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Hasse-Schmidt Derivations on Banach-Jordan Pairs

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Abstract. The aim of this paper consists in establishing the automatic continuity of Hasse-Schmidt derivations on Banach-Jordan Pairs and Banach-Jordan Algebras satisfying some algebraic conditions. Namely, higher derivations on semiprimitive Banach-Jordan Pairs and semiprimitive Banach-Jordan Algebras are continuous.

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1. Introduction

Higher derivations were introduced first by Hasse and Schmidt [12], that's why algebraist sometimes call them Hasse-Schmidt derivations. For further algebraic properties about these operators, the reader is referred to [5,7,11,17,27,28] where they are studied in other context. Higher derivations are used in [30] to study generic solving of higher differential equations. Loy proved in [22] that if A is an (F)-algebra which is a subalgebra of a Banach algebra B of power series, then every higher derivation $\{d_n\} : A \longrightarrow B$ $(n = 0, 1, 2, ...)$ is automatically continuous. Jewell showed in [15] that any higher derivation from a Banach algebra into a semisimple Banach algebra is continuous provided $\ker(d_0) \subseteq \ker(d_n)$, for all $n \geq 1$. S. Hejazian and T.L. Shatery show in [13] that every higher derivation $\{d_n\}$ from a JB^*-algebra A into a JB^*-algebra B is continuous provided that d_0 is a *-homomorphism. They also prove that every higher derivation from a commutative C^*-algebra or from a C^*-algebra which has minimal idempotents and is the closure of its socle is continuous. M. Mirzavaziri gives in [24] a characterization of higher derivations on algebras.

In this paper, we deal with higher derivations on Banach-Jordan pairs. We intend to settle the automatic continuity of these operators provided that some algebraic conditions are satisfied. Our approach to this result consists in intensive use of local algebras theory frequently used by authors in Jordan structures. Let us note that Jordan pairs are a natural

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extension of Jordan algebras and arise as well in a natural way in the geometry of bounded symmetric domains. Loos proved in [21] a strong dependence between homogeneous circled domains, in finite complex vector spaces, and Jordan pairs.

2. Preliminaries

In this paper we shall deal with Jordan pairs and Jordan algebras over a commutative ring of scalars $\mathcal{R}$ of characteristic not two. The reader is referred to [18] for further details. However, we shall record in this section some notations and results.

A *Jordan pair* over a commutative ring $\mathcal{R}$ of characteristic not two is a pair of $\mathcal{R}$-modules $P = (P^+, P^-)$ endowed with a couple (Q^+, Q^-) of quadratic operators $Q^\sigma : P^\sigma \longrightarrow Hom_{\mathcal{R}}(P^{-\sigma}, P^\sigma)$ such that the following identities hold for all $(x, y) \in P^\sigma \times P^{-\sigma}$ $(\sigma = \pm)$

$$V^\sigma_{(x,y)}Q^\sigma_x = Q^\sigma_x V^{-\sigma}_{(y,x)}, \quad V^\sigma_{(Q^\sigma_x y,x)} = V^\sigma_{(x,Q^{-\sigma}_y x)},$$

where $V^\sigma_{(x,y)}z = Q^\sigma_{(x,z)}y = \{x, y, z\}_\sigma$, $Q^\sigma_{(x,z)} = Q^\sigma_{x+z} - Q^\sigma_x - Q^\sigma_z$ and $\{x, y, x\}_\sigma = 2Q^\sigma_x y$.

An example of Jordan pairs over a field $\mathcal{K}$ is given by taking $P = A(M, R, \varphi)^J$, where $M = (M^+, M^-)$ is a pair of R-vector spaces such that M^+ is a left R-module and M^- is a right R-module over an associative $\mathcal{K}$-algebra R and $\varphi : M^+ \times M^- \longrightarrow R$ is an R-bilinear form in the sense that $\varphi(ax, yb) = a\varphi(x, y)b$. The product of $P = A(M, R, \varphi)^J$ is defined by:

$$Q_x y = \varphi(x, y)x \text{ and } Q_y x = y\varphi(x, y) \;\; \forall (x, y) \in M^+ \times M^-.$$

A Jordan pair $P = (P^+, P^-)$ is said to be *normed (Banach)* provided the vector spaces P^+ and P^- are endowed with norms (complete), both denoted by $\|.\|$, making continuous the triple products $\{x, y, z\}_\sigma$ of P, merely denoted $\{x, y, z\}$.

A typical example of Banach-Jordan pairs is given by taking

$$P^+ = \mathcal{BL}(\mathcal{X}, \mathcal{Y}), \, P^- = \mathcal{BL}(\mathcal{Y}, \mathcal{X}),$$

the pair of linear bounded operators between real or complex Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ with the multiplication $Q_u v = uvu$. Such pair is frequently denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

A (linear) *Jordan algebra* is a vector space J endowed with a binary product $(a, b) \longmapsto ab$ satisfying the identities: $ab = ba$, and $a^2(ba) = (a^2b)a$. If a complete norm is defined on J and makes continuous its product ab, J is said to be a Banach-Jordan algebra. Jordan pairs are known by their intimate relationship with Jordan algebras. Indeed, Any associative, alternative or Jordan algebra A gives rise to a Jordan pair (A, A) with a quadratic multiplication xyx or $U_x y$, with U denoting the usual U-operator of a Jordan algebra defined by $U_x y = 2x(xy) - x^2 y$.

In the opposite direction, given a Jordan pair $V = (V^+, V^-)$ and an element $u \in V^{-\sigma}$, the vector space V^σ gives rise to a Jordan algebra by defining the U-operator $U_a = U^{(u)}_a = Q_a Q_u$, and the square $a^{(2,u)} = Q_a u$. This Jordan algebra, denoted by $V^{\sigma(u)}$, is called the *u-homotope* of V at u. If V is a linear Jordan pair, we just need to define the linear product in $V^{\sigma(u)}$ as follows: $a.b = \frac{1}{2}\{a, u, b\}$.

Local algebras of a Jordan pair. Let V be a Jordan pair and $0 \neq u \in V^{-\sigma}$. By [18, 4.19] the set $\ker(u)$ whose elements are those $x \in V^{\sigma}$ such that $Q_u x = Q_u Q_x u = 0$, turns out to be an ideal of $V^{\sigma(u)}$ and the quotient $V^{\sigma(u)}/\ker(u)$ is a Jordan algebra called the *local algebra* of V at u which we denote by V_u. As pointed out in [9, 1.2.4(ii)] the condition $Q_u Q_x u = 0$ is superfluous if V is linear or nondegenerate: $Q_x = 0$ implies $x = 0$.

If V is a normed Jordan pair, Then $V^{\sigma(u)}$ is a normed Jordan algebra for the norm $|x| = \|x\|_{\sigma} \|u\|_{-\sigma}$. Moreover, by [23, §II. Lemma 3.1] , the local algebra V_u is also normed for the quotient norm $\|x + \ker(u)\| = \inf_{z \in \ker(u)} |x + z|$ which is complete if so are the norms of V.

Socle and capacity. For a nondegenerate Jordan pair V, its *socle,* denoted by $Soc(V)$, is the ideal $Soc(V)$ =$(Soc(V^+), Soc(V^-))$, where $Soc(V^{\sigma})$ denotes the sum of all minimal inner ideals of $V^{\pm}$.

(2.1) Let V be a nondegenerate Jordan pair and $u \in V^{\sigma}$. Then $u \in Soc(V^{\sigma})$ if and only if V_u has finite capacity [25, 0.7(b)].

A nondegenerate Jordan pair V has a *finite capacity* if it contains an orthogonal system $\{e_1, ..., e_n\}$ of *division idempotents* ($V_2(e_i)$ is a division Jordan pair) such that $\cap_{i=1}^{n} V_0(e_i) = 0$, equivalently the lengths of its chains of principal inner ideals are bounded

Primitive Jordan pairs and Jacobson radical. A Jordan pair $V = (V^+, V^-)$ is said to be *primitive* at $b \in V^{-\sigma}$ if there exists a proper inner ideal K of V^{σ} such that:

i) K is a c-modular inner ideal of the homotope $V^{\sigma(b)}$ for some $c \in V^{\sigma}$,

ii) K complements the (σ)-parts of nonzero ideals: $I^{\sigma} + K = V^{\sigma}$ for any nonzero ideal $I = (I^+, I^-)$ of V. .

Anquela and Cortés proved in [1] and [2] the following results:

(2.2) V is primitive at $b \in V^{-\sigma}$ if and only if V_b is a primitive Jordan algebra and V is strongly prime.

(2.3) If V is primitive at some $0 \neq b_0 \in V^{-\sigma}$ then so is V at every element $0 \neq b \in V^{\pm}$.

Further results on primitive Jordan pairs can be found in [1] , [2] and [3] .

Following [18], the *Jacobson radical* of a Jordan pair V is defined as the the ideal $Rad(V) = (Rad(V^+), Rad(V^-))$, where $Rad(V^{\sigma})$ is the set of *properly quasi-invertible* elements of V^{σ}, that is, those elements which are quasi-invertible in every homotope $V^{\sigma(u)}$. A Jordan pair is said to be semiprimitive is $Rad(V) = 0$.

As in the case of associative algebras, an ideal P of a Jordan system (algebra or pair) V is called *primitive* if the factor system (algebra or pair) V/P is primitive. Moreover, it follows from [14, A.4.8] , or either [31].

(2.4) The Jacobson radical of a Jordan pair is the intersection of all its primitive ideals.

3. Technical results

Recall that we can measure the continuity of a linear operator acting between two normed spaces by considering its so called separating subspace. Indeed, if T is a linear operator defined between two real or complex normed vector spaces X and Y, then its separating subspace $S(T)$ is defined by:

$$S(T) = \{y \in Y : \exists \{x_n\}_n \subset X \text{ such that } \lim x_n = 0 \text{ and } \lim T(x_n) = y\}.$$

It is easily seen that the separating subspace of T is a closed subspace of Y. Moreover, by the closed graph Theorem, if both X and Y are Banach spaces, then T is continuous if and only if $S(T) = 0$.

Let V and W be two Jordan pairs. By a *higher derivation of rank k* (k may be infinite), we mean a family of linear mappings $\{\varphi_n = (\varphi_n^+, \varphi_n^-)\}_{n=1}^k$ from V into W such that

$$\varphi_n^\sigma \{x, y, z\} = \sum_{i+j+h=n} \left\{\varphi_i^\sigma x, \varphi_j^{-\sigma} y, \varphi_h^\sigma z\right\}, \ (x, z \in V^\sigma, \ y \in V^{-\sigma}, \ n = 0, 1, 2, ..., k),$$

where $\varphi_0^\sigma = Id_{V^\sigma}$ ($\sigma = \pm$).

Let $D = (D_+, D_-)$ be a *derivation* from V into W, that is a pair of linear operators $D_\sigma : V^\sigma \longrightarrow V^\sigma$ satisfying

$$D_\sigma \{x, y, z\} = \{D_\sigma x, y, z\} + \{x, D_{-\sigma} y, z\} + \{x, y, D_\sigma z\}, \text{ for all} (x, z \in V^\sigma, \ y \in V^{-\sigma}.$$

Any derivation $D = (D_+, D_-)$ from V into W gives rise to a standard example of higher derivations $\{\varphi_n = (\varphi_n^+, \varphi_n^-)\}_{n \geq 0}$ from V into W by setting

$$\varphi_n^+ = \frac{1}{n!} D_+^n, \text{ and } \varphi_n^- = \frac{1}{n!} D_-^n.$$

Remark 1. *i) It follows from the last definitions that* $\varphi_1 = (\varphi_1^+, \varphi_1^-)$ *is a derivation.*
ii) In order to simplify notations, the index $\sigma = \pm$ *in expressions like* $D_i^\pm(x)$, $\varphi_i^\pm(x), ...$ *will be sometimes suppressed if there is no confusion.*

Lemma 1. *Let V be a normed Jordan pair and let $k \geq 2$ be a fixed positive integer. If $\varphi_n = (\varphi_n^+, \varphi_n^-)\}$ is a higher derivation on V such that φ_i^σ is continuous for every $i \in \{0, 1, ..., k-1\}$ ($\sigma = +, -$), then the separating subspace $S(\varphi_k) = (S(\varphi_k^+), S(\varphi_k^-))$ of φ_k is a closed ideal of V.*

Proof. Since the characteristic of the ground field is zero, it suffices to prove that $S(\varphi_n)$ is an outer ideal of V. That is

$$Q_{V^{-\sigma}} S(\varphi_k^\sigma) \subset S(\varphi_k^\sigma) \text{ and } \{S(\varphi_k^\sigma), V^{-\sigma}, V^\sigma\} \subset S(\varphi_k^\sigma).$$

Let s be an element of $S(\varphi_k^\sigma)$ and a be an arbitrary element of $V^{-\sigma}$. Then there exist a sequence $\{x_n\}_n \subset V^\sigma$ such that $\lim x_n = 0$ and $\lim \varphi_k^\sigma(x_n) = s$. Consider the sequence $\{Q_a x_n\}$. By continuity of the operator Q_a, we get $\lim Q_a x_n = Q_a \lim x_n = 0$. Moreover, using the continuity of the triple product of V and that of φ_j^σ such that $0 \leq j \leq k-1$, we see that the terms $\left\{\varphi_i^\sigma a, \varphi_j^{-\sigma} x_n, \varphi_h^\sigma a\right\}$ converge to zero when n tends to ∞ and consequently we have

$$\begin{aligned}\lim \varphi_k^\sigma Q_a x_n &= \frac{1}{2} \lim \varphi_k^\sigma \{a, x_n, a\} \\ &= \frac{1}{2} \lim \sum_{i+j+h=k} \left\{\varphi_i^\sigma a, \varphi_j^{-\sigma} x_n, \varphi_h^\sigma a\right\} \\ &= \frac{1}{2} \left\{a, \lim \varphi_k^{-\sigma} x_n, a\right\} \\ &= \frac{1}{2} \{a, s, a\} \\ &= Q_a s,\end{aligned}$$

which establishes $Q_{V^{-\sigma}} S(\varphi_k^\sigma) \subset S(\varphi_k^\sigma)$. On the other hand, for arbitrary pair (u, v) of elements in $V^{-\sigma} \times V^\sigma$, the sequence $\{x_n, u, v\}$ converges to 0. Using again the continuity of the triple product of V as well as that of φ_i^σ such that $0 \leq i \leq k-1$ and $i + j + h = k$, we see that, for arbitrary pair (u, v) of elements in $V^{-\sigma} \times V^\sigma$, the terms $\left\{\varphi_i^\sigma x_n, \varphi_j^\sigma u, \varphi_h^\sigma v\right\}$ converge to zero when n tends to ∞. Consequently, we do have

$$\begin{aligned}\lim \varphi_k^\sigma(\{x_n, u, v\}) &= \lim \sum_{i+j+h=k} \{\varphi_i^\sigma(x_n), \varphi_j^{-\sigma}(u), \varphi_h^\sigma(v)\} \\ &= \sum_{i+j+h=k} \{\lim \varphi_i^\sigma(x_n), \varphi_j^{-\sigma}(u), \varphi_h^\sigma(v)\} \\ &= \{\lim \varphi_k^\sigma(x_n), u, v\} \\ &= \{s, u, v\},\end{aligned}$$

which establishes $\{S(\varphi_k^\sigma), V^{-\sigma}, V^\sigma\} \subset S(\varphi_k^\sigma)$ as required. Finally, $S(\varphi_k)$ is an ideal of V which is closed since the separating subspace of any linear operator is closed as it is pointed out.

Remark 2. *Let $\{D_n = (D_n^+, D_n^-)\}$ be a higher derivation on a normed Jordan pair $V = (V^+, V^-)$ and let b be a nonzero element in $V^{-\sigma}$. Let us note that $\{D_n^\sigma\}$ is not a higher derivation on the Jordan algebra $V^{\sigma(b)}$ even if D_n^σ vanishes at b for all positive integers n. However, the behavior of D_n^σ towards $V^{\sigma(b)}$ conserves nice properties as it is clarified in the following.*

Lemma 2. *Let $V = (V^+, V^-)$ be a normed Jordan pair and let b be a nonzero element in $V^{-\sigma}$. If $\{D_n = (D_n^+, D_n^-)\}_{n\geq 0}$ is a higher derivation on V such that D_i^σ is continuous for every $i \in \{0, 1, ..., k-1\}$ where k is a fixed positive integer greater than 2. Then for every T in the multiplication algebra $\mathcal{M}(V^{\sigma(b)})$ of the Jordan algebra $V^{\sigma(b)}$, the linear operator $[D_k^\sigma, T]$ is continuous.*

Proof. Consider the set

$$B = \left\{T \in \mathcal{M}(V^{\sigma(b)}) : \quad [D_k^\sigma, T] \text{ is continuous}\right\}.$$

It is clear that B is a subspace of $\mathcal{M}(V^{\sigma(b)})$. Moreover, a simple computation shows that the formula

$$T\,[D_k^\sigma, S] + [D_k^\sigma, T]\,S = [D_k^\sigma, TS]$$

holds for all T, S in $\mathcal{M}(V^{\sigma(b)})$. This proves that B is a subalgebra of $\mathcal{M}(V^{\sigma(b)})$. On the other hand, for all $a \in V^{\sigma(b)}$, the left multiplication L_a lies in B. Indeed, since $L_a = \frac{1}{2}V_{(a,b)}$, for all $x \in V^\sigma$ we have

$$\begin{aligned}
[D_k^\sigma, L_a]\,x &= D_k^\sigma L_a x - L_a D_k^\sigma x \\
&= \frac{1}{2}(D_k^\sigma\,\{a,b,x\} - \{a,b,D_k^\sigma x\}) \\
&= \frac{1}{2}(\sum_{i+j+h=k} \{D_i^\sigma a, D_j^{-\sigma} b, D_h^\sigma x\} - \{a,b,D_k^\sigma x\}) \\
&= \frac{1}{2}(\sum_{\substack{i+j+h=k \\ h\le k-1}} \{D_i^\sigma a, D_j^{-\sigma} b, D_h^\sigma x\}).
\end{aligned}$$

This shows that

$$[D_k^\sigma, L_a] = \frac{1}{2}(\sum_{\substack{i+j+h=k \\ 1\le h\le k-1}} V_{(D_i^\sigma a, D_j^{-\sigma} b)} D_h^\sigma + \sum_{i+j=k} V_{(D_i^\sigma a, D_j^{-\sigma} b)}),$$

which shows that the operator $[D_k^\sigma, L_a]$ is continuous since so are $V_{(D_i^\sigma a, D_j^{-\sigma} b)}$ and D_h^σ for all $h \in \{1, ..., k-1\}$. Finally, since $\mathcal{M}(V^{\sigma(b)})$ is generated by all left multiplications L_a, we see that $\mathcal{M}(V^{\sigma(b)}) = B$.

The first automatic continuity result concerns higher derivations on nondegenerate Banach-Jordan pairs with nonzero socle.

Theorem 1. *Let $V = (V^+, V^-)$ be a nondegenerate Banach-Jordan pair with nonzero socle. If $D_n = (D_n^+, D_n^-)\}_{n\ge 0}$ is a higher derivation on V, then D_k^σ is continuous for every positive integer k.*

Proof. By the closed graph Theorem, it suffices to prove that $S(D_k^\sigma) = 0$. We proceed by induction on k. For $k = 0$, the identity operator $D_0^\sigma = Id_{V^\sigma}$ is obviously continuous. Assume that D_i is continuous for $i = 1, 2, ..., k-1$ and prove that so is D_k. In virtue of Lemma 1, it is known that $S(D_k)$ is an ideal of V. We claim that $Soc(V^+) \cap S(D_k^+) = 0$. Assume that this is not the case. We follow the pattern given in [10, Theorem 3.6] to look for a contradiction. By [10, Lemma 3.5], there exists a nonzero element r in $S(D_k^+) \cap Soc(V^+)$ such that r is *reduced*: $Q_r V^- = \mathbb{C}.r$. By von Neumann regularity of $Soc(V)$, there exists a nonzero element v in V^- such that $r = Q_r v$. Replace v by $u = Q_v r$ to see that, using JP_3 in [18] ,

$$(1)\quad Q_r u = Q_r Q_v r = Q_r Q_v Q_r v = Q_{Q_r v} v = Q_r v = r.$$

By idealness of $S(D_k)$, u lies in $S(D_k^-)$ and u is nonzero because otherwise $r = 0$, which is a contradiction. Hence, there exists a sequence $\{x_n\}$ in V^- such that $\lim x_n = 0$ and $\lim D_k^- x_n = u$. Since r is reduced, we have $Q_r V^- = \mathbb{C}.r$ and consequently, for every non negative integer n, there exists a complex number λ_n such $Q_r x_n = \lambda_n r$. Now the boundedness of the operator Q_r shows that $\lim Q_r x_n = Q_r \lim x_n = 0$. This makes the sequence $\{\lambda_n\}$ converging to zero in the complex field $\mathbb{C}$. It follows that

$$(2) \quad \lim D_k^+(Q_r x_n) = \lim D_k^+(\lambda_n r) = \lim \lambda_n D_k^+(r) = 0.$$

On the other hand, by making use of the triple product of V and that of D_j^-, such that $1 \leq j \leq k-1$, we see that all terms like $\{D_i^+ r, D_j^- x_n, D_h^+ r\}$ converge to zero when n tends to ∞. That is

$$\lim\{D_i^+ r, D_j^- x_n, D_h^+ r\} = \{D_i^+ r, \lim D_j^- x_n, D_h^+ r\} = \{D_i^+ r, D_j^- \lim x_n, D_h^+ r\} = 0.$$

It follows that, taking into account (1),

$$\begin{aligned}
\lim D_k^+(Q_r x_n) &= \frac{1}{2} \lim D_k^-(\{r, x_n, r\}) \\
&= \frac{1}{2} \lim \sum_{i+j+h=k} \{D_i^+ r, D_j^- x_n, D_h^+ r\} \\
&= \frac{1}{2}\{r, \lim D_k^- x_n, r\} \\
&= \frac{1}{2}\{r, u, r\} \\
&= Q_r u \\
&= r,
\end{aligned}$$

which contradicts (2) since r is nonzero. Now, by idealness of $S(D_k)$ and $Soc(V)$, we see that for all $s \in Soc(V^-)$

$$Q_s(S(D_k^+)) \subset Soc(V^-) \cap S(D_k^-) = 0.$$

This shows that $S(D_k^+) \subseteq \ker(Q_s)$ for every s in $Soc(V^-)$, that is $S(D_k^+) \subseteq \bigcap_{s \in Soc(V^-)} \ker(Q_s)$. But in virtue of [18, Theorem 4.13] , we see

$$\bigcap_{s \in Soc(V^-)} \ker(Q_s) \subseteq rad(Soc(V^+)) \text{ and } rad(Soc(V^+)) = Soc(V^+) \cap rad(V^+).$$

But, the McCrimmon radical $rad(V)$ is reduced to zero by nondegeneracy of V. This proves that $S(D_k^+) = 0$ and, by the closed graph Theorem, D_k^+ is continuous. By the symmetry of the argument we see that D_k^- is analogously continuous.

As a fundamental example of Jordan pairs having nonzero socle, $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ the Jordan pair of bounded linear operators between two Banach spaces X and Y. So we have the following.

Corollary 1. *Any higher derivation $D_n = (D_n^+, D_n^-)$ on the Banach-Jordan pair $B(X,Y)$ consists of continuous operators.*

Proof. It is known that the Banach-Jordan pair $\mathcal{B}(\mathcal{X},\mathcal{Y})$ of bounded linear operators between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is nondegenerate and has

$$Soc(\mathcal{B}(\mathcal{X},\mathcal{Y})) = (\mathcal{FL}(\mathcal{X},\mathcal{Y}), \mathcal{FL}(\mathcal{Y},\mathcal{X}),$$

the Banach-Jordan pair consisting in bounded linear operators of finite rank. Now the continuity of $\{D_n = (D_n^+, D_n^-)\}$ follows immediately from Theorem 1.

4. Main result

Before going on the proof the main Theorem in this paper, we recall the following technical results which seem to be useful in the sequel.

Lemma 3. [29]. *Let X be a Banach space, $\{T_i\}_i$ a sequence of continuous linear operators defined on X and let $\{R_i\}_i$ be a sequence of linear continuous operators whose domain is X but which may map into other Banach spaces. Let T be a possibly discontinuous map from X to itself. If the operator $R_nTT_1...T_m$ is continuous for m greater than n then $R_nTT_1...T_n$ is continuous when n is sufficiently large.*

Proposition 1. [10]. *Let J be a Banach-Jordan algebra and I be a primitive ideal of J. If D is a linear operator defined on J such that $[D,T]$ is continuous for all T in $\mathcal{M}(J)$, then the primitive Jordan algebra $(S(D)+I)/I$ has finite capacity.*

Lemma 4. *Let V a nondegenerate Jordan pair and let $P_1, ..., P_n$ be nonzero ideals of V. If H is an ideal of V such that $H \cap P_1 \cap ... \cap P_n = 0$ then. $H = 0$.*

Proof. We proceed by induction. For $n = 1$, by idealness of H and P_1, we have, for all $u \in P_1^\sigma$, $Q_u H^{-\sigma} \subseteq H^\sigma \cap P_1^\sigma = 0$. Then, by [18, Proposition 4.19] together with [18, Theorem 4.13]

$$H-^\sigma \subseteq \cap_{u \in P_1^\sigma} Ker(u) \subset rad(P_1^\sigma) = rad(V^\sigma) \cap P_1^\sigma,$$

and hence $H^\sigma = 0$ by nondegeneracy of V: $rad(V) = 0$. Suppose the statement is true for some natural integer n and let $P_1, ..., P_n, P_{n+1}$ be nonzero ideals of V satisfying the condition stated in the lemma. Then the ideals P_1 and $K = P_2 \cap ... \cap P_{n+1} \cap H$ also satisfy the same condition. Therefore, by we have just proved in the case $n = 1$, $K = 0$ and hence $H = 0$ by induction.

Given a Banach space X, we denote by $Cl(E)$ the closure of a subset E of X.
We can now state our main result in this paper.

Theorem 2. *Let $\{D_n = (D_n^+, D_n^-)\}$ be a higher derivation on a Banach-Jordan pair $V = (V^+, V^-)$. If V is semiprimitive, then D_k^σ is continuous for every non negative integer k.*

Proof. We proceed by induction. For if $n = 0$, $D_0^\sigma = Id_{V^\sigma}$ is trivially continuous. Suppose that $D_1, ..., D_k$ are continuous and show that this is also the case for D_{k+1}, that is $S(D_{k+1}) = 0$. Suppose that D_{k+1} is discontinuous. Then, there exists a primitive ideal P such that $S(D_{k+1})$ is not contained in P. As a first step we show that all primitive ideals contain $S(D_{k+1})$ except finitely primitive ideals $P_1, ..., P_n$ for which the quotient pairs V/P_i have finite capacity. In other words de set

$$\Gamma = \left\{P = (P^+, P^-) \text{ primitive ideal of } V : \; S(D_{k+1}) \;\nsubseteq P\right\}$$

is finite and, for any $P \in \Gamma$, the quotient pair V/P has finite capacity.

Take $P = (P^+, P^-)$ in Γ and $b \in V^-$ such that $b \notin P^-$. Since P^σ is closed in V^σ (see [14, A.5.2]), V/P is a Banach-Jordan pair and hence by (2.2) $(V/P)_{\bar{b}}$ is a primitive Banach-Jordan algebra where $\bar{b} = b + P^-$ is the image of b under the canonical projection $V^- \longmapsto V^-/P^-$. The algebra $(V/P)_{\bar{b}}$ is known to be isomorphic to $V^{+(b)}/I$ where $I = Q_b^{-1}(P^-)$ is so a primitive ideal of the Banach-Jordan algebra $V^{+(b)}$. Moreover, by Lemma 2 the linear operator D_{k+1} and the ideal I satisfy the conditions required in Proposition 1 with respect to the Banach-Jordan algebra $V^{+(b)}$. Therefore, $(S(D_{k+1})+I)/I$ has nonzero finite capacity. This implies that $(V/P)_{\bar{b}}$ has itself nonzero finite capacity [26, Theorem 18]. Thus by (2.1), $Soc(V/P) = V/P$ and hence, by completeness, V/P has nonzero finite capacity.

Suppose that the set Γ is infinite, then we can take an infinite sequence $\{P_n\}$ of distinct primitive ideals in Γ. By we have just proved, V/P_n is simple with finite capacity and hence has finite spectrum (see [19, Theorem 1] and [20, Theorem 3.8]). By a similar process used in [6, Lemma 2.8], we show the existence of an element b in V^- and a sequence $\{a_n\}$ in V^+ such that $b \notin \cup_n P_n^-$, $\pi_m(a_n)$ is invertible in $(V/P_m)_{\bar{b}}$ for $n < m$ and $\pi_m(a_n) = 0$ for $m < n$ where $\pi_m : V^+ \longmapsto V^+/P_m^+$ is the natural projection. Indeed, take b_1 in V^- such that $b_1 \notin P_1^-$. By induction we can construct the sequences $\{b_n\}$ in V^- and $\{\lambda_n\}$ in the complex field such that $\lambda_1 = 1$. Having defined $b_1, ..., b_{n-1}$ and $\lambda_1, ..., \lambda_{n-1}$, we take b_n in $\bigcap_{i=1}^{n-1} P_i^-$ with $\|b_n\| = 1,\ 1 < \lambda_n < \frac{1}{2^n}$ and $\sum_{i=1}^{n} \lambda_i b_i \notin P_n^-$. This last condition is satisfied since $\bigcap_{i=1}^{n-1} P_i$ is not contained in P_n. Since the series $\sum_{i=1}^{n} \lambda_i b_i$, converges in V^-, we write $b = \sum_{i=1}^{\infty} \lambda_i b_i$. We see that $\bar{b} = \sum_{i=1}^{n} \lambda_i \overline{b_i}$ is nonzero in V^-/P_n^- and hence $b \notin \cup_n P_n^-$. Now take u_1 in V^+ such that $u_1 \notin P_1^+$. We proceed by choosing $\{u_n\}$ in V^+ and , for any natural number k, the scalars $\left\{\lambda_n^k\right\}_{n=k}^{\infty}$ such that $\lambda_k^k = 1$. Having selected them up to $n-1$, we take u_n and λ_n^k such that $u_n \in \bigcap_{i=1}^{n-1} P_i^+$, $\pi_n(u_n)$ is the unit of the Banach-Jordan algebra $(V/P_m)_{\bar{b}}$, $0 < \lambda_n^k < \frac{1}{2^n \|u_n\|}$ and $\pi_n(\sum_{i=k}^{n} \lambda_i^k u_i)$ is invertible. If we take $a_n = \sum_{i=n}^{\infty} \lambda_i^n u_i$, then we will have $\pi_m(a_n)$ is invertible in $(V/P_m)_{\bar{b}}$ for $m > n$ and $\pi_m(a_n) = 0$ for $m < n$ as required.

Now consider an arbitrary x in V^+ and positive integers m, n. We compute in $(V/P_n)_{\bar{b}}$

to have

$$\begin{aligned}
\pi_n D_{k+1} U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_m}^{(b)}(x) &= \frac{1}{4}\pi_n D_{k+1}\left\{a_1, \left\{b, U_{a_2}^{(b)} ... U_{a_m}^{(b)}(x), b\right\}, a_1\right\} \\
&= \frac{1}{4}\pi_n \sum_{i+j+h=k+1} \left\{D_i a_1, D_j\left\{b, U_{a_2}^{(b)} ... U_{a_m}^{(b)}(x), b\right\}, D_h a_1\right\} \\
&= \frac{1}{4}\pi_n \sum_{\substack{i+j+h=k+1 \\ j\le k}} \left\{D_i a_1, D_j\left\{b, U_{a_2}^{(b)} ... U_{a_m}^{(b)} x, b\right\}, D_h a_1\right\} \\
&\quad + \frac{1}{4}\pi_n \left\{a_1, D_{k+1}\left\{b, U_{a_2}^{(b)} ... U_{a_m}^{(b)} x, b\right\}, a_1\right\} \\
&= \varphi(x) + \frac{1}{2}\pi_n Q_{a_1}\Big(\sum_{i+j+h=k+1} \left\{D_i b, D_j U_{a_2}^{(b)} ... U_{a_m}^{(b)} x, D_h b\right\}\Big) \\
&= \varphi(x) + \frac{1}{2}\pi_n Q_{a_1}\Big(\sum_{\substack{i+j+h=k+1 \\ j\le k}} \left\{D_i b, D_j U_{a_2}^{(b)} ... U_{a_m}^{(b)} x, D_h b\right\}\Big) \\
&\quad + \pi_n Q_{a_1} Q_b D_{k+1} U_{a_2}^{(b)} ... U_{a_m}^{(b)} x \\
&= \psi_1(x) + \pi_n U_{a_1}^{(b)} D_{k+1} U_{a_2}^{(b)} ... U_{a_m}^{(b)}(x),
\end{aligned}$$

where

$$\begin{aligned}
\psi_1(x) &= \varphi(x) + \frac{1}{2}\pi_n Q_{a_1}\Big(\sum_{\substack{i+j+h=k+1 \\ j\le k}} \left\{D_i b, D_j U_{a_2}^{(b)} U_{a_3}^{(b)} ... U_{a_m}^{(b)} x, D_h b\right\}\Big) \\
\varphi(x) &= \frac{1}{4}\pi_n \sum_{\substack{i+j+h=k+1 \\ j\le k}} \left\{D_i a_1, D_j\left\{U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_m}^{(b)}(x)\right\}, D_h a_1\right\}
\end{aligned}$$

are clearly continuous operators. By iterating the same process, we show that there exits a continuous linear operator ψ_m such that

$$\pi_n D_{k+1} U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_m}^{(b)}(x) = \psi_m(x) + \pi_n U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_m}^{(b)} D_{k+1}(x).$$

But we have $\pi_n U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_m}^{(b)} D_{k+1} = 0$ when $n < m$. It follows that the operator $\pi_n D_{k+1} U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_m}^{(b)}$ is continuous. Now, Lemma 3 applies to the sequences $\{R_i\}$ and $\{T_i\}$ with $R_i = \pi_i$ and $T_i = U_{a_i}^{(b)}$ to obtain the continuity of the operator $\pi_n D_{k+1} U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_n}^{(b)}$ when the integer n is sufficiently large. That is

$$S(\pi_n D_{k+1} U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_n}^{(b)}) = 0.$$

But since $\pi_n(a_i)$ is invertible for $i \le n$, we have

$$S(\pi_n D_{k+1} U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_n}^{(b)}) = S(\pi_n U_{a_1}^{(b)} U_{a_2}^{(b)} ... U_{a_n}^{(b)} D_{k+1})$$

$$\begin{aligned} &= Cl(U^{(\bar{b})}_{\pi_1 a_1} U^{(\bar{b})}_{\pi_2 a_2} ... U^{(\bar{b})}_{\pi_n a_n} S(\pi_n D_{k+1})) \\ &= Cl(S(\pi_n D_{k+1})), \end{aligned}$$

which is a contradiction because $Cl(S(\pi_n D_{k+1})) \neq 0$ since otherwise we will have $S(D_{k+1}) \subseteq P_n$ for any positive integer n. The set Γ is actually finite, say $\Gamma = \{P_1, ..., P_n\}$. Set $H = \underset{P \notin \Gamma}{\cap} P$. The ideals $P_1, P_2, ..., P_n$ and H satisfy the requirements of Lemma 4 since $(\overset{n}{\underset{i=1}{\cap}} P_i) \cap H = Rad(V)$ is the intersection of all primitive ideals of V (2.4) and $Rad(V) = 0$. We conclude that $H = 0$. But $S(D_{k+1}) \subseteq P$ for any primitive ideal P not contained in Γ, then $S(D_{k+1}) \subseteq \underset{P \notin \Gamma}{\cap} P = H = 0$, which is a contradiction. D_{k+1} is finally continuous.

As it is pointed out, any Jordan algebra gives rise to a Jordan pair (J, J) with the quadratic map $Q_a = U_a$ defined by $U_a b = 2a(ab) - a^2 b$.

A family $\{d_n\}$ ($n = 0, 1, 2, ..., k,$ k may be ∞) of linear operators defined on J is said to be a *higher derivation* if, for all a, b in J, we have

$$d_n(ab) = \sum_{k=1}^{k=n} d_k(a) d_{n-k}(b).$$

A tedious computation enables to prove that any higher derivation $\{d_n\}_{n \geq 0}$ on a Jordan algebra gives rise to a higher derivation $\{(d_n, d_n)\}_{n \geq 0}$ on the Jordan pair $(J, \bar{J})$ with respect to the triple product

$$\{x, y, z\} = (xy)z + (yz)x - (zx)y.$$

The Jordan pair is semiprimitive if so is J. Hence, according to Theorem 2, we have the following.

Corollary 2. . *Any higher derivation $\{d_n\}_{n \geq 0}$ on a semiprimitive Banach-Jordan algebra consists of continuous operators.*

5. Higher derivations on Banach alternative pairs and JB^*-triples

The reader is referred to [18] for definitions and basic results on alternative pairs. Given an alternative pair $A = (A^+, A^-)$, we write $(x, y, z) \longmapsto \langle xyz \rangle$ to denote the triple product of (x, y, z) in $A^\sigma \times A^{-\sigma} \times A^\sigma$ $(\sigma = \pm)$.

By a *normed alternative pair* we mean a complex alternative pair $A = (A^+, A^-)$, where the vector spaces A^+ and A^- are equipped with norms $\|.\|_\sigma$ making continuous the triple product $\langle xyz \rangle$. $A = (A^+, A^-)$ is said to b Banach alternative pair provided the norms $\|.\|_\sigma$ are complete. The Banach spaces $\mathcal{M}_{p,q}(\mathbb{C})$, $\mathcal{M}_{q,p}(\mathbb{C})$ of rectangular matrices with entries in the complex field $\mathbb{C}$ define a Banach alternative pair $A = (\mathcal{M}_{p,q}(\mathbb{C}), \mathcal{M}_{q,p}(\mathbb{C}))$, with respect to the triple product $\langle RST \rangle = RST$, the usual matrices product.

A *higher derivation* on an alternative pair $A = (A^+, A^-)$ is a sequence $\{D_n = (D_n^+, D_n^-)\}_{n\geq 0}$ of linear operators $D_n^\sigma : A^\sigma \longmapsto A^\sigma$ satisfying the formula

$$D_n^\sigma(< xyz >) = \sum_{i+j+h=n} < D_i^\sigma(x) D_j^{-\sigma}(y) D_h^\sigma(z) >, \ (x, z \in A^\sigma, \ y \in A^{-\sigma}, \ n = 0, 1, 2, ...$$

with $D_0^\sigma = Id_{A^\sigma}$.

Corollary 3. *Let $\{D_n = (D_n^+, D_n^-)\}_{n\geq 0}$ be a higher derivation on a Banach alternative pair $A = (A^+, A^-)$. If A is semiprimitive, then D_k^σ is continuous for every positive integer k.*

Proof. It is known that any alternative pair $A = (A^+, A^-)$ gives rise to a Jordan pair frequently denoted by A^J (see [18, Theorem 7.1]) by considering the quadratic operators

$$Q_x y = \langle xyx \rangle \text{ for all } (x, y) \text{ in } A^\sigma \times A^{-\sigma}.$$

Clearly A^J is a Banach-Jordan pair whenever A is a Banach alternative pair. By [18, 7.9(1)], A^J is semiprimitive if and only if so is A. Moreover, a simple computation enables to verify that every higher derivation $\{D_n = (D_n^+, D_n^-)\}$ on A induces a higher derivation on A^J with respect to its triple product defined by

$$\{x, y, z\} = Q_{(x,z)} y = \langle xyz \rangle + \langle zyx \rangle \text{ for all } (x, y, z) \text{ in } A^\sigma \times A^{-\sigma} \times A^\sigma.$$

Actually, Theorem 2 applies to deduce that D_n^σ is continuous for every natural number n.

Following [4], we mean by a *higher derivation* on a JB^*-triple E, a sequence $\{\delta_n\}_{n\geq 0}$ of linear operators $\delta_k : E \longrightarrow E$ satisfying

$$\delta_n(\{x, y, z\}) = \sum_{i+j+k=n} \{\delta_i x, \delta_j y, \delta_k z\}, \text{ for all } x, y, z \text{ in } E,$$

where $\delta_0 = Id_E$.

Corollary 4. *Any higher derivation $\{\delta_n\}_{n\geq 0}$ on a JB^*-triple E is continuous.*

Proof. Since every JB^*-triple E gives rise to a complex semiprimitive Banach-Jordan pair $V = (V^+, V^-)$, where $V^+ = E$ as vector space and V^- is the conjugate complex vector space of E that is the vector space with the new scalar multiplication $\lambda.x = \overline{\lambda} x$ for $x \in E$ and $\lambda \in \mathbb{C}$. Moreover, $\{\delta_n\}_{n\geq 0}$ defines a higher derivation $\{(\delta_n, \delta_n)\}_{n\geq 0}$ on the complex semiprimitive Banach-Jordan pair $V = (V^+, V^-)$. Thus the continuity of δ_n holds by Theorem 2.

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