Generalization of Dunkl Dini Lipschitz Functions

Salah El Ouadih\textsuperscript{1,*}, Radouan Daher\textsuperscript{1}

\textsuperscript{1} Department of Mathematics, Faculty of Sciences Aïn Chock, University Hassan II
Morocco

\textbf{Abstract.} Using a generalized spherical mean operator, we obtain a generalization of Younis’s Theorem 5.2 in [12] for the Dunkl transform for functions satisfying the \(d\)-Dunkl Dini Lipschitz condition in the space \(L^p(\mathbb{R}^d, w_\lambda(x)dx), 1 < p \leq 2\), where \(w_\lambda\) is a weight function invariant under the action of an associated reflection group.

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1. Introduction and Preliminaries

Younis’s Theorem 5.2 [12] characterized the set of functions in \(L^2(\mathbb{R})\) satisfying the Dini Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

\textbf{Theorem 1.} [12] Let \(f \in L^2(\mathbb{R})\). Then the following are equivalents

\begin{enumerate}[(i)]
\item \(\|f(x+h) - f(x)\|_2 = O\left(\frac{h^n}{(\log \frac{1}{h})^\delta}\right)\), as \(h \to 0, 0 < \eta < 1, \delta \geq 0\)
\item \(\int_{|\lambda| \geq \eta} \vert \hat{f}(\lambda) \vert^2 d\lambda = O\left(\frac{s^{-2\eta}}{(\log s)^25}\right), \text{ as } s \to \infty,\)
\end{enumerate}

where \(\hat{f}\) stands for the Fourier transform of \(f\).

In this paper, we obtain a generalization of Theorem 1.1 for the Dunkl transform on \(\mathbb{R}^d\) in the space \(L^p(\mathbb{R}^d, w_\lambda(x)dx), 1 < p \leq 2\). For this purpose, we use a generalized spherical mean operator.

We consider the Dunkl operators \(D_j, 1 \leq j \leq d,\) on \(\mathbb{R}^d\) which are the differential-difference operators introduced by Dunkl in [3]. These operators are very important in pure mathematics and in physics. The theory of Dunkl operators provides generalizations of various multivariable analytic structures, among others we cite the exponential function,

\textsuperscript{*}Corresponding author.

\textit{Email addresses:} salahwadih@gmail.com (S. El Ouadih), rjdaher024@gmail.com (R. Daher)
the Fourier transform and the translation operator. For more details about these operators see [6, 5]. The Dunkl Kernel $E_l$ has been introduced by Dunkl in [4]. This Kernel is used to define the Dunkl transform.

Let $R$ be a root system in $\mathbb{R}^d$, $W$ the corresponding reflection group, $R_+$ a positive subsystem of $R$ (see [6, 5, 1, 8, 9]) and $l$ a non-negative and $W$-invariant function defined on $R$. The Dunkl operator is defined for $f \in C^1(\mathbb{R}^d)$ by

$$D_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in R_+} l(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{<\alpha, x>}, x \in \mathbb{R}^d (1 \leq j \leq d).$$

Here $<,>$ is the usual Euclidean scalar product on $\mathbb{R}^d$ with the associated norm $|.|$ and $\sigma_\alpha$ the reflection with respect to the hyperplane $H_\alpha$ orthogonal to $\alpha$, and $\alpha_j = <\alpha, e_j>$, $(e_1, e_2, ..., e_d)$ being the canonical basis of $\mathbb{R}^d$.

We consider the weight function

$$w_l(x) = \prod_{\zeta \in R_+} |<\zeta, x>|^{2l(\alpha)}, x \in \mathbb{R}^d,$$

where $w_l$ is $W$-invariant and homogeneous of degree $2\gamma$ where

$$\gamma = \gamma(R) = \sum_{\zeta \in R_+} l(\zeta) \geq 0.$$

The Dunkl kernel $E_l$ on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C. F. Dunkl in [4]. For $y \in \mathbb{R}^d$, the function $x \mapsto E_l(x, y)$ is the unique solution on $\mathbb{R}^d$ of the following initial problem

$$\left\{ \begin{array}{l}
D_j u(x, y) = y_j u(x, y) \quad \text{si } 1 \leq j \leq d \\
u(0, y) = 0 \quad \text{for all } y \in \mathbb{R}^d
\end{array} \right.$$

$E_l$ is called the Dunkl kernel.

**Lemma 1.** [6] Let $z, w \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$

1. $E_l(z, 0) = 1$, $E_l(z, w) = E_l(w, z)$, $E_l(\lambda z, w) = E_l(z, \lambda w)$.
2. For all $\nu = (\nu_1, ..., \nu_d) \in \mathbb{N}^d, x \in \mathbb{R}^d, z \in \mathbb{C}^d$, we have

$$|\partial^\nu z E_l(x; z)| \leq |x|^\nu \exp(||x|| Re z),$$

where

$$\partial^\nu z = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} ... \partial x_d^{\nu_d}}, |\nu| = \nu_1 + ... + \nu_d.$$

In particular $|\partial^\nu z E_l(ix; z)| \leq |x|^\nu$ for all $x, z \in \mathbb{R}^d$.

We denote by $L^p_l(\mathbb{R}^d) = L^p(\mathbb{R}^d, w_l(x) dx), 1 < p \leq 2$, the space of measurable functions on $\mathbb{R}^d$ with the norm

$$\|f\|_{p,l} = \left( \int_{\mathbb{R}^d} |f(x)|^p w_l(x) dx \right)^{\frac{1}{p}} < \infty.$$
The Dunkl transform is defined for \( f \in L^1_l(\mathbb{R}^d) = L^1(\mathbb{R}^d, w_l(x)dx) \) by

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = c_l^{-1} \int_{\mathbb{R}^d} f(x) E_l(-i\xi, x) w_l(x)dx,
\]

where the constant \( c_l \) is given by

\[
c_l = \int_{\mathbb{R}^d} e^{-|z|^2/2} w_l(z)dz.
\]

The Dunkl transform shares several properties with its counterpart in the classical case, we mention here in particular that Plancherel’s Theorem holds in \( L^2_l(\mathbb{R}^d) \), when both \( f \) and \( \hat{f} \) are in \( L^1_l(\mathbb{R}^d) \), we have the inversion formula

\[
f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) E_l(ix, \xi) w_l(\xi)d\xi, x \in \mathbb{R}^d.
\]

By Plancherel’s Theorem and the Marcinkiewicz interpolation theorem (see [10]), we get for \( f \in L^p_l(\mathbb{R}^d) \) with \( 1 < p \leq 2 \) and \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \),

\[
\|\mathcal{F}(f)\|_{q,l} \leq K\|f\|_{p,l},
\]

where \( K \) is a positive constant.

The generalized spherical mean value of \( f \in L^p_l(\mathbb{R}^d) \) is defined by

\[
M_h f(x) = \frac{1}{d_l} \int_{S^{d-1}} \tau_x f(hy)d\mu_l(y), x \in \mathbb{R}^d, h > 0.
\]

where \( \tau_x \) Dunkl translation operator (see [9, 11]), \( \mu \) be the normalized surface measure on the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \) and set \( d\mu_l(y) = w_l(y)d\mu(y), \mu_l \) is a \( W \)-invariant measure on \( S^{d-1} \) and \( d_l = \mu_l(S^{d-1}) \).

We see that \( M_h f \in L^p_l(\mathbb{R}^d) \) whenever \( f \in L^p_l(\mathbb{R}^d) \) and

\[
\|M_h f\|_{p,l} \leq \|f\|_{p,l}.
\]

for all \( h > 0 \).

For \( \beta \geq -\frac{1}{2} \), we introduce the Bessel normalized function of the first kind \( j_\beta \) defined by

\[
j_\beta(z) = \Gamma(\beta + 1) \sum_{n=0}^{\infty} \frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n + \beta + 1)}, z \in \mathbb{C}.
\]

Lemma 2. (Analog of lemma 2.9 in [2]) The following inequality is true

\[
|1 - j_\beta(x)| \geq c,
\]

with \( |x| \geq 1 \), where \( c > 0 \) is a certain constant which depend only on \( \beta \).
Moreover, from (1) we see that

$$\lim_{z \to 0} \frac{j_{\gamma + \frac{d}{2} - 1}(z) - 1}{z^2} \neq 0.$$  \hspace{1cm} (3)

**Lemma 3.** [7] Let $f \in L^p_l(\mathbb{R}^d)$. Then

$$\hat{M}_h f(\xi) = j_{\gamma + \frac{d}{2} - 1}(h|\xi|) \hat{f}(\xi).$$

The first and higher order finite differences of $f(x)$ are defined as follows

$$Z_h f(x) = (M_h - I) f(x),$$

where $I$ is the identity operator $L^p_l(\mathbb{R}^d)$.

$$Z^k_h f(x) = Z_h(Z^{k-1}_h f(x)) = (M_h - I)^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} M_i^h f(x),$$

where $M_i^0 f(x) = f(x)$, $M_i^1 f(x) = M_h(M_i^{h-1} f(x))$, $i = 1, 2, ..$ and $k = 1, 2, ..$

From Lemma 3, we obtain

$$\hat{Z}^k_h f(\xi) = (j_{\gamma + \frac{d}{2} - 1}(h|\xi|) - 1)^k \hat{f}(\xi).$$

By (1), we have

$$\int_{\mathbb{R}^d} |1 - j_{\gamma + \frac{d}{2} - 1}(h|\xi|)|^q |\hat{f}(\xi)|^q w_l(\xi) d\xi \leq K^q \|Z^k_h f(x)\|_{p,l}^q,$$  \hspace{1cm} (4)

where $\frac{1}{p} + \frac{1}{q} = 1$.

2. Dunkl Dini Lipschitz Condition

**Definition 1.** Let $f \in L^p_l(\mathbb{R}^d)$, and define

$$\|Z^k_h f(x)\|_{p,l} \leq C \frac{h^{\eta}}{(\log \frac{1}{h})^{\delta}}, \quad \delta \geq 0,$$

i.e.,

$$\|Z^k_h f(x)\|_{p,l} = O \left( \frac{h^{\eta}}{(\log \frac{1}{h})^{\delta}} \right),$$

for all $x$ in $\mathbb{R}^d$ and for all sufficiently small $h$, $C$ being a positive constant. Then we say that $f$ satisfies a $d$-Dunkl Dini Lipschitz of order $\eta$, or $f$ belongs to $Lip(\eta, \delta)$. 
**Definition 2.** If however
\[
\frac{\|Z_h^k f(x)\|_{p,l}}{(\log \frac{1}{h})^\delta} \to 0, \quad \text{as} \quad h \to 0,
\]
i.e.,
\[
\|Z_h^k f(x)\|_{p,l} = O\left(\frac{h^\eta}{(\log \frac{1}{h})^\delta}\right), \quad \text{as} \quad h \to 0, \delta \geq 0,
\]
then \( f \) is said to belong to the little \( d \)-Dunkl Dini Lipschitz class \( \text{lip}(\eta, \delta) \).

**Remark.** It follows immediately from these definitions that
\[
\text{lip}(\eta, \delta) \subset \text{Lip}(\eta, \delta).
\]

**Theorem 2.** Let \( \eta > 1 \). If \( f \in \text{Lip}(\eta, \delta) \), then \( f \in \text{lip}(1, \delta) \).

**Proof.** For \( x \in \mathbb{R}^d \), \( h \) small and \( f \in \text{Lip}(\eta, \delta) \) we have
\[
\|Z_h^k f(x)\|_{p,l} \leq C \frac{h^\eta}{(\log \frac{1}{h})^\delta}.
\]
Then
\[
(\log \frac{1}{h})^\delta \|Z_h^k f(x)\|_{p,l} \leq Ch^\eta.
\]
Therefore
\[
\frac{(\log \frac{1}{h})^\delta}{h} \|Z_h^k f(x)\|_{p,l} \leq Ch^{\eta-1},
\]
which tends to zero with \( h \to 0 \). Thus
\[
\frac{(\log \frac{1}{h})^\delta}{h} \|Z_h^k f(x)\|_{p,l} \to 0, \quad h \to 0.
\]
Then \( f \in \text{lip}(1, \delta) \).

**Theorem 3.** If \( \eta < \nu \), then \( \text{Lip}(\eta, 0) \supset \text{Lip}(\nu, 0) \) and \( \text{lip}(\eta, 0) \supset \text{lip}(\nu, 0) \).

**Proof.** We have \( 0 \leq h \leq 1 \) and \( \eta < \nu \), then \( h^\nu \leq h^\eta \).
Then the proof of the theorem is immediate.

### 3. New Results on Dunkl Dini Lipschitz Class

**Theorem 4.** Let \( \eta > 2k \). If \( f \) belong to the \( d \)-Dunkl Dini Lipschitz class, i.e.,
\[
f \in \text{Lip}(\eta, \delta), \quad \eta > 2k, \delta \geq 0.
\]
Then \( f \) is equal to the null function in \( \mathbb{R}^d \).
Proof. Assume that \( f \in \text{Lip}(\eta, \delta) \). Then
\[
\|Z_h f(x)\|_{p,l} \leq C\frac{h^\eta}{(\log \frac{1}{h})^\delta}.
\]
From (4), we have
\[
\int_{\mathbb{R}^d} \left|1 - j_{\gamma + \frac{d-1}{2}}(h|\xi|)\right|^{2k} |\hat{f}(\xi)|^q w_l(\xi) d\xi \leq K^q C^q \frac{h^{\eta q}}{(\log \frac{1}{h})^{q \delta}}.
\]
Then
\[
\int_{\mathbb{R}^d} \frac{\left|1 - j_{\gamma + \frac{d-1}{2}}(h|\xi|)\right|^{2k} |\hat{f}(\xi)|^q w_l(\xi) d\xi}{h^{2qk}} \leq K^q C^q \frac{h^{\eta q - 2qk}}{(\log \frac{1}{h})^{q \delta}}.
\]
Since \( \eta > 2k \) we have
\[
\lim_{h \to 0} \frac{h^{\eta q - 2qk}}{(\log \frac{1}{h})^{q \delta}} = 0.
\]
Thus
\[
\lim_{h \to 0} \int_{\mathbb{R}^d} \frac{\left|1 - j_{\gamma + \frac{d-1}{2}}(h|\xi|)\right|^{2qk} |\hat{f}(\xi)|^q w_l(\xi) d\xi}{|\xi|^q} = 0.
\]
and also from the formula (3) and Fatou’s theorem, we obtain
\[
\int_{\mathbb{R}^d} |\xi|^{2qk} |\hat{f}(\xi)|^q w_l(\xi) d\xi = 0.
\]
Hence \( |\xi|^{2k} \hat{f}(\xi) = 0 \) for all \( \xi \in \mathbb{R}^d \), then \( f(x) \) is the null function.

Analog of the Theorem 4, we obtain this theorem.

Theorem 5. Let \( f \in L^p_1(\mathbb{R}^d) \). If \( f \) belong to \( \text{lip}(2, 0) \), i.e.,
\[
\|Z_h f(x)\|_{p,l} = O(h^2), \quad \text{as} \quad h \to 0.
\]
Then \( f \) is equal to null function in \( \mathbb{R}^d \).

Now, we give another the main result of this paper analog of Theorem 1.

Theorem 6. Let \( f \in L^p_1(\mathbb{R}^d) \). If \( f(x) \) belong to \( \text{Lip}(\eta, \delta) \), then
\[
\int_{|\xi| \geq s} |\hat{f}(\xi)|^q w_l(\xi) d\xi = O\left(\frac{s^{-\eta q}}{(\log s)^{q \delta}}\right), \quad s \to \infty,
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. Suppose that \( f \in \text{Lip}(\eta, \delta) \). Then

\[
\| Z_h^k f(x) \|_{p,l} = O \left( \frac{h^\eta}{(\log \frac{1}{h})^{\delta}} \right), \quad h \to 0.
\]

From (4), we have

\[
\int_{\mathbb{R}^d} |1 - j_{\gamma + \frac{q}{2} - 1}(h|\xi|)|^q |\hat{f}(\xi)|^q w_l(\xi) d\xi \leq K^q \| Z_h^k f(x) \|_{p,l}^q.
\]

If \(|\xi| \in [\frac{1}{h}, \frac{2}{h}]\) then \( h|\xi| \geq 1 \) and Lemma 2 implies that

\[
1 \leq \frac{1}{c^q k} |1 - j_{\gamma + \frac{q}{2} - 1}(h|\xi|)|^q.
\]

Then

\[
\int_{\frac{1}{h} \leq |\xi| \leq \frac{2}{h}} |\hat{f}(\xi)|^q w_l(\xi) d\xi \leq \frac{1}{c^q k} \int_{\frac{1}{h} \leq |\xi| \leq \frac{2}{h}} |1 - j_{\gamma + \frac{q}{2} - 1}(h|\xi|)|^q |\hat{f}(\xi)|^q w_l(\xi) d\xi
\]
\[
\leq \frac{1}{c^q k} \int_{\mathbb{R}^d} |1 - j_{\gamma + \frac{q}{2} - 1}(h|\xi|)|^q |\hat{f}(\xi)|^q w_l(\xi) d\xi
\]
\[
\leq \frac{K^q}{c^q k} \| Z_h^k f(x) \|_{p,l}^q
\]
\[
= O \left( \frac{h^\eta}{(\log \frac{1}{h})^{\delta}} \right).
\]

So we obtain

\[
\int_{|\xi| \geq s} |\hat{f}(\xi)|^q w_l(\xi) d\xi \leq C' \frac{s^{-\eta}}{(\log s)^{q\delta}},
\]

where \( C' \) is a positive constant. Now, we have

\[
\int_{|\xi| \geq s} |\hat{f}(\xi)|^q w_l(\xi) d\xi = \sum_{i=0}^{\infty} \int_{2^i s}^{2^{i+1} s} |\hat{f}(\xi)|^q w_l(\xi) d\xi
\]
\[
\leq C' \frac{s^{-\eta}}{(\log s)^{q\delta}} + \frac{(2s)^{-\eta}}{(\log 2s)^{q\delta}} + \frac{(4s)^{-\eta}}{(\log 4s)^{q\delta}} + \cdots
\]
\[
\leq C' \frac{s^{-\eta}}{(\log s)^{q\delta}} \left( 1 + 2^{-\eta} + (2^{-\eta})^2 + (2^{-\eta})^3 + \cdots \right)
\]
\[
\leq K \eta \frac{s^{-\eta}}{(\log s)^{q\delta}},
\]

where \( K = C'(1 - 2^{-\eta})^{-1} \) since \( 2^{-\eta} < 1 \). Consequently

\[
\int_{|\xi| \geq s} |\hat{f}(\xi)|^q w_l(\xi) d\xi = O \left( \frac{s^{-\eta}}{(\log s)^{q\delta}} \right), \quad \text{as} \quad s \to \infty.
\]
References


