{\mathfrak{M}}D_{\mathfrak{M}} \subseteq N_{\mathfrak{M}}$. Since N is a $P(N)$-locally weakly primary submodule of M, then $N_{\mathfrak{M}}$ is a weakly primary submodule of $M_{\mathfrak{M}}$. By [1, Theorem 3.6], either $I_{\mathfrak{M}} \subseteq rad(N_{\mathfrak{M}} : M_{\mathfrak{M}})$ or $D_{\mathfrak{M}} \subseteq N_{\mathfrak{M}}$. Since $P(N) \subseteq \mathfrak{M}$, then $I \subseteq rad(N : M)$ or $D \subseteq N$.

Let $\mathfrak{M}$ be a maximal ideal of R with $P(N) \subseteq \mathfrak{M}$. Since N is a proper ideal of R, then there is an $a \in M \setminus N$, but $\frac{a}{1} \in M_{\mathfrak{M}}$. If $\frac{a}{1} \in N_{\mathfrak{M}}$, then $qa \in N$ such that $q \in R \setminus \mathfrak{M}$. As $a \in M \setminus N$, then $q \in P(N)$, that is, $q \in \mathfrak{M}$, which is a contradiction. So $\frac{a}{1} \in M_{\mathfrak{M}} \setminus N_{\mathfrak{M}}$. Hence $N_{\mathfrak{M}}$ is a proper ideal of $R_{\mathfrak{M}}$. Let $\overline{I}$ be an ideal of $R_{\mathfrak{M}}$ and $\overline{D}$ be a submodule of $M_{\mathfrak{M}}$ with $0_{\mathfrak{M}} \neq \overline{ID} \subseteq N_{\mathfrak{M}}$. By [4, Proposition 2.16], $\overline{I} = I_{\mathfrak{M}}$, for some ideal I of R and $\overline{D} = D_{\mathfrak{M}}$, for some submodule D of M. So $0_{\mathfrak{M}} \neq I_{\mathfrak{M}}D_{\mathfrak{M}} \subseteq N_{\mathfrak{M}}$, that is, $0_{\mathfrak{M}} \neq (ID)_{\mathfrak{M}} \subseteq N_{\mathfrak{M}}$. Since $P(N) \subseteq \mathfrak{M}$, then $ID \subseteq N$. Also $0 \neq ID$. On the contrary, $(ID)_{\mathfrak{M}} = 0_{\mathfrak{M}}$. By the hypothesis, we have either $I \subseteq rad(N : M)$ or $D \subseteq N$. If $I \subseteq rad(N : M)$, then $\overline{I} = I_{\mathfrak{M}} \subseteq rad((N : M)_{\mathfrak{M}})$. If $D \subseteq N$, then $\overline{D} = D_{\mathfrak{M}} \subseteq N_{\mathfrak{M}}$. From [1, Theorem 3.6], $N_{\mathfrak{M}}$ is a weakly primary submodule of $M_{\mathfrak{M}}$. Therefore, N is a $P(N)$-locally weakly primary submodule of M. □

Corollary 5. *Let N be an $\mathfrak{M}$-primal submodule of an R-module M with $P(0) \subseteq P(N)$. Then N is a $P(N)$-locally weakly primary submodule of M if and only if N is a weakly primary submodule of M.*

Proof. It is clear from Theorem 2 and [1, Theorem 3.6]. □

Theorem 3. *Let M be an R-module and N be an $\mathfrak{M}$-primal submodule of M with $P(0) \subseteq P(N)$. Then the following statements are equivalent:*

(i) N is a $P(N)$-locally weakly primary submodule of M.

(ii) For any $m \in M \setminus N$, $rad(N : Rm) = rad(N : M) \cup (0 : Rm)$.

(iii) For any $m \in M \setminus N$, $rad(N : Rm) = rad(N : M)$ or $rad(N : Rm) = (0 : Rm)$.

Proof. $(i) \Longrightarrow (ii)$: Let N be a $P(N)$-locally weakly primary submodule of M and let $m \in M \setminus N$. Since N is $\mathfrak{M}$-primal, then $P(N)$ is an ideal of R. As $i \notin P(N)$, then $P(N)$ is a proper ideal. So we have $P(N) \subseteq \mathfrak{M}$ for some maximal ideal $\mathfrak{M}$ of R. Hence $N_{\mathfrak{M}}$ is a weakly primary submodule of $M_{\mathfrak{M}}$. As $m \in M$, then $\frac{m}{1} \in M_{\mathfrak{M}}$, but $\frac{m}{1} \in M_{\mathfrak{M}} \setminus N_{\mathfrak{M}}$. If $\frac{m}{1} \in N_{\mathfrak{M}}$, then $pm \in N$ for some $p \in R \setminus \mathfrak{M}$. Since $p \notin P(N)$, then $m \in N$, this is a contradiction. By [2, Theorem 2.15], $rad(N_{\mathfrak{M}} : R_{\mathfrak{M}}\frac{m}{1}) = rad(N_{\mathfrak{M}} : M_{\mathfrak{M}}) \cup (0_{\mathfrak{M}} : R_{\mathfrak{M}}\frac{m}{1})$ and from [4, Corollary 2.9], $rad(N_{\mathfrak{M}} : (Rm)_{\mathfrak{M}}) = rad(N_{\mathfrak{M}} : M_{\mathfrak{M}}) \cup (0_{\mathfrak{M}} : (Rm)_{\mathfrak{M}})$. Then by Proposition 3, Proposition 4 and Corollary 4, $rad((N : Rm)_{\mathfrak{M}}) = rad((N : M)_{\mathfrak{M}}) \cup (0 : Rm)_{\mathfrak{M}}$. Let $r \in rad(N : Rm)$. Then $\frac{r}{1} \in rad((N : Rm)_{\mathfrak{M}})$ and so $\frac{r}{1} \in rad((N : M)_{\mathfrak{M}})$ or $\frac{r}{1} \in (0 : Rm)_{\mathfrak{M}}$. If $\frac{r}{1} \in rad((N : M)_{\mathfrak{M}})$, then

$\frac{r^n}{1} \in (N:M)_{\mathfrak{M}}$ for some positive integer n and thus $qr^n \in (N:M)$ for some $q \in R \setminus \mathfrak{M}$, that is, $qr^nM \subseteq N$. Assume that $r^nM \nsubseteq N$. Then $r^nm \notin N$ for some $m \in M$, however $qr^nm \in N$. Hence $q \in P(N)$. Then $q \in \mathfrak{M}$, which is a contradiction. So $r^nM \subseteq N$ for some positive integer n, that is, $r \in rad(N:M)$. If $\frac{r}{1} \in (0:Rm)_{\mathfrak{M}}$, then $pr \in (0:Rm)$ for some $p \in R \setminus \mathfrak{M}$. Thus $prRm = 0$. Assume that $rRm \neq 0$. Then $rsm \neq 0$ for some $s \in R$, but $prsm = 0$. Therefore $p \in P(0)$. As $P(0) \subseteq \mathfrak{M}$, then $p \in \mathfrak{M}$, which is a contradiction. So $rRm = 0$. Then $r \in (0:Rm)$. Hence $r \in rad(N:M) \cup (0:Rm)$. Conversely, let $r \in rad(N:M) \cup (0:Rm)$. If $r \in rad(N:M)$, then $r^nM \subseteq N$ for some positive integer n and so we get $r^nRm \subseteq r^nM \subseteq N$. Thus $r \in rad(N:Rm)$. If $r \in (0:Rm)$, then $rRm = 0 \subseteq N$. Thus $r \in (N:Rm) \subseteq rad(N:Rm)$.

$(ii) \Rightarrow (iii)$: Clear.

$(iii) \Rightarrow (i)$: Let $\mathfrak{M}$ be a maximal ideal of R with $P(N) \subseteq \mathfrak{M}$. Let $\frac{m}{p} \in M_{\mathfrak{M}} \setminus N_{\mathfrak{M}}$ where $m \in M$, $p \in R \setminus \mathfrak{M}$. Then $m \in M \setminus N$. By the condition of the theorem, $rad(N:Rm) = rad(N:M)$ or $rad(N:Rm) = (0:Rm)$ for some $m \in M \setminus N$. If $rad(N:Rm) = rad(N:M)$, then $rad((N:Rm)_{\mathfrak{M}}) = rad((N:M)_{\mathfrak{M}})$ and from Proposition 3 and Proposition 4
$rad(N_{\mathfrak{M}}:(Rm)_{\mathfrak{M}}) = rad(N_{\mathfrak{M}}:M_{\mathfrak{M}})$. By [4, Proposition 2.8], $rad(N_{\mathfrak{M}}:R_{\mathfrak{M}}\frac{m}{p}) = rad(N_{\mathfrak{M}}:M_{\mathfrak{M}})$. If $rad(N:Rm) = (0:Rm)$, then $rad((N:Rm)_{\mathfrak{M}}) = (0:Rm)_{\mathfrak{M}}$ and by Proposition 4 and Corollary 4, $rad(N_{\mathfrak{M}}:(Rm)_{\mathfrak{M}}) = (0_{\mathfrak{M}}:(Rm)_{\mathfrak{M}})$. By [4, Proposition 2.8],
$rad(N_{\mathfrak{M}}:R_{\mathfrak{M}}\frac{m}{p}) = (0_{\mathfrak{M}}:R_{\mathfrak{M}}\frac{m}{p})$. By [2, Theorem 2.15], $N_{\mathfrak{M}}$ is a weakly primary submodule of $M_{\mathfrak{M}}$. Thus N is a $P(N)$-locally weakly primary submodule of M. □

Theorem 4. *Let M be an R-module and N be an $\mathfrak{M}$-primal submodule of M with $P(0) \subseteq P(N)$. Then the following statements are equivalent:*

(i) *N is a $P(N)$-locally weakly primary submodule of M.*

(ii) *$0 \neq ID \subseteq N$ for any ideal I of R and any submodule D of M implies either $I \subseteq rad(N:M)$ or $D \subseteq N$.*

(iii) *$rad(N:Rm) = rad(N:M) \cup (0:Rm)$ for any $m \in M \setminus N$.*

(iv) *$rad(N:Rm) = rad(N:M)$ or $rad(N:Rm) = (0:Rm)$ for any $m \in M \setminus N$,*

Proof. It is clear from Theorem 2 and Theorem 3. □

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