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On $(1+u)$-Cyclic and Cyclic Codes over $F_2+uF_2+vF_2$

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Abstract. It is studied codes over the ring $R=F_2+uF_2+vF_2$, $u^2=0, v^2=v, uv=vu=0$ which contains the two ring $F_2+uF_2, u^2=0$ and $F_2+vF_2, v^2=v$. It is introduced $(1+u)$-cyclic codes and cyclic codes over $F_2+uF_2+vF_2$. It is characterized codes over F_2+vF_2 which are the images of $(1+u)$-cyclic codes and cyclic codes over $F_2+uF_2+vF_2$. It is obtained a representation of a linear code of length n over R by means of C_1 and C_2 which are linear codes of length n over F_2+uF_2. It is also characterized codes over F_2 which are the Gray images of $(1+u)$-cyclic codes or cyclic codes over $F_2+uF_2+vF_2$.

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1. Introduction

It was introduced linear $(1+u)$ constacyclic codes and cyclic codes over F_2+uF_2 and characterized codes over F_2 which are the Gray images of $(1+u)$ constacyclic codes or cyclic codes over F_2+uF_2, in [6]. In [1], they extended the result of [6] to codes over the commutative ring $F_{p^k}+uF_{p^k}$ where p is a prime, $k\in N$ and $u^2=0$.

In [5], it was introduced $(1-u^2)$-cyclic codes over $F_2+uF_2+u^2F_2$ and characterized codes over F_2 which are the Gray images of $(1-u^2)$-cyclic codes or cyclic codes over $F_2+uF_2+u^2F_2$.

In [2], it was defined a distance preserving map from $F_2+uF_2+u^2F_2+u^3F_2+\ldots+u^mF_2$ to F_2 and characterized codes over F_2 which are the Gray images of $(1-u^m)$-cyclic codes or cyclic codes over $F_2+uF_2+u^2F_2+u^3F_2+\ldots+u^mF_2$. In [8], Udomkavanich and Jitman generalized these results to the ring $F_{p^k}+uF_{p^k}+\ldots+u^mF_{p^k}$. The Gray images of $(1-u^m)$-constacyclic and cyclic codes over $F_{p^k}+uF_{p^k}+\ldots+u^mF_p^k$ were studied in the mentioned paper.

In [4], $(1+v)$-constacyclic codes over $R_2=F_2+uF_2+vF_2+uvF_2, u^2=v^2=0, uv-vu=0$ were studied. $(1+v)$-constacyclic codes over R_2 of odd length were characterized with help of cyclic codes over R_2.

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In [3], it is studied $(1+u)$-cyclic codes over a finite commutative ring $F_2+uF_2+vF_2+uvF_2, u^2=0, v^2=0, uv-vu=0$. A set of generator of such constacyclic codes for an arbitrary length was determined.

In [7], they studied linear codes over a new ring $S=F_2+uF_2+vF_2+uvF_2, u^2=0, v^2=v, uv=vu$. It is obtained MacWilliams identities for Lee weight enumerator of linear codes over this ring using a Gray map from S^n to $(F_2+uF_2)^n$. Moreover, they studied self dual and cyclic codes over S.

Liu Xiusheng and Liu Hualu gave rise to a new ring $R=F_2+uF_2+vF_2, u^2=0, v^2=v, uv=vu=0$ in [9]. It is Frobenius ring. They defined a Gray map. The MacWilliams identity over F_2 and the MacWilliams identities for the Lee weight enumerators of linear codes over the ring $F_2+uF_2+vF_2$ were given. Moreover, they gave some examples.

In this paper, it is given some definitions in section 2. It is seen that the image of a $(1+u)$-cyclic code of length n over R under the map $\phi_{1,1}$ is a distance invariant cyclic code of length $2n$ over F_2+vF_2. It is shown that if n is odd, then the image of a cyclic code of length n over R under the map $\phi_{1,1}$ is a permutation equivalent to cyclic code of length $2n$ over F_2+vF_2. In section 3, it is given a representation of a linear code of length n over R by means of C_1 and C_2 which are linear codes of length n over F_2+uF_2. In section 4, it is characterized codes over F_2 which are the Gray images of $(1+u)$-cyclic codes or cyclic codes over $F_2+uF_2+vF_2$. It is proved that the Gray image of a linear $(1+u)$-cyclic code over $F_2+uF_2+vF_2$ of length n is a binary permutation equivalent to quasi-cyclic codes of index 3 and length $3n$ over F_2. It is also proved that if n is odd, then every code over F_2 which is the Gray image of a linear cyclic code of length n over $F_2+uF_2+vF_2$ is permutation equivalent to a quasi-cyclic code of index 3.

2. Preliminaries

In [9], the commutative ring $R=F_2+uF_2+vF_2, u^2=0, v^2=v, uv=vu=0$ is given. Then R is a finite, principal ideal and semilocal ring with two maximal ideals I_{u+v} and I_{1+v}. The quotient rings R/I_{u+v} and R/I_{1+v} are isomorphic to F_2. A direct decomposition of R is $R=I_v\oplus I_{1+v}$. The set of units of R is $R^*=\{1,1+u\}$.

Let the C be a code of length n over R and $P(C)$ be its polynomial representation, i.e, $P(C)=\{\sum_{i=0}^{n-1} r_i x^i | (r_0,\ldots,r_{n-1})\in C\}$ Let σ and ν be maps from R^n to R^n given by

$$\sigma(r_0,\ldots,r_{n-1})=(r_{n-1},r_0,\ldots,r_{n-2})$$

and

$$\nu(r_0,\ldots,r_{n-1})=((1+u)r_{n-1},r_0,\ldots,r_{n-2})$$

Then C is said to be cyclic if $\sigma(C)=C$ and $(1+u)-$ cyclic if $\nu(C)=C$.

A code C of length n over R is cyclic if and only if $P(C)$ is an ideal of $R[x]/\langle x^n-1\rangle$. A code C of length n over R is $(1+u)-$ cyclic if and only if $P(C)$ is an ideal of $R[x]/\langle x^n-(1+u)\rangle$.

Let $a\in F_2^{3n}$ with $a=(a_0,a_1,\ldots,a_{3n-1}) = (a^{(0)}|a^{(1)}|a^{(2)})$, $a^{(i)}\in F_2^n$ for all $i=0,1,2$. Let $\sigma^{\otimes 3}$ be the map from F_2^{3n} to F_2^{3n} given by $\sigma^{\otimes 3}(a)=(\tilde{\sigma}(a^{(0)})|\tilde{\sigma}(a^{(1)})|\tilde{\sigma}(a^{(2)}))$ where $\tilde{\sigma}$ is the usual cyclic shift

$$(c_0, \dots, c_{n-1}) \longmapsto (c_{n-1}, c_0, \dots, c_{n-2})$$

on F_2^n. A code $\tilde{C}$ of length $3n$ over F_2 is said to be quasi-cyclic of index 3 if $\sigma^{\otimes 3}(\tilde{C}) = \tilde{C}$.

The Hamming weight $w_H(x)$ of a codeword x is the number of nonzero components in x. The Hamming distance $d(x, y)$ between two codewords x and y is the Hamming weight of the codewords $x - y$. The minimum Hamming distance d_H of C is defined as

$$min\{d_H(x, y) | x, y \in C, x \neq y\}.$$

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two vectors of R^n. The Euclidean inner product of x and y is defined

$$xy = \sum_{i=1}^{n} x_i y_i.$$

The dual code $C^\perp$ of C is defined as $C^\perp = \{x \in R^n | xc = 0 \text{ for all } c \in C\}$. C is said to be self orthogonal if $C \subseteq C^\perp$ and C is said to be self dual if $C = C^\perp$.

Recall that the Gray map ϕ_1 on $F_2 + uF_2, u^2 = 0$ is defined as $\phi_1(z) = (r, r + q)$ where $z = q + ur$ with $r, q \in F_2$ and the Gray map ϕ_2 on $F_2 + vF_2$, $v^2 = v$ is defined as $\phi_2(s) = (m, m + t)$ where $s = m + vt$ with $m, t \in F_2$. The maps ϕ_1 and ϕ_2 can be extended to $(F_2 + uF_2)^n$ and $(F_2 + vF_2)^n$, respectively as follows,

$$\begin{aligned}
\phi_1 :& (F_2 + uF_2)^n \to F_2^{2n} \\
& (z_0, \dots, z_{n-1}) \mapsto (r_0, \dots, r_{n-1}, r_0 \oplus q_0, \dots, r_{n-1} \oplus q_{n-1}) \\
\phi_2 :& (F_2 + vF_2)^n \to F_2^{2n} \\
& (s_0, \dots, s_{n-1}) \mapsto (m_0, \dots, m_{n-1}, m_0 \oplus t_0, \dots, m_{n-1} \oplus t_{n-1})
\end{aligned}$$

where $z_i = r_i + uq_i, s_i = m_i + vt_i$ and $q_i, r_i, m_i, t_i \in F_2$ for $0 \leq i \leq n-1$ and $\oplus$ is componentwise addition in F_2.

Each element $c \in R = F_2 + uF_2 + vF_2$ can be expressed $c = a + ub$ where $a, b \in F_2 + vF_2$. The map $\phi_{1,1}$ is defined as

$$\begin{aligned}
\phi_{1,1} :& R^n \to (F_2 + vF_2)^{2n} \\
& (c_0, \dots, c_{n-1}) \mapsto (b_0, \dots, b_{n-1}, b_0 + a_0, \dots, b_{n-1} + a_{n-1})
\end{aligned}$$

where $c_i = a_i + ub_i$ with $a_i, b_i \in F_2 + vF_2$ for $0 \leq i \leq n - 1$.

Each element $c \in R = F_2 + uF_2 + vF_2$ can be also expressed $c = a' + vb'$ where $a', b' \in F_2 + uF_2$. The map $\phi_{2,1}$ is defined as

$$\begin{aligned}
\phi_{2,1} :& R^n \to (F_2 + uF_2)^{2n} \\
& (c_0, \dots, c_{n-1}) \mapsto (a'_0, \dots, a'_{n-1}, b'_0 + a'_0, \dots, b'_{n-1} + a'_{n-1})
\end{aligned}$$

where $c_i = a'_i + vb'_i$ with $a'_i, b'_i \in F_2 + uF_2$ for $0 \leq i \leq n - 1$.

A Gray map ϕ from R to F_2^m which is the composition of $\phi_{1,1}$ and ϕ_2 or $\phi_{2,1}$ and ϕ_1 can be obtained.

The Lee weights of $0, 1, u, 1+u \in F_2 + uF_2$ are $0, 1, 2, 1$ respectively. The Lee weights of $0, 1, v, 1+v \in F_2 + vF_2$ are $0, 2, 1, 1$ respectively. These Lee weights can be extended to $(F_2+uF_2)^n$ and $(F_2+vF_2)^n$. It is known that ϕ_1 and ϕ_2 are distance-preserving map from $(F_2+uF_2)^n$ (Lee distance) to F_2^{2n} (Hamming distance) and $(F_2+vF_2)^n$ (Lee distance) to F_2^{2n} (Hamming distance), respectively. For any element $a+vb \in R$ with $a, b \in F_2+uF_2$, it is defined Lee weight, denoted by w_L as $w_L(a+vb) = w_L(b, b+a)$. The Lee distance of a linear code over R, denoted by $d_L(C)$ is defined as minimum Lee weight of nonzero codewords of C.

$$\begin{aligned}
\phi_1 &:(F_2+uF_2)^n \text{ (Lee distance) } \to F_2^{2n} \text{ (Hamming distance)}\\
\phi_2 &:(F_2+vF_2)^n \text{ (Lee distance) } \to F_2^{2n} \text{ (Hamming distance)}\\
\phi_{1,1} &:R^n \text{ (Lee distance) } \to (F_2+vF_2)^{2n} \text{ (Lee distance)}\\
\phi_{2,1} &:R^n \text{ (Lee distance) } \to (F_2+uF_2)^{2n} \text{ (Lee distance)}
\end{aligned}$$

Now, it will be characterized codes over F_2+vF_2 which are the images of $(1+u)$-cyclic and cyclic codes over R.

Proposition 1. *Let $\phi_{1,1}$ be defined as above. Let ν be $(1+u)$-cyclic shift on R^n and σ the cyclic shift on $(F_2+vF_2)^{2n}$. Then $\phi_{1,1}\nu = \sigma\phi_{1,1}$.*

Proof. Let $z = (z_0, \dots, z_{n-1}) \in R^n$ where $c_i = q_i + ur_i$ and $q_i, r_i \in F_2+vF_2$ for $0 \le i \le n-1$. From definition, we get,

$$\phi_{1,1}(z) = (r_0, \dots, r_{n-1}, r_0+q_0, \dots, r_{n-1}+q_{n-1})$$

and

$$\sigma(\phi_{1,1}(z)) = (r_{n-1}+q_{n-1}, r_0, \dots, r_{n-1}, r_0+q_0, \dots, r_{n-2}+q_{n-2})$$

On the other hand,

$$\nu(z) = ((1+u)z_{n-1}, z_0, \dots, z_{n-2}) = (q_{n-1}+u(q_{n-1}+r_{n-1}), q_0+ur_0, \dots, q_{n-2}+ur_{n-2})$$

and

$$\phi_{1,1}(\nu(z)) = (q_{n-1}+r_{n-1}, r_0, \dots, q_{n-2}+r_{n-2}).$$

□

Theorem 1. *A linear code C of length n over R is a $(1+u)$-cyclic code iff $\phi_{1,1}(C)$ is a cyclic code of length $2n$ over F_2+vF_2.*

Proof. If C is $(1+u)$-cyclic code, from Proposition 1 we get $\phi_{1,1}(\nu(C)) = \sigma(\phi_{1,1}(C))$. So $\phi_{1,1}(C)$ is a cyclic code of length $2n$ over F_2+vF_2. Conversely, if $\phi_{1,1}(C)$ is a cyclic code of length $2n$ over F_2+vF_2, from Proposition 1, we get $\phi_{1,1}(\nu(C)) = \sigma(\phi_{1,1}(C)) = \phi_{1,1}(C)$. By using $\phi_{1,1}$ is injection, hence $\nu(C) = C$. □

Corollary 1. *The image of a $(1+u)$-cyclic code of length n over R under the map $\phi_{1,1}$ is a distance invariant cyclic code of length $2n$ over F_2+vF_2.*

Note that $(1+u)^n = 1+u$ if n is odd, $(1+u)^n = 1$ if n is even. In here, it is studied the properties of $(1+u)$ cyclic codes of odd length in this section.

Let μ be the map of $R[x]/\langle x^n-1\rangle$ into $R[x]/\langle x^n-(1+u)\rangle$ defined by $\mu(c(x)) = c((1+u)x)$. If n is odd, then μ is a ring isomorphism. Hence I is an ideal of $R[x]/\langle x^n-1\rangle$ if and only if $\mu(I)$ is an ideal of $R[x]/\langle x^n-(1+u)\rangle$. If $\bar{\mu}'$ is the map

$$\bar{\mu}' : R^n \rightarrow R^n$$
$$z \mapsto (z_0, (1+u)z_1, (1+u)^2 z_2, \ldots, (1+u)^{n-1} z_{n-1})$$

where $z_i = q_i + ur_i$ and $r_i, q_i \in F_2 + vF_2$ for $0 \leq i \leq n-1$, then it also follows that:

Proposition 2. *The set $C \subseteq R^n$ is a linear cyclic code if and only if $\bar{\mu}'(C)$ is a linear $(1+u)$-cyclic code.*

Let τ' be the following permutation of $\{0,1,2,\ldots,2n-1\}$ with n odd: $\tau' = (1,n+1)(3,n+3)\ldots(n-2,2n-2)$. The Nechaev permutation π' of $(F_2+vF_2)^{2n}$ is defined by

$$\pi'(r_0, r_1, \ldots, r_{2n-1}) = (r_{\tau'(0)}, r_{\tau'(1)}, \ldots, r_{\tau'(2n-1)}).$$

Proposition 3. *Assume n odd, let $\bar{\mu}'$ be the permutation of R^n such that*

$$\bar{\mu}'(z_0, \ldots, z_{n-1}) = (z_0, (1+u)z_1, \ldots, (1+u)^{n-1} z_{n-1}).$$

Then $\phi_{1,1}\bar{\mu}' = \pi'\phi_{1,1}$.

Corollary 2. *If $\tilde{C}$ is the Gray image of a linear cyclic code of length n over R, then $\tilde{C}$ is permutation equivalent to a cyclic code and length $2n$ over F_2+vF_2.*

Proof. From Proposition 2, a code C of length n over R is linear cyclic code if and only if $\bar{\mu}'(C)$ is linear $(1+u)$-cyclic. From Theorem 1, this is also so if and only if $\phi_{1,1}(\bar{\mu}'(C))$ is permutation equivalent to a linear cyclic code over F_2+vF_2. From Proposition 3, $\phi_{1,1}(C)$ is permutation equivalent to linear cyclic over F_2+vF_2. □

3. A Representation of a Code over R

In this section, it will be obtained a representation of a linear code of length n over R by means of C_1 and C_2 which are linear codes of length n over F_2+uF_2.

Theorem 2. *The map $\phi_{2,1}$:$R^n \rightarrow (F_2+uF_2)^{2n}$ is a linear isometry.*

Proof. For any $m,k \in R^n$ and $s,t \in F_2+uF_2$, it is verified that

$$\phi_{2,1}(sm+tk) = s\phi_{2,1}(m) + t\phi_{2,1}(k),$$

so $\phi_{2,1}$ is linear. For isometry, we get

$$d_L(\phi_{2,1}(m), \phi_{2,1}(k)) = w_L(\phi_{2,1}(m-k)) = w_L(m-k) = d_L(m,k).$$

□

Theorem 3. *If C is a linear code of length n over R, then $\phi_{2,1}(C)$ is a linear code of length $2n$ over $F_2 + uF_2$.*

Proof. It is seen from linearity of $\phi_{2,1}$. □

Let A and B be two codes. The direct product and sum of A and B are defined by, respectively

$$A \otimes B = \{(a,b) | a \in A, b \in B\}$$
$$A \oplus B = \{a + b | a \in A, b \in B\}.$$

Theorem 4. *If C be a linear code of length n over R, then $C = (1+v)C_1 \oplus vC_2$, $\phi_{2,1}(C) = C_1 \otimes C_2$ and $|C| = |C_1||C_2|$ where $C_1 = \{m \in (F_2 + uF_2)^n | m + vt \in C$ for some $t \in (F_2 + uF_2)^n\}$ and $C_2 = \{m + t \in (F_2 + uF_2)^n | m + vt \in C$ for some $m \in (F_2 + uF_2)^n\}$.*

Proof. Let $c = m + vt \in C$ for some $m, t \in (F_2 + uF_2)^n$. So $m \in C_1, m + t \in C_2$. Hence $c = (1+v)m + v(m+t) \in (1+v)C_1 \oplus vC_2$. We have $C \subseteq (1+v)C_1 \oplus vC_2$. On the other hand, $(1+v)m + v(m+t) \in (1+v)C_1 \oplus vC_2$ where $m \in C_1$ and $t \in C_2$, there exist $a, b \in C$ and $r, q \in (F_2 + uF_2)^n$ such that $a = m + vr$ and $b = m + t + (1+v)q$. As C is linear over R, from $c = (1+v)a + vb \in C$ we have $(1+v)C_1 \oplus vC_2 \subseteq C$. □

Theorem 5. *A linear code $C = (1+v)C_1 \oplus vC_2$ cyclic over R if and only if C_1 and C_2 are all cyclic codes over $F_2 + uF_2$.*

Proof. Let $(r_0, \dots, r_{n-1}) \in C_1$ and $(s_0, \dots, s_{n-1}) \in C_2$. Suppose that $c_i = (1+v)r_i + vs_i$ for $i = 0, \dots, n-1$. Let $c = (c_0, \dots, c_{n-1}) \in C$. As C is cyclic, it follows that $(c_{n-1}, c_0, \dots, c_{n-2}) \in C$. Note that $(c_{n-1}, c_0, \dots, c_{n-2}) = (1+v)(r_{n-1}, r_0, \dots, r_{n-2}) + v(s_{n-1}, s_0, \dots, s_{n-2})$. So $(r_{n-1}, r_0, \dots, r_{n-2}) \in C_1$, $(s_{n-1}, s_0, \dots, s_{n-2}) \in C_2$, that is C_1, C_2 are cyclic codes over $F_2 + uF_2$.

Conversely, let C_1, C_2 be cyclic codes over $F_2 + uF_2$. Let $(c_{n-1}, c_0, \dots, c_{n-2}) \in C$ where $c_i = (1+v)r_i + vs_i$ for $i = 0, \dots, n-1$. Then $(r_0, \dots, r_{n-1}) \in C_1$ and $(s_0, \dots, s_{n-1}) \in C_2$. Note that $(c_{n-1}, c_0, \dots, c_{n-2}) = (1+v)(r_{n-1}, r_0, \dots, r_{n-2}) + v(s_{n-1}, s_0, \dots, s_{n-2}) \in (1+v)C_1 \oplus vC_2 = C$. So C is a cyclic code. □

Theorem 6. *Let C be a linear code of length n over R. Then $\phi_{2,1}(C^\perp) = (\phi_{2,1}(C))^\perp$.*

Proof. By using $\phi_{2,1}(C^\perp) \subseteq (\phi_{2,1}(C))^\perp$ and $|\phi_{2,1}(C^\perp)| = |(\phi_{2,1}(C))^\perp|$, we have expected result. □

Theorem 7. *If C is a linear code of length n over R such that $C = (1+v)C_1 \oplus vC_2$, then $C^\perp = (1+v)C_1^\perp \oplus vC_2^\perp$. Moreover C is self dual if and only if C_1, C_2 are self dual over $F_2 + uF_2$.*

Theorem 8. *Let $C = (1+v)C_1 \oplus vC_2$ be a linear code of length n over R. Then $d_{min}(C) = min\{d_1, d_2\}$ where d_{min}, d_1 and d_2 are minimum Lee distance of C, C_1 and C_2, respectively.*

4. The Gray Images of $(1+u)-$ Cyclic Codes and Cyclic over $F_2+uF_2+vF_2$

In this section, by using the Gray map which is defined by Liu Xiusheng, Liu Hualu, we will characterize codes over F_2 which are the Gray images of $(1+u)$-cyclic and cyclic codes over R.

In [9], it was defined the Gray map ϕ on R^n as follows

$$\begin{aligned}\phi :& R \to F_2^3\\ & a+ub+vc \mapsto (c, b+c, a+b+c).\end{aligned}$$

This map can be extended to R^n in a natural way. For $z=(z_0,\ldots,z_{n-1}) \in R^n$,

$$\begin{aligned}&\phi : R^n \to F_2^{3n}\\ &z=(z_0,\ldots,z_{n-1}) \mapsto (s_0,\ldots,s_{n-1}, s_0\oplus q_0,\ldots,s_{n-1}\oplus q_{n-1}, r_0\oplus q_0\oplus s_0,\ldots,r_{n-1}\oplus q_{n-1}\oplus s_{n-1})\end{aligned}$$

where $z_i=r_i+uq_i+vs_i$, for $0\le i\le n-1$ and $\oplus$ is componentwise addition in F_2.

In [9], they extended the definition of the Lee weight from F_2+vF_2 to the ring $F_2+uF_2+vF_2$. The Lee weight $w_L(x)$ of a codeword $x=(x_1,\ldots,x_n)$ was defined as $\sum_{i=1}^{n} w_L(x_i)$ where

$$w_L(x)=\begin{cases}0 & \text{if } x_i=0\\ 1 & \text{if } x_i=1, 1+u, u+v\\ 2 & \text{if } x_i=u, 1+v, 1+u+v\\ 3 & \text{if } x_i=v\end{cases}$$

The Lee distance $d_L(x,y)$ between two codewords x and y is the Lee weight of $x-y$. The Gray map ϕ is an isometry from (R^n, d_{Lee}) to F_2^{3n} under the Hamming distance.

Proposition 4. *$\phi\nu=\rho\sigma^{\otimes 3}\phi$ where ρ is a permutation of $\{0,\ldots,3n-1\}$ which is defined $\rho=(n+1, 2n+1)$.*

Proof. Let $z=(z_0,z_1,\ldots,z_{n-1})\in R^n$. Let $r_i,q_i,s_i\in F_2$ such that $z_i=r_i+uq_i+vs_i$, for $0\le i\le n-1$. We have

$$\phi(z)=(s_0,\ldots,s_{n-1}, s_0\oplus q_0,\ldots,s_{n-1}\oplus q_{n-1}, r_0\oplus q_0\oplus s_0,\ldots,r_{n-1}\oplus q_{n-1}\oplus s_{n-1}).$$

Then

$$\begin{aligned}\sigma^{\otimes 3}(\phi(z)) =&(s_{n-1},s_0,\ldots,s_{n-2}, s_{n-1}\oplus q_{n-1}, s_0\oplus q_0,\ldots,s_{n-2}\oplus q_{n-2}\\ &, r_{n-1}\oplus q_{n-1}\oplus s_{n-1}, r_0\oplus q_0\oplus s_0,\ldots,r_{n-2}\oplus q_{n-2}\oplus s_{n-2}).\end{aligned}$$

On the other hand, $\nu(z)=((1+u)z_{n-1}, z_0,\ldots,z_{n-2})$ where $(1+u)z_{n-1}=r_{n-1}+u(r_{n-1}+q_{n-1})+vs_{n-1}$. We have

$$\begin{aligned}\phi(\nu(z)) =&(s_{n-1},s_0,\ldots,s_{n-2}, r_{n-1}\oplus q_{n-1}\oplus s_{n-1}, q_0\oplus s_0,\ldots,q_{n-2}\oplus s_{n-2}, r_{n-1}\oplus q_{n-1}\\ &, r_0\oplus q_0\oplus s_0,\ldots,r_{n-2}\oplus q_{n-2}\oplus s_{n-2}).\end{aligned}$$

Hence $\phi\nu=\rho\sigma^{\otimes 3}\phi$. □

So we have the following theorem.

Theorem 9. *A code C of length n over R is $(1+u)$—cyclic if and only if $\phi(C)$ is permutation equivalent to quasi-cyclic of index* 3 *and length* $3n$ *over* F_2.

Proof. Suppose C is $(1+u)$—cyclic. As $\rho(\sigma^{\otimes 3}(\phi(C))) = \phi(\nu(C))$, $\phi(C)$ is permutation equivalent to a quasi-cyclic of index 3. Conversely, if $\phi(C)$ is permutation equivalent to quasi-cyclic of index 3, then $\phi(\nu(C)) = \rho(\sigma^{\otimes 3}(\phi(C))) = \phi(C)$. Since ϕ is isometry, so $\nu(C) = C$, that is C is $(1+u)$— cyclic code. □

Note that $(1+u)^n = 1+u$ if n is odd, $(1+u)^n = 1$ if n is even. In here, it is studied the properties of $(1+u)$ cyclic codes of odd length in this section.

Let μ be the map of $R[x]/\langle x^n - 1\rangle$ into $R[x]/\langle x^n - (1+u)\rangle$ defined by $\mu(c(x)) = c((1+u)x)$. If n is odd, then μ is a ring isomorphism. Hence I is an ideal of $R[x]/\langle x^n - 1\rangle$ if and only if $\mu(I)$ is an ideal of $R[x]/\langle x^n - (1+u)\rangle$. If $\bar{\mu}$ is the map

$$\begin{aligned} \bar{\mu} :& R^n \rightarrow R^n \\ & z \mapsto (z_0, (1+u)z_1, (1+u)^2 z_2, \ldots, (1+u)^{n-1} z_{n-1}) \end{aligned}$$

where $z_i = s_i + ut_i + vy_i$ and $s_i, t_i, y_i \in F_2$ for $0 \leq i \leq n-1$, then it also follows that:

Proposition 5. *The set $C \subseteq R^n$ is a linear cyclic code if and only if $\bar{\mu}(C)$ is a linear $(1+u)$-cyclic code.*

Let τ be the following permutation of $\{0, 1, 2, \ldots, 3n-1\}$ with n odd:

$$\tau = (n+1, 2n+1)(n+3, 2n+3)(n+5, 2n+5)\ldots(2n-2, 3n-2)$$

The permutation π of F_2^{3n} is defined by

$$\pi(r_0, r_1, \ldots, r_{3n-1}) = (r_{\tau(0)}, r_{\tau(1)}, \ldots, r_{\tau(3n-1)})$$

Proposition 6. *Assume n odd, let $\bar{\mu}$ be the permutation of R^n such that*

$$\bar{\mu}(z_0, \ldots, z_{n-1}) = (z_0, (1+u)z_1, \ldots, (1+u)^{n-1} z_{n-1}).$$

Then $\phi\bar{\mu} = \pi\phi$.

Corollary 3. *If $\tilde{C}$ is the Gray image of a linear cyclic code of length n over R, then $\tilde{C}$ is permutation equivalent to a quasi-cyclic code of index* 3 *and length* $3n$ *over* F_2.

Proof. From Proposition 5, a code C of length n over R is linear cyclic code if and only if $\bar{\mu}(C)$ is linear $(1+u)$-cyclic. From Theorem 9, this is also so if and only if $\phi(\bar{\mu}(C))$ is permutation equivalent to a linear quasi-cyclic code of index 3 over F_2. From Proposition 6, $\phi(C)$ is permutation equivalent to linear quasi cyclic of index 3 over F_2. □

5. Conclusion

It is introduced $(1+u)$-cyclic codes and cyclic codes over R. Firstly, it is characterized codes over $F_2 + vF_2$ which are the Gray images of $(1+u)$-cyclic codes and cyclic codes over R. It is obtained a representation of a linear code of length n over R by means of C_1 and C_2 which are linear codes of length n over $F_2 + uF_2$. It is characterized codes over F_2 which are the Gray images of $(1+u)$cyclic codes or cyclic codes over R.

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