EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 9, No. 3, 2016, 244-249
ISSN 1307-5543 – www.ejpam.com

Comultiplication Modules

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Abstract. Let R be a commutative ring. An R-module M is comutiplication if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$. This paper is devoted to study some properties of comultiplication rings and modules.

2010 Mathematics Subject Classifications: 13C13
Key Words and Phrases: Multiplication modules, Comultiplication modules

1. Introduction

Throughout this paper, R will denote a commutative ring with identity. We recall that R-module M is comutiplication if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$. It was shown that M is comultiplication if and only if for each submodule N of M, $N = (0 :_M Ann_R(N))$ [4]. Also a Noetherian local ring R is a gorenstein ring if $injdimR < \infty$ [6]. In this article, among other results, we will show that if R is a local Artinian ring, then R is comultiplication if and only if R is gorenstein. An R-module M is called generalized hopfian, if every surjective endomorphism of M has a small kernel. It is proved that every comultiplication module is generalized hopfian. At last but not at least, we consider the direct sum of comultiplication modules, it is shown that $M = \bigoplus_{i\in I} M_i$, is comultiplication if and only if for each $i \in I, M_i$ is a comultiplication module and for each submodule N of $M, N = \bigoplus_{i\in I}(N \bigcap M_i)$.

2. Auxiliary Results

In this section we will provide the definitions and results which are necessary in the next section.

Definition 1.

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(1) *Let M be an $R-$ module. A submodule N of M is said to be large (resp. small) if for every non-zero submodule K of M, we have $N\bigcap K\neq 0$ (resp. $N+K\neq M$).*

(2) *An R-module M is called generalized hopfian, if every subjective endomorphism of M has a small kernel.*

(3) *An R-module M is called weakly co-hopfian, if every injective endomorphism of M has a large image.*

(4) *Let I be an ideal of R. We say that I is a second ideal of R, if for each $r\in R$, $rI=0$ or $rI=I$.*

(5) *An R-module M is called uniform, if every submodule of M is large.*

(6) *An ideal I of R is a pure ideal if for each ideal J of R, $IJ=I\cap J$.*

(7) *A submodule N of M is a copure submodule if for each ideal I of R, $(N:_M I)=N+(0:_M I)$.*

(8) *An R-module M is called weak comultiplication if for every prime submodule N of M, there exists an ideal I of R such that $N=(0:_M I)$*

Theorem 1. *Let R be a discrete valuation ring with the unique maximal ideal m. If R-module M is comultiplication, then $M\cong E(R/m)$ or $M\cong R/m^n$, for some $n\in N$.*

Proof. See [1] and [2]. □

Theorem 2 ([4]). *Let R be a Noetherian ring, and M be a comultiplication R-module so M is Artinian.*

Theorem 3 ([3]). *Let R be a Noetherian ring, and M be an injective multiplication R-module, so M is comultiplication.*

Lemma 1 ([5]). *If M is a comultiplication R -module, then for each endomorphism f of M, $Imf=Ann_R(kerf)M$.*

3. Main Results

Lemma 2. *Let R be a Noetherian ring. Then the following statements are equivalent:*

(1) *R is a comultiplication ring;*

(2) *For all $P\in Spec(R)$, R_P is a comultiplication ring;*

(3) *For all $P\in Max(R)$, R_P is a comultiplication ring.*

Proof. (1→2) Let J be an ideal of R_P, then there exists an ideal I of R such that $J=I_P$. Now $I=AnnAnnI$ and so $J=I_P=AnnAnnI_P=AnnAnnJ$.

(2→3) It is clear.

(3→1) Let I be an ideal of R. For all $P\in Max(R)$, we have $I_P=AnnAnnI_P=(AnnAnnI)_P$ and so $I=AnnAnnI$. □

Theorem 4. *Let M be a comultiplication R-module then*

(1) If I is a second ideal of R, then $N = (0 :_M I)$ is a prime submodule of M.

(2) If N is a second submodule of M, then $Ann_R N$ is a prime ideal of R.

(3) If M is faithful, and N a submodule of M such that $Ann_R N$ is a large ideal of R, then N is a small submodule of M.

(4) If N be a submodule of M such that $AnnN$ is a pure ideal of R, then N is a copure submodule of M.

Proof.

(1) Let $r \in R$ and $m \in M$ be elements such that $rm \in N$ and $r \notin (N :_R M)$. Therefore $rM \nsubseteq N$ and so $rMI \neq 0$. This shows that $rI \neq 0$, and by hypothesis $rI = I$. Since $rm \in N = (0 :_M I)$, it follows that $rmI = 0$ and so $mI = 0$, that implies $m \in (0 :_M I) = N$.

(2) Let N be a second submodule of M. Set $I := Ann_R N$ and so $N = (0 :_M I)$. Suppose that x, y be two elements of R such that $xy \in I$ but $x \notin I$ and $y \notin I$. Now $xy \in I$, implies that $xyN = 0$ and hence $(xy)^n N \neq N$ for each $n \in N$. Since $x, y \notin I$, it follows that there exists $n \in N$ such that $x^n N = N$ and $y^n N = N$. Consequently $(xy)^n N = x^n y^n N = N$, which is a contradiction.

(3) Let there exists a submodule K of M such that $M = N + K$. So

$$M = N + K = (0 :_M AnnN) + (0 :_M AnnK) = (0 :_M AnnN \bigcap AnnK).$$

Therefore $AnnN \bigcap AnnK \subseteq AnnM = 0$, and consequently $AnnK = 0$ which implies that $K = (0 :_M AnnK) = M$.

(4) We show that for each ideal I of R, $(N :_M I) = N + (0 :_M I)$. Note that for each ideal I of R there exists a submodule K of M such that $(0 :_M I) = (0 :_M AnnK)$, so

$$\begin{aligned}(N :_M I) =&((0 :_M AnnN) :_M I) = ((0 :_M I) :_M AnnN) = ((0 :_M AnnK) :_M AnnN)\\ =&(0 :_M AnnKAnnN) = (0 :_M AnnK \bigcap AnnN)\\ =&(0 :_M AnnN) + (0 :_M AnnK) = N + (0 :_M I).\end{aligned}$$

□

Theorem 5. *Every comultiplication module is a generalized hopfian and weakly co-hopfian module.*

Proof. Let M be a comultiplication module and f be a surjective endomorphism of M. Suppose that there exists a submodule N of M such that $M = kerf + N$. In this case $f(M) = f(kerf + N) = f(N)$. Therefore $M = f(N) = (0 :_M (0 :_R f(N)))$.

Since M is a comultiplication R-module, it follows that $f(N) \subseteq N$ and so we have:

$$M = (0 :_M (0 :_R f(N))) \subseteq (0 :_M (0 :_R N)) = N,$$

and hence $kerf$ is a small submodule of M. Now let f be an injective endomorphism of M and N be a submodule of M such that $Imf \bigcap N = 0$. By the previous lemma, $Imf = Ann_R(kerf)M$, and so $Ann_R(kerf)M \bigcap N = 0$. But $kerf = 0$ and hence $N = M \bigcap N = 0$. □

Theorem 6. *let (R,m) be a local Artinian ring, then the following statements are equivalent:*

(1) R is a comultiplication ring;

(2) R is a gorenstein ring;

(3) $soc(R) \approx R/m$;

(4) $E(R/m)$ is a multiplication R module.

Proof. (1→ 2) It is enough to show that $r(R) = 1$. Suppose on the contrary that $r(R) \neq 1$, so $r(R) = 0$ or $r(R) > 1$. If $r(R) = 0$, then $r(R) = dim_k Hom_R(R/m, R) = 0$ and so $(0 :_R m) \approx Hom_R(R/m, R) = 0$. Which is a contradiction, because the annihilator of any proper ideal of an Artinian local ring is non-zero. Now suppose that$r(R) > 1$, so there exist two ideals I and J of R such that $(0 :_R m) = I \bigoplus J = (0 :_R AnnI) \bigoplus (0 :_R AnnJ) = (0 :_R AnnI \bigcap AnnJ)$, this means that $(0 :_R AnnI \bigcap AnnJ) \neq 0$. On the other hand

$$I \bigcap J = AnnAnnI \bigcap AnnAnnJ = Ann(AnnI + AnnJ) \neq 0,$$

Which is a contradiction.

(2→3) Since R is gorenstein, it follows from [6], for all non-zero ideals I and J of R, $I \bigcap J \neq 0$. Now let S_1 and S_2 be two simple submodules of R, then $S_1 \bigcap S_2 \neq 0$ and consequently $S_1 = S_2$.

(3→4) Since $soc(R) = (0 :_R m) \approx R/m$, it follows that $r(R) = 1$ and so R is a gorenstein ring. On the other hand $dimR = injdimR = 0$. Thus R is an injective R module and so $R \approx E(R/m)$, by [6].

(4→1) $E(R/m)$ is multiplication and Artinian, it follows that $E(R/m)$ is cyclic and so $E(R/m) \approx R$. Now R is an injective and multiplication R module, then by [3] R is comultiplication. □

Theorem 7. *Let (R,m) be a local Artinian ring, and M be a faithful comultiplication R-module. Then M is uniform.*

Proof. Let (R,m) be a local Artinian ring, and M be a faithful comultiplication R-module and N be a submodule of M such that $N \cap K = 0$, for some submodule K of M. Then we have

$$N \cap K = (0 :_M Ann_R(N)) \cap (0 :_M Ann_R(K)) = (0 :_M Ann_R(N) + Ann_R(K)).$$

Now if $Ann_R(N)+Ann_R(K)=0$, then

$$N\cap K=(0:_M 0)=M\neq 0$$

So suppose that $Ann_R(N)+Ann_R(K)\neq 0$, then $Ann_R(N)+Ann_R(K)\subseteq m$. Therefore

$$0=N\cap K=(0:_M Ann_R(N)+Ann_R(K)). \tag{1}$$

Hence

$$Ann(Ann_R(N)+Ann_R(K))\subseteq Ann(M)=0 \tag{2}$$

which is a contradiction, because R is Artinian. □

Lemma 3. *Let (R,m) be an Artinian local ring. Then R is comultiplication if and only if*

$$(0:_R m^{n+1})/(0:_R m^n)\simeq m^n/m^{n+1} \text{ for all } n\geq 0.$$

Proof. Let R be a comultiplication ring, then by Theorem 6, $soc(R)\approx R/m$ and so we have $(0:_R m)\approx R/m$. Now let $n\geq 1$, consider the following exact sequence:

$$0\to m^n/m^{n+1}\to R/m^{n+1}\to R/m^n\to 0$$

Since R is gorenstein, it follows that $0=dimR\leq injdimR=depthR\leq dimR=0$, and so R is injective R-module. Therefore we have the following exact sequence.

$$0\to(0:_R m^n)\to(0:_R m^{n+1})\to Hom(m^n/m^{n+1},R)\to 0$$

On the other hand $r(R)=1$ and we have

$$Hom(m^n/m^{n+1},R)\approx Hom(\bigoplus_{i=1}^{t}R/m,R)\approx\bigoplus_{i=1}^{t}Hom(R/m,R)\approx\bigoplus_{i=1}^{t}R/m=m^n/m^{n+1}.$$

Therefore by the last exact sequence $(0:_R m^n)/(0:_R m^{n+1})\approx m^n/m^{n+1}$. □

Lemma 4. *Let M_1,M_2 be two submodules of a comultiplication R-module M such that $M=M_1\bigoplus M_2$. Then $Hom_R(M_1,M_2)=Hom_R(M_2,M_1)=0$.*

Proof. Let $f:M_1\to M_2$ be a homomorphism. Since M is comultiplication, $f(M_1)\subseteq M_1$, by [2]. On the other hand $f(M_1)\subseteq M_2$ and so $f(M_1)\subseteq M_1\bigcap M_2=0$. This shows that $f=0$. □

Theorem 8. *Let R be a Dedekind domain, and M be a comultiplication module, then there exist distinct maximal ideals ${P_i}_{i\in I}$ of R and submodules M_i, $i\in I$ of M, such that $M=\bigoplus_{i\in I}M_i$ and for each $i\in I$, $M_i\cong E(R/P_i)$ or $M_i\cong R/P_i^{n_i}$, for some $n_i\in N$.*

Proof. Let R be a Dedekind domain and M be comultiplication, so R is Noetherian and M is Artinian by [4]. Set $M(P) := \{m \in M \mid \exists n \in N,\ P^n m = 0\}$ for $P \in Spec(R)$. There exist distinct maximal ideal $\{P_i\}_{i\in I}$ such that $M = \bigoplus_{i\in I} M(P_i)$. Let $M_i = M(P_i)$, since M is comultiplication, it follows that each M_i is also comultiplication. On the other hand each M_i is an R_{P_i}-module. So by Theorem 8 for each $i \in I$, $M_i \cong E(R_{P_i}/P_i R_{P_i})$ or $M_i \cong R_{Pi}/P_i R_{P_i}^{n_i}$, since $P_i \bigcap (R \setminus P_i) = \emptyset$, it follows that $E(R_{P_i}/P_i R_{P_i}) \cong E(R/P_i)$, also $R_{P_i}/P_i R_{P_i}^{n_i} \cong R/P_i^{n_i}$. □

Theorem 9. *Let $R \subseteq \overline{R}$ be an integral extension and $\overline{R}$ be weak comultiplication, then R is weak comultiplication.*

Proof. Let $P \in Spec(R)$ so there exists a prime ideal q of $\overline{R}$ such that $p = q^c$ so

$$p = q^c = (0 :_R Ann_R(q))^c \supseteq (0 :_R Ann_R(q^c)) = (0 :_R Ann_R(p)). \tag{3}$$

□

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