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A module whose second spectrum has the surjective or injective natural map

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Abstract. Let R be a commutative ring and M be an R-module. Let $Spec^s(M)$ be the set of all second submodules of M. In this article, we topologize $Spec^s(M)$ with Zariski and classical Zariski topologies and study the classes of all modules whose second spectrum have the surjective or injective natural map. Moreover, we investigate the interplay between the algebraic properties of M and the topological properties of $Spec^s(M)$.

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1. Introduction

Throughout this article, R denotes a commutative ring with identity and all modules are unitary. Also the notation $\mathbb{Z}$ (resp. $\mathbb{Q}$) will denote the ring of integers (resp. the field of fractions of $\mathbb{Z}$). If N is a subset of an R-module M, then $N \leq M$ denotes N is an R-submodule of M. For any ideal I of R containing $Ann_R(M)$, $\bar{R}$ and $\bar{I}$ denote $R/Ann_R(M)$ and $I/Ann_R(M)$, respectively. The *colon ideal of* M *into* N is defined to be $(N : M) = \{r \in R : rM \subseteq N\} = Ann_R(M/N)$.

Let M be an R-module. A proper submodule N of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in N$, we have $m \in N$ or $r \in (N :_R M)$. This implies that $(N :_R M) = p$ is prime ideal of R and we say that N is a *p-prime* submodule of M.

The dual notion of prime submodules (i.e., second submodules) was introduced and studied in [30]. A non-zero submodule S of M is said to be *second* if for each $a \in R$ the homomorphism $S \xrightarrow{a} S$ is either surjective or zero. This implies that $Ann_R(S) = p$ is a prime ideal of R and S is said to be *p-second*. More information about this class of modules can be seen in [4, 5, 7, 8, 14, 16, 15].

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The concept of prime submodule has led to the development of topologies on the spectrum of modules. A brief history of this development can be seen in [23, p. 808]. More information concerning the spectrum of rings, posets, and modules can be found in [1, 2, 7, 8, 12, 14, 20, 22, 26].

The second spectrum of M is defined as the set of all second submodules of M and denoted by $Spec^s(M)$ or X^s. We call the map $\psi : X^s \to Spec(\overline{R})$ given by $S \mapsto \overline{Ann_R(S)}$ as the *natural map* of X^s.

Let N be a submodule of M. Define $V^s(N) := \{S \in Spec^s(M) : Ann_R(N) \subseteq Ann_R(S)\}$ and set $\zeta^s(M) := \{V^s(N) : N \leq M\}$. Then there exists a topology, τ^s say, on $Spec^s(M)$ having ζ^s as the family of all its closed sets. This topology is called the *Zariski topology on* $Spec^s(M)$ (see [7]).

For any submodule N of M, define $V^{s*}(N) = \{S \in Spec^s(M) : S \subseteq N\}$. Set $\zeta^{s*}(M) = \{V^{s*}(N) : N \subseteq M\}$. Then $\zeta^{s*}(M)$ contains the empty set and $Spec^s(M)$, and it is closed under arbitrary intersections. In general $\zeta^{s*}(M)$ is not closed under finite unions. A module M is called a cotop module if $\zeta^{s*}(M)$ is closed under finite unions. In this case, $\zeta^{s*}(M)$ is called the *quasi Zariski topology* (see [7]).

Now for a submodule N of M, define $W^s(N) = Spec^s(M) - V^{s*}(N)$ and set $\Omega^s(M) = \{W^s(N) : N \leq M\}$. Let $\eta^s(M)$ be the topology on $Spec^s(M)$ by the sub-basis $\Omega^s(M)$. In fact $\eta^s(M)$ is the collection U of all unions of finite intersections of elements of $\Omega^s(M)$. We call this topology the *classical Zariski topology* on $Spec^s(M)$ (or *second classical Zariski topology*) (see [8]). It is clear that if M is a cotop module, then its related topology, as it was mentioned in the above paragraph, coincide with the second classical Zariski topology.

The dual notion of the prime radical of a submodule (i.e, second socle or second radical) have been introduced by H. Ansari-Toroghy and F. Farshadifar in [4]. For a submodule N of M, the *second radical* of N is defined as the sum of all second submodules of M contained in N and denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) . Also, $N \neq (0)$ is said to be a *socle submodule* of M if $sec(N) = N$. One can see that M is cotop if and only if for every two socle submodules N and K of M and every second submodule S of M with $S \subseteq N + K$, we have $S \subseteq N$ or $S \subseteq K$.

In this article, we introduce the concept of X^s-injective modules and investigate some of their basic properties. We say that M is X^s*-injective* if the natural map of X^s is injective (see Definition 3.9).

The primeful R-modules was introduced and studied by C.P. Lu in several papers. The dual of this notion (i.e., secondful modules) was studied by H. Ansari-toroghy and F. Farshadifar in [3] and [17]. An R-module M is *secondful* if the natural map of X^s is surjective. In section two, we explore more properties of this class of modules. In Theorem 2.3, we provide a useful characterization for Artinian secondful modules and by using this, we prove that if M is an Artinian secondful module with Noetherian spectrum, then $Hom_R(R_p, M)$ has also a Noetherian second spectrum with Zariski topology. Moreover, in Theorem 2.9 we provide a useful characterization for secondful modules.

In section three, among other results, we show that if every closed subset of ($Spec^s(M)$, τ^{s*}) has a finite number of irreducible components, then every submodule of M has a finite number of maximal second submodules. We provide an example which shows the converse is not true in general. In [8, Proposition 3.13], it is proved that if M has *dcc* condition on its socle submodules, then every irreducible closed subset of $Spec^s(M)$ (with second classical Zariski topology) has a generic point. In Theorem 3.2, we remove this restriction and prove that every irreducible closed subset of $Spec^s(M)$ has a generic point. Further it is shown that if $Spec^s(M)$ is a Noetherian space, then it is a spectral space (see Corollary 3.3). Proposition 3.13 states that if $(M_i)_{i\in I}$ is a family of R-modules and $M = \bigoplus_{i\in I} M_i$ is an X^s-injective module, then $Spec^s(M)$ can be specified in terms of second submodules of M_i. Theorem 3.15 says that $M = \bigoplus_{i\in I} M_i$ is an X^s-injective (resp. a cotop) R-module if and only if $(M_i)_{i\in I}$ is a family of second-compatible X^s-injective (resp. cotop) R-modules. Moreover, in Theorem 3.18, we prove that if R is a perfect ring, then the classes of cotop, weak comultiplication, and X^s-injective modules are all equal.

In the rest of this article, $X^s := Spec^s(M)$ will denote the set of all second submodules of M. Also the map $\psi : X^s \to Spec(\overline{R})$ given by $S \mapsto \overline{Ann_R(S)}$ is called the natural map of X^s.

2. Secondful modules

We recall that an R-module M is secondful if the natural map X^s is surjective. For example, for each positive integer n $(n > 1)$, $\mathbb{Z}_n$ as $\mathbb{Z}$-module is secondful.

Throughout this section, we assume that $Spec^s(M)$ is topologized with Zariski topology.

Remark 2.1. Let M be an R-module. Then we have the following.

(a) Let M be an Artinian module. Then M is secondful $\Leftrightarrow Hom_R(R_p, (0 :_M p)) \neq (0)$ for every $p \in V(Ann_R(M))$ [7, Theorem 3.8].

(b) Let M be a secondful R-module. Then M has a Noetherian second spectrum if and only if the ring $\overline{R}$ has Noetherian spectrum [17, Theorem 4.3].

Lemma 2.2. Let M be an R-module and p, q be two prime ideals of R such that $q \subseteq p$. Then

$$Hom_R(R_q, (0 :_M q)) \cong Hom_{R_p}((R_p)_{qR_p}, (0 :_{Hom_R(R_p,M)} qR_p)).$$

Proof.

$$\begin{aligned} Hom_R(R_q, (0 :_M q)) &\cong Hom_R(R_p \otimes_{R_p} R_q, (0 :_M q)) \\ &\cong Hom_{R_p}(R_q, Hom_R(R_p, Hom_R(R/q, M))) \\ &\cong Hom_{R_p}(R_q, Hom_R(R_p \otimes_R R/q, M)) \\ &\cong Hom_{R_p}(R_q, (0 :_{Hom_R(R_p,M)} qR_p)) \\ &\cong Hom_{R_p}((R_p)_{qR_p}, (0 :_{Hom_R(R_p,M)} qR_p)). \end{aligned}$$

Theorem 2.3. *Let M be an Artinian R-module. Then the following statements are equivalent:*

(a) M is a non-zero secondful R-module;

(b) $Hom_R(R_p, M)$ is non-zero secondful R_p-module for every $p \in V(Ann_R(M))$.

Proof. $(a) \Rightarrow (b)$ Let p be a prime ideal of R such that $p \supseteq Ann_R(M)$. Since M is a non-zero secondful module, $Hom_R(R_p, (0 :_M p)) \neq (0)$ by Remark 2.1 (a). Hence we have $Hom_R(R_p, M) \neq (0)$. To prove that $Hom_R(R_p, M)$ is secondful R_p-module, let $qR_p \in V(Ann_{R_p}(Hom_R(R_p, M)))$. Since R_p is quasi-local ring,

$$pR_p \supseteq qR_p \supseteq Ann_{R_p}(Hom_R(R_p, M)) \supseteq Ann_R(M)R_p.$$

Taking the contraction of each term of this sequence, we have that

$$p \supseteq q \supseteq S_p(Ann_R(M)) \supseteq Ann_R(M).$$

Hence $Hom_R(R_q, (0 :_M q)) \neq (0)$ by Remark 2.1 (a). It follows that $Hom_R(R_p, M)$ is a secondful R_p-module by Remark 2.1 (a) and Lemma 2.2.

$(b) \Rightarrow (a)$ Let $p \in V(Ann_R(M))$. By part (b), $Hom_R(R_p, M) \neq (0)$. Thus $Ann_{R_p}(Hom_R(R_p, M)) \neq R_p$. Therefore, $Ann_{R_p}(Hom_R(R_p, M)) \subseteq pR_p$ and so $pR_p \in V(Ann_{R_p}(Hom_R(R_p, M)))$. Now by Remark 2.1 (a),

$$Hom_{R_p}((R_p)_p, (0 :_{Hom_R(R_p,M)} pR_p)) \neq (0),$$

and so $Hom_R(R_p, (0 :_M p)) \neq (0)$. Thus M is a secondful R-module.

Corollary 2.4. Let M be a non-zero Artinian secondful R-module and suppose $p \in V(Ann_R(M))$. Then if M has Noetherian second spectrum, so does the R_p-module $Hom_R(R_p, M)$.

Proof. Let $p \in Spec(R)$ and $Ann_R(M) \subseteq p$. By Theorem 2.3, $Hom_R(R_p, M)$ is a secondful R_p-module. Hence, in order to show that $Hom_R(R_p, M)$ has Noetherian second spectrum, we need to prove that $R_p/Ann_{R_p}(Hom_R(R_p, M))$ has Noetherian spectrum by Remark 2.1 (b). Let $S = R \backslash p$. Then $\overline{S} = \{s + Ann(M) : s \in S\}$ is a multiplicative closed subset of $\overline{R} = R/Ann_R(M)$. Let $\phi : \overline{R} \to (\overline{R})_{\overline{S}}$ be the natural homomorphism and $\phi^* : Spec((\overline{R})_{\overline{S}}) \to Spec(\overline{R})$ the associated mapping. Set $\Sigma = \{\overline{p} \in Spec(\overline{R}) : \overline{p} \cap \overline{S} = \emptyset\}$. Since M is secondful and has Noetherian spectrum, $Spec(\overline{R})$ is Noetherian by Remark 2.1 (b). Further Σ as a subspace of $Spec(\overline{R})$ is a Noetherian space and it is homeomorphic to $Spec((\overline{R})_{\overline{S}})$ by [21, p. 81, Proposition 4.12 (b)]. Hence $Spec((\overline{R})_{\overline{S}})$ is a Noetherian space. Also, we have $(\overline{R})_{\overline{S}} \cong R_p/(Ann_R(M))_p$. Thus $Spec(R_p/(Ann_R(M))_p)$ is a Noetherian space. On other hand, we have $(Ann_R(M))_p \subseteq Ann_{R_p}(Hom_R(R_p, M))$. Now

$$\frac{R_p}{Ann_{R_p}(Hom_R(R_p, M))} \cong \left(\frac{R_p}{(Ann_R(M))_p}\right) / \left(\frac{Ann_{R_p}(Hom_R(R_p, M))}{(Ann_R(M))_p}\right).$$

From this, we deduce that $R_p/Ann_{R_p}(Hom_R(R_p, M))$ has Noetherian spectrum.

Let T be a topological space and $T' \subseteq T$. Then T is irreducible if $T \neq \varnothing$ and for every decomposition $T = A_1 \cup A_2$ with closed subsets $A_i \subseteq T$, $i = 1, 2$, we have $A_1 = T$ or $A_2 = T$. The subset T' of T is irreducible if it is irreducible as a space with the relative topology. Equivalently, T' is irreducible if and only if for every pair of sets F, G which are closed in T, it holds that $T' \subsetneq F \cup G$, $T' \subsetneq F$ or $T' \subsetneq G$ [13, p. 94].

An irreducible component of T is a maximal irreducible subset of T. Every irreducible subset of T is contained in an irreducible component of T, and T is the union of its irreducible components.

Let Z be a subset of a topological space W. Then the notion $cl(Z)$ will denote the closure of Z in W.

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point* of Y if $Y = cl(\{y\})$. If the topological space is a T_0-space, then a generic point of every closed subset is unique.

Proposition 2.5. Let M be a secondful module over R and $N \leq M$.

(a) Let Y be a nonempty subset of $V^s(N)$. Then Y is an irreducible closed subset of $V^s(N)$ if and only if Y has a generic point in $V^s(N)$.

(b) The mapping $\rho : S \mapsto V^s(S)$ is a surjection of $V^s(N)$ onto the set of irreducible closed subsets of $V^s(N)$.

(c) The mapping $\phi : V^s(S) \mapsto \overline{Ann_R(S)} \in Spec(\overline{R})$ is a bijection of the set of irreducible components of $V^s(N)$ onto the set of minimal prime divisors of $\overline{Ann_R(N)}$ in $\overline{R} = R/Ann_R(M)$.

Proof.

(a) This is clear.

(b) This follows directly from part (a).

(c) Let $\phi : V^s(S) \mapsto \overline{Ann_R(S)}$. Then ϕ is a well defined injective mapping by part (a). We show that ϕ is surjective. Let $\overline{p}$ be a minimal prime divisor of $\overline{Ann_R(N)}$ in $\overline{R}$ and let p be the prime ideal of R such that $p/Ann_R(M) = \overline{p}$. Then $Ann_R(M) \subseteq Ann_R(N) \subseteq p$. Since M is secondful, there exists a p-second submodule $S \in X^s$. Now $Ann_R(N) \subseteq p = Ann_R(S)$ implies that $S \in V^s(N)$, and so $V^s(S) \subseteq V^s(N)$. Thus $V^s(S)$ is an irreducible closed subset of $V^s(N)$. Note that the minimality of $\overline{p} \in V(\overline{Ann_R(N)})$ implies the maximality of $V^s(S)$ among all irreducible closed subsets $V^s(S')$, $S' \in V^s(N)$, as $\overline{Ann_R(N)} \subseteq \overline{Ann_R(S')}$. Therefore, $V^s(S)$ is an irreducible component of $V^s(N)$. This proves that ϕ is surjective.

Proposition 2.6. Let M be a secondful R-module. Then $Spec^s(M)$ has a chain of irreducible closed subsets of length r if and only if $\overline{R}$ has a chain of prime ideals of length r.

Proof.
Let $Z_0 \subsetneq Z_1 \subsetneq ... \subsetneq Z_r$ be a strictly increasing chain of irreducible closed subset Z_i of $Spec^s(M)$ of length r. By [7, Theorem 5.8], each Z_i has a generic point, so that $Z_i = V^s(S_i)$ for some $S_i \in Spec^s(M)$. Hence, we have $V^s(S_0) \subsetneq V^s(S_1) \subsetneq ...V^s(S_r)$. Thus $\overline{Ann_R(S_0)} \supsetneq \overline{Ann_R(S_1)} \supsetneq ... \supsetneq \overline{Ann_R(S_r)}$, a strictly decreasing chain of prime ideals $\overline{R}$ of length r. Conversely, let $\overline{q_0} \supsetneq \overline{q_1} \supsetneq ... \supsetneq \overline{q_r}$ be a strictly decreasing chain of prime ideals in $\overline{R}$ of length r and, for each i $(1 \leq i \leq r)$, let q_i be a prime ideal of R containing $Ann_R(M)$ such that $\overline{q_i} = q_i/Ann_R(M)$. Since M is a secondful R-module, there exists a q_i-second submodule Q_i of M for each i $(1 \leq i \leq r)$. Hence we have $Ann_R(Q_0) \supsetneq Ann_R(Q_1) \supsetneq ... \supsetneq Ann_R(Q_r)$ so that $V^s(Q_0) \subsetneq V^s(Q_1) \subsetneq ... \subsetneq V^s(Q_r)$. Thus, we obtain a strictly increasing chain of length r of irreducible closed subsets of $Spec^s(M)$.

We consider strictly decreasing (or strictly increasing) chain $Z_0, Z_1, ..., Z_r$ of length r of irreducible closed subsets Z_i of T. The supremum of the lengths, taken over all such chains, is called the *combinatorial dimension* of T and denoted by $dimT$ (for $T = \emptyset$, define $dimT = -1$).

A submodule N of M is said to be *cocyclic* if $N \subseteq E(R/m)$ for some maximal ideal m of R (here $E(R/m)$ denote the injective envelope of R/m) (see [29]). The *cosupport* of M, denoted by $Cosupp(M)$, is defined as the set of all prime ideals p of R such that $p \supseteq Ann_R(N)$ for some cocyclic homomorphic image N of M (see [28]).

For every finitely cogenerated module M, $Cosupp(M) = V(Ann_R(M))$ [28, Lemma 2.3]. The equality holds also if M is a secondful module by [17, Theorem 2.5]. Using this fact and Proposition 2.6, we can establish the next theorem.

Theorem 2.7. *Let M be a secondful R-module and equip $Spec^s(M)$ and $Spec(R)$ with their Zariski topologies. Then the combinatorial dimension of $Spec^s(M)$, the Krull dimension of $\overline{R} = R/Ann_R(M)$, and the combinatorial dimension of the closed subspace $Cosupp(M) = V(Ann_R(M))$ of $Spec(R)$ are all equal.*

As a result of Theorem 2.7, the definition of the classical Krull dimension of modules, which is independent of $Spec^s(M)$, becomes more significant for secondful modules M.

Let p be a prime ideal of R. For an R-module M, $Spec^s_p(M)$ denotes the set of all p-second submodules of M.

Corollary 2.8. Let M be a secondful R-module such that $Spec^s(M)$ has zero combinatorial dimension. Then we have the following.

(a) Every irreducible closed subset of $Spec^s(M)$ is an irreducible component of $Spec^s(M)$.

(b) For every $p \in V(Ann_R(M))$ and for every p-second submodule S of M, $Spec^s_p(M) = V^s(S)$.

(c) If M has Noetherian second spectrum, then the set of irreducible components of $Spec^s(M)$ is $\{V^s((0:_M m_1)), V^s((0:_M m_2)), ..., V^s((0:_M m_k))\}$ for some positive integer k, where m_i, for $i = 1, 2, ..., k$, are all the minimal prime divisors of $Ann_R(M)$.

Proof.

(a) Let $V^s(N)$ be an irreducible closed subset of $Spec^s(M)$. By Proposition 2.5 (a), there exists $S \in Spec^s(M)$ such that $V^s(N) = V^s(S)$. Now if $V^s(S)$ is not component, then $V^s(S)$ is contained properly in some component $V^s(S')$ for some $S' \in Spec^s(M)$ by [13, p. 95, Proposition 5]. It follows that $Ann_R(S) \supsetneq Ann_R(S')$ and hence $dim(Spec^s(M)) \geq 1$, a contradiction.

(b) Let $p \in V(Ann_R(M))$ and let S be a p-second submodule of M. Since $dim(Spec^s(M)) = 0$, we have $dim(\overline{R}) = 0$. Hence $Spec(\overline{R}) = Max(\overline{R})$. Now if $S' \in Spec^s(M)$, then $Ann_R(S')$ is also a maximal ideal. Therefore, we have $S' \in V^s(S) \Leftrightarrow Ann_R(S') = Ann_R(S) = p \Leftrightarrow S' \in Spec^s_p(M)$. Thus $V^s(S) = Spec^s_p(M)$.

(c) Since $Spec^s(M)$ is a Noetherian space with $dim(Spec^s(M)) = 0$, we have $dim(\overline{R}) = 0$ and $\overline{R}$ has a Noetherian spectrum. Now $Spec(\overline{R}) = Max(\overline{R})$ and $Spec(\overline{R})$ has only finitely many elements $\overline{m_1}, \overline{m_2}, ..., \overline{m_k}$, each of which is both maximal and minimal prime ideal of $\overline{R}$ by [21, p. 41, Examples 1.4 (c) and (d)]. Then m_i is a minimal prime divisor of $Ann_R(M)$, which is also a maximal ideal of R for every i $(1 \leq i \leq k)$. Thus $\{m_1, m_2, ..., m_k\}$ is the set of all minimal prime divisor of $Ann_R(M)$. Since M is a secondful R-module, $(0:_M m_i) \neq (0)$ and $Ann_R((0:_M m_i)) = m_i$ is a maximal ideal of R. Thus $(0:_M m_i)$ is an m_i-second submodule and $V^s((0:_M m_i))$ is an irreducible component of $Spec^s(M)$ for every i by part (a). Applying Proposition 2.5 (c), we can conclude that $\{V^s((0:_M m_1)), V^s((0:_M m_2)), ..., V^s((0:_M m_k))\}$ is the set of all irreducible components of $Spec^s(M)$.

Theorem 2.9. *Let M be a non-zero secondful R-module. Then the following are equivalent:*

(a) $Spec^s(M) = Min(M)$ (here $Min(M)$ denotes the set of all minimal submodules of M);

(b) $(Spec^s(M), \tau^s)$ is a T_1-space;

(c) $(Spec^s(M), \tau^s)$ is a T_4-space;

(d) $(Spec^s(M), \tau^s)$ is a T_0-space and $Cosupp(M) \approx Spec(\overline{R}) = Max(\overline{R})$;

(e) $Spec^s(M) = \{(0:_M p) \mid p \in V(Ann_R(M)) \cap Max(R)\}$.

Proof.

$(a) \Rightarrow (b)$. Let $\{S\}$ be a singleton subset of $Spec^s(M)$. Let $S' \in V^s(S)$ with $Ann_R(S') = q$. Then by [30, Proposition 1.6], $q \in Max(R)$ and $(0:_M q) \neq (0)$, so

$Ann_R(0:_M q) = q$. By [30, Proposition 1.4], it follows that $(0:_M q) \in Spec^s(M)$ and hence $(0:_M q) \in Min(M)$. But $Ann_R(S) = Ann_R(S') = q$ implies that $S \subseteq (0:_M q)$ and $S' \subseteq (0:_M q)$. Hence $S = S' = (0:_M q)$. It turns out that $\{S\} = V^s(S)$ is a closed subset, as required.

$(b) \Rightarrow (c)$. Let $Spec^s(M)$ be a T_1-space. Then $Spec^s(M)$ is homeomorphic to $Spec(\overline{R})$ by [7, Theorem 6.3]. Thus $Spec(\overline{R})$ is a T_1-space. Now by [10, p.44, Exer. 11], $Spec(\overline{R})$ is a T_1-space if and only if it is a Hausdorff topological space (i.e. T_2-space). Hence by the above arguments, $Spec^s(M)$ is a Hausdorff topological space. On the other hand, by [27, chap. 3, Exer. 3.5], every Hausdorff quasi-compact topological space is a T_4-space. Therefore, the result follows from the fact that $(Spec^s(M), \tau^s)$ is a compact topological space [7, Theorem 4.4].

$(c) \Rightarrow (d)$. Since $(Spec^s(M), \tau^s)$ is a T_0-space, it is homeomorphic to $Spec(\overline{R})$ by [7, Theorem 6.3]. Thus $Spec(\overline{R})$ is T_4-space. This implies that $Spec(\overline{R}) = Max(\overline{R})$ by [10, p. 44, Exer. 11], but $Spec(\overline{R})$ is homeomorphic to $V(Ann_R(M))$ by [10, p. 13, Exer. 21]. On the other hand, $V(Ann_R(M)) = Cosupp(M)$ by [17, Theorem 2.5]. It follows that $Cosupp(M) \approx Spec(\overline{R}) = Max(\overline{R})$, as desired.

$(d) \Rightarrow (e)$. Set $T = \{(0:_M p) \mid p \in V(Ann_R(M)) \cap Max(R)\}$. Let $p \in V(Ann_R(M)) \cap Max(R)$. Since M is secondful, $(0:_M p) \neq (0)$. Thus $(0:_M p) \in Spec^s(M)$ by [30, Proposition 1.4]. Therefore, $T \subseteq Spec^s(M)$. Conversely, let $S \in Spec^s(M)$. Then $\overline{Ann_R(S)} \in Spec(\overline{R})$. This implies that $Ann_R(S) \in V(Ann_R(M)) \cap Max(R)$. Set $p = Ann_R(S)$. Then $S \subseteq (0:_M p)$, so $Ann_R(0:_M p) = Ann_R(S) = p \in Max(R)$. Thus $(0:_M p) \in Spec^s(M)$. Now since $(Spec^s(M), \tau^s)$ is a T_0-space, the natural map of $Spec^s(M)$ is injective by [7, Theorem 6.3]. It follows that $S = (0:_M p)$ as required.

$(e) \Rightarrow (a)$. Let $S \in Spec^s(M)$. Then there is $p \in V(Ann_R(M)) \cap Max(R)$ such that $(0:_M p) = S$. Now let N be a non-zero submodule of M with $N \subseteq S$. Then $Ann_R(S) = Ann_R(N) = p \in Max(R)$. Hence we have $N \in Spec^s_p(M)$ by [30, Proposition 1.4], so $N = (0:_M p) = S$. It follows that S is a minimal submodule of M and the proof is completed.

Corollary 2.10. Let M be a non-zero secondful R-module. Then $Spec^s(M) = Min(M)$ if and only if $Spec^s(M)$ is a singleton set provided that $\overline{R}$ contains no idempotent other than $\overline{0}$ and $\overline{1}$; $\overline{R}$ is such a ring if R is a quasi-local ring or $Ann_R(M) \in Spec(R)$. (We recall that R is a quasi-local ring if $|Max(R)| = 1$.)

Proof. Let $Spec^s(M)$ be a singleton set. Then $Spec^s(M)$ is a T_1-space. Therefore, $Spec^s(M) = Min(M)$ by Theorem 2.9. Conversely if $Spec^s(M) = Min(M)$, then we have $Spec(\overline{R}) = Max(\overline{R})$ by Theorem 2.9. But by [10, p. 44, Exer. 11], $Spec(\overline{R})$ is totally disconnected topological space. Thus $Spec^s(M) \approx Spec(\overline{R})$ is a totally disconnected space. On the other hand, $Spec^s(M)$ is a connected space by [7, Corollary 3.13]. Thus $Spec^s(M)$ is a single set and the proof is completed.

3. On the second classical Zariski topology

Throughout this section, $Spec^s(M)$ is equipped with classical Zariski topology. We recall that if M is a cotop module, then its related topology (i.e. quasi-Zariski topology) coincide with classical Zariski topology.

A spectral space is a topological space homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology.

Spectral spaces have been characterized by M. Hochster as quasi-compact T_0-space having a quasi-compact open base closed under finite intersections and each irreducible closed subset has a generic point (see [19]).

Lemma 3.1. Let M be an R-module and Y be a nonempty subset of $Spec^s_R(M)$. Then $cl(Y) = cl(\bigcup_{S\in Y} V^{s*}(S))$. In particular, when Y is closed we have $Y = \bigcup_{S\in Y} V^{s*}(S)$.

Proof. This is straightforward.

Theorem 3.2. *For any R-module M, every irreducible closed subset of $Spec^s_R(M)$ has a generic point. In particular, this is true when M is a cotop module.*

Proof. Let Y be an irreducible closed subset of $Spec^s(M)$ and $\sum_{S\in Y} S = S_1$. Then by [8, Theorem 3.5 (a)], S_1 is a second submodule of M. We claim that $Y = V^{s*}(S_1)$. By Lemma 3.1, it is enough to show that $\bigcup_{S\in Y} V^{s*}(S) = V^{s*}(S_1)$. Clearly, $\bigcup_{S\in Y} V^{s*}(S) \subseteq V^{s*}(S_1)$. To see the reverse inclusion, let F be a closed subset of $Spec^s(M)$ containing $\bigcup_{S\in Y} V^{s*}(S)$. Since F is closed, $F = \bigcap_{i\in\Lambda} \bigcup_{j=1}^{n_i} V^{s*}(N_{i,j})$ for some submodules $N_{i,j}$ of M. Without loss of generality, We can assume $F = V^{s*}(N_1) \cup V^{s*}(N_2)$, where N_1 and N_2 are submodules of M. Now by Lemma 3.1, we have $\bigcup_{S\in Y} V^{s*}(S) = Y$ is irreducible. Since $\bigcup_{S\in Y} V^{s*}(S) \subseteq V^{s*}(N_1) \cup V^{s*}(N_2)$, we have $\bigcup_{S\in Y} V^{s*}(S) \subseteq V^{s*}(N_1)$ or $\bigcup_{S\in Y} V^{s*}(S) \subseteq V^{s*}(N_2)$. It follows that $S_1 = \sum_{S\in Y} S \subseteq N_1$ or $S_1 = \sum_{S\in Y} S \subseteq N_2$. Thus $V^{s*}(S_1) \subseteq N_1$ or $V^{s*}(S_1) \subseteq N_2$. Therefore $V^{s*}(S_1) \subseteq F$. This in turn implies that $V^{s*}(S_1) = \bigcup_{S\in Y} V^{s*}(S)$. Therefore $Y = V^{s*}(S_1) = \bigcup_{S\in Y} V^{s*}(S)$.

Corollary 3.3. Let M be an R-module and suppose $Spec^s(M)$ is a Noetherian space. Then $Spec^s(M)$ is a spectral space. In particular, this is true when M is a cotop module.

Proof. Since $Spec^s(M)$ is Noetherian, it is quasi-compact and the quasi-compact open subsets of $Spec^s(M)$ are closed under finite intersection and form an open base by [10, p. 79, Exer. 6]. Also $Spec^s(M)$ is a T_0 space by [8, Lemma 3.7 (a)]. Now the result follows from Theorem 3.2 and Hochster's characterizations.

Let N be a non-zero submodule of an R-module M and let S be a second submodule of M such that $S \leq N$. S is said to be a *maximal second* submodule of N if there doesn't exist $S' \in Spec^s(M)$ with $S \lneqq S' \lneqq N$ (see [4]).

Definition 3.4. Let N be a non-zero submodule of an R-module M and let S be a second submodule of M. We say that S is a *second summand* (resp. *maximal second summand*) of N if $S \in V^{s*}(N)$ (resp. $S \in Max(V^{s*}(N))$). If $V^{s*}(N) \neq \emptyset$, then by using Zorn's lemma, one can see that N contains a maximal second summand.

The *second submodule dimension* of an R-module M, denoted by $S.dimM$, is defined to be the supremum of the length of chains of second submodules of M if $Spec^s(M) \neq \emptyset$ and -1 otherwise (see [6]).

Theorem 3.5. *Let M be an R-module.*

(a) Let Y be a closed subset of $Spec^s(M)$. Then Z is an irreducible component of Y if and only if $Z = V^{s*}(S)$ for some maximal element S of Y.

(b) If every closed subset of $Spec^s(M)$ has a finite number of irreducible components, then every submodule of M has a finite number of maximal second summands. However, the converse is not true in general.

(c) $dim(Spec^s(M)) = S.dimM$.

Proof.

(a) Let Y be a closed subset of $Spec^s(M)$ and let Z be an irreducible component of Y. Since every irreducible component is closed, Z is closed in Y. But every irreducible closed subset of Y is an irreducible closed subset of $Spec^s(M)$. Therefore $Z = V^{s*}(S_1)$ for some second submodule S_1 of M by Theorem 3.2. Since $Z = V^{s*}(S_1) \subseteq Y$ is an irreducible component of Y, it follows that S_1 is a maximal element in Y (note that if $S_1, S_2 \in Spec^s(M)$, then $S_1 \leq S_2 \Leftrightarrow V^{s*}(S_1) \subseteq V^{s*}(S_2)$ by [8, Corollary 3.2 (b)]). Conversely, suppose S is a maximal element of Y. Then $V^{s*}(S)$ is an irreducible closed subset of $Spec^s(M)$. Since $S \in Y$, $V^{s*}(S) \subseteq cl(Y) = Y$ by Lemma 3.1. Now let $V^{s*}(S) \subseteq T$, where T is an irreducible subset of Y. This implies that $cl(T)$ be an irreducible closed subset of $Spec^s(M)$ and so $(T) = V^{s*}(S_1)$ for some second submodule of M by Theorem 3.2. It follows that $S \leq S_1$ so that $S = S_1$. Hence $T = V^{s*}(S)$ is a maximal irreducible subset of Y.

(b) Let N be a non-zero submodule of M and let S be a maximal second summand of N. Then $V^{s*}(S)$ is an irreducible component of $V^{s*}(N)$ by part (a). Hence every submodule of M has a finite number of maximal second submodules by hypothesis. To see the second assertion, set $M = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots$ and regard M as $\mathbb{Q}$-module. The second submodules of a vector space are just the non-zero submodules, so every submodule of M has a finite number of maximal second summands. Now let $S_1 = (0) \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots, S_2 = \mathbb{Q} \oplus (0) \oplus \mathbb{Q} \oplus \cdots, S_3 = \mathbb{Q} \oplus \mathbb{Q} \oplus (0) \oplus \mathbb{Q} \oplus \cdots, \cdots$, $S'_1 = \mathbb{Q} \oplus (0) \oplus (0) \oplus \cdots, S'_2 = (0) \oplus \mathbb{Q} \oplus (0) \oplus \cdots, S'_3 = (0) \oplus (0) \oplus \mathbb{Q} \oplus (0) \oplus \cdots, \cdots$, and $Y = \bigcap_{i \in \mathbb{N}} (V^{s*}(S_i) \cup V^{s*}(S'_i))$. One can see that S'_1, S'_2, $\cdots$ are maximal elements of Y. Hence Y is a closed subset of $Spec^s(M)$ with infinitely many irreducible components by part (a).

(c) Let $Z_0 \subsetneq Z_1 \subsetneq ... \subsetneq Z_t$ be a strictly increasing chain of irreducible closed subsets Z_i of $Spec^s(M)$ of length t. By Theorem 3.2, for each i, $0 \leq i \leq t$, we have $Z_i = V^{s*}(S_i)$ for some $S_i \in Spec^s(M)$. On the other hand $V^{s*}(S_i) \subsetneq V^{s*}(S_j)$ if and only if $S_i \subsetneq S_j$.

Hence $S_0 \supsetneq S_1 \supsetneq ... \supsetneq S_t$, is an strictly decreasing chain of second submodules of M of length t. Conversely, for every strictly decreasing chain $S_0 \supsetneq S_1 \supsetneq ... \supsetneq S_t$ of second submodules of M of length t, $V^{s*}(S_0) \subsetneq V^{s*}(S_1) \subsetneq ... \subsetneq V^{s*}(S_t)$ is a strictly increasing chain of irreducible closed subsets of $Spec^s(M)$ of length t. This in turn implies that $dim(Spec^s(M)) = S.dimM$ and the proof is completed.

A proper submodule N of an R-module M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$ (see [18]).

Let p be a prime ideal of R and let N be a submodule of an R-module M. Then $N^{ec} = \{m \in M : cm \in N$ for some $c \in R\backslash p\}$ and it is called the p-closure of N and denoted by $cl_p(N)$ (see [24, p. 92]). The dual of this notion, i.e., p-interior of N relative to M is defined as the set $I_p^M(N) := \bigcap\{L| \, L$ *is a completely irreducible submodule of* M *and* $rN \subseteq L$ *for some* $r \in R\backslash p\}$ (see [4]). It is easy to see that $I_p^M(N) = \bigcap_{r\in R\backslash p} rN$.

Let R be an integral domain. A submodule N of an R-module M is said to be *cotorsion-free* (resp. *cotorsion*) if $I_0^M(N) = N$ (resp. $I_0^M(N) = (0)$) (see [5]).

We need the following lemma.

Lemma 3.6. Let p and q be prime ideals of R with $q \subseteq p$. Let M be an R-module with $cl_p(0) = (0)$ and let $\phi : Hom_R(R_p, M) \to M$ be the natural homomorphism given by $f \mapsto f(1/1)$. Then we have the following.

(i) If S is an R_p-submodule of $Hom_R(R_p, M)$, then we have $S^{ec} = S = Hom_R(R_p, L)$, where $L = S^e$. (Here T^e, where $T \subseteq Hom_R(R_p, M)$, and N^c, where $N \subseteq M$, denote $\phi(T)$ and $\phi^{-1}(N)$, respectively.)

(ii) If M is an Artinian R-module and K is a q-second submodule of M, then $Hom_R(R_p, K)$ is a qR_p-second submodule of $Hom_R(R_p, M)$ and $K^c = Hom_R(R_p, K)$. Further we have $(Hom_R(R_p, K))^e = \phi(Hom_R(R_p, K)) = I_p(K) = K$ and $K^{ce} = K$.

Proof.

(i) Clearly, $S \subseteq S^{ec}$. So let $g \in S^{ec}$. Then there exists $f \in S$ such that $g(1/1) = f(1/1) \in S^e$ and so $f(r/1) = g(r/1)$ for every $r \in R$. Now let $\lambda \in R_p$, then $\lambda = r/s$ for some $r \in R$ and $s \in R - p$. Then we have $s(f(\lambda) - g(\lambda)) = 0$. Since $cl_p(0) = 0$, $f(\lambda) = g(\lambda)$, so $g \in S$. Now $Hom_R(R_p, L) = S^{ec}$, where $L = S^e$, follows directly from the above arguments.

(ii) Let K be a q-second submodule of M. Then by [25, Theorem 3.1 (2)], $\phi : Hom_R(R_p, K) \to K$ given by $f \mapsto f(1/1)$ is a surjective homomorphism, so $Hom_R(R_p, K) \neq (0)$. To see $Hom_R(R_p, K)$ is a qR_p-second, let $\frac{r}{s} \in R_P\backslash qR_p$ and so $r \in R\backslash q$. Thus $rK = K$ because K is q-second. Since $cl_p(0) = (0)$ and K is an Artinian R-module, we see that $Hom_R(R_p, rK) = rHom_R(R_p, K)$ by [25, Proposition 2.4]. This implies that
$$\frac{r}{s}(Hom_R(R_p, K)) = \frac{1}{s}(Hom_R(R_p, rK)) = Hom_R(R_p, K).$$

If $\frac{r}{s} \in qR_p$, then $\frac{r}{s}(Hom_R(R_p, K)) = \frac{1}{s}(Hom_R(R_p, rK)) = (0)$. Further $cl_p(0) = (0)$ implies that $K^c = Hom_R(R_p, K)$. The last assertion follows from [7, Lemma 2.15].

Proposition 3.7. Let p be a prime ideal of R and let M be a cotop R- module with $cl_p(0) = 0$. Then $Hom_R(R_p, M)$ is a cotop R_p-module.

Proof. Let S be a second submodule of the R_p-module $Hom_R(R_p, M)$. Then we have $S^e = \phi(S) \neq (0)$. Clearly, S^e is a second submodule of M. Now let S_1 and S_2 be socle submodules of R_p-module $Hom_R(R_p, M)$ with $S \subseteq S_1 + S_2$. Then S_1^e and S_2^e are socle submodules of M with $S^e \subseteq (S_1 + S_2)^e = S_1^e + S_2^e$. Since M is a cotop R-module, we have $S^e \subseteq S_1^e$ or $S^e \subseteq S_2^e$. Thus we have $S = S^{ec} \subseteq S_1^{ec} = S_1$ or $S = S^{ec} \subseteq S_2^{ec} = S_2$ by Lemma 3.6 (i). This implies that $Hom_R(R_p, M)$ is a cotop R_p-module.

Example 3.8. Let R be an integral domain and let Q be the field of fractions of R. Then clearly, Q is a torsion-free cotop R-module and hence for every prime ideal p of R, $Hom_R(R_p, Q)$ is a cotop R_p-module by Proposition 3.7. Note that since Q is a torsion-free R-module, $cl_p(0) = (0)$ for every prime ideal p of R.

Let M be an R-module. The class of X-injective modules, where X is the set of all prime submodules of M, was studied by H. Ansari-Toroghy and R. Ovlyaee-Sarmazdeh in [9]. M is X-injective if the map $\phi : X \to \bar{R}$ given by $P \mapsto \overline{(P :_R M)} = (P :_R M)/Ann_R(M)$ is injective.

Definition 3.9. Let M be an R-module. We say that M is an X^s*-injective* if the natural map of X^s is injective. Equivalently, M is X^s-injective if and only if $Ann_R(S_1) = Ann_R(S_2)$, $S_1, S_2 \in X^s$, implies that $S_1 = S_2$ if and only if for every $p \in Spec(R)$, $|Spec_p^s(M)| \leq 1$.

Let M be an R-module. M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ (see [3]).

Example 3.10. Let p be a prime integer.

(a) Every comultiplication module is X^s-injective. In particular $\mathbb{Z}_{p^\infty} = E(\mathbb{Z}/p\mathbb{Z})$ is an X^s-injective $\mathbb{Z}$-module. Note that $Spec^s(\mathbb{Z}_{p^\infty}) = \{\langle 1/p + \mathbb{Z}\rangle, \mathbb{Z}_{p^\infty}\}$.

(b) Let $M = \mathbb{Q} \oplus \mathbb{Z}_p$. Then M is X^s-injective $\mathbb{Z}$-module. Note that $Spec^s(M) = \{\mathbb{Q} \oplus (0), (0) \oplus \mathbb{Z}_p\}$.

(c) Every submodule of an X^s-injective R-module is an X^s-injective module.

The following example shows that not every homomorphic image of an X^s-injective R-module is X^s-injective.

Example 3.11. Let p be a positive prime integer and let $M = \mathbb{Q} \oplus \mathbb{Z}_p$. Then M is an X^s-injective $\mathbb{Z}$-module but its homomorphic image $\mathbb{Z}_p^\infty \oplus \mathbb{Z}_p$ is not. note that

$$Spec^s(\mathbb{Z}_p^\infty \oplus \mathbb{Z}_p) = \{\mathbb{Z}_p^\infty \oplus (0), (0) \oplus \mathbb{Z}_p, < 1/p + \mathbb{Z} >\}.$$

Remark 3.12. Let S be a commutative ring with identity. S is said to be a *perfect ring* if it satisfies DCC on principal ideals. Clearly, every Artinian ring is perfect. Note that if S is a perfect ring and $p \in Spec(S)$, then by [11, Lemma 2.2], R/p is a perfect domain so that it is a field. Hence $dim(S) = 0$. Furthermore, every perfect ring is a semilocal ring by [11, Theorem P or p. 475, Examples (6)].

Proposition 3.13. (i) If M is a cotop module over a semilocal (for example a perfect) ring R. Then $(0 :_M Jac(R))$ is cyclic.

(ii) Let $(M_i)_{i\in I}$ be a family of R-modules and let $M = \bigoplus_{i\in I} M_i$. If M is an X^s-injective module, then

$$Spec^s(M) = \left\{ S \oplus \left(\bigoplus_{j\neq i\in I} (0) \right) \mid j \in I,\ S \in Spec^s(M_j) \right\}.$$

Proof.

(i) Suppose that M is a cotop module. Let $m_1, m_2, ..., m_n$ denote the distinct maximal ideals of R, where n is a positive integer. By [7, Corollary 2.6 (e)], $(0 :_M m_i)$ is cyclic for each $1 \leq i \leq n$. We show that $(0 :_M Jac(R)) = \bigoplus_{i=1}^n (0 :_M m_i)$. Set $I = m_2 \cap m_3 \cap ... \cap m_n$, so that $Jac(R) = m_1 \cap I$. Note that $m_1 + I = R$ implies that $0 = (0 :_M R) = (0 :_M m_1 + I) = (0 :_M m_1) \cap (0 :_M I)$. Moreover,

$$\begin{aligned}(0 :_M m_1) + (0 :_M I) &= ((0 :_M m_1) + (0 :_M I) :_M m_1 + I) \\ &\supseteq ((0 :_M m_1) :_M I) \cap ((0 :_M I) :_M m_1) \\ &= (0 :_M m_1 I) = (0 :_M Jac(R)) \\ &\supseteq (0 :_M m_1) + (0 :_M I).\end{aligned}$$

Therefore $(0 :_M m_1) + (0 :_M I) = (0 :_M Jac(R))$. By induction, this implies that $(0 :_M Jac(R)) = \bigoplus_{i=1}^n (0 :_M m_i)$. Without loss of generality, we may assume that $(0 :_M m_i) \neq 0$. Thus we have

$$(0 :_M Jac(R)) = \bigoplus_{i=1}^{n} (0 :_M m_i) = \bigoplus_{i=1}^{n} R/m_i \cong R/Jac(R).$$

Hence $(0 :_M Jac(R))$ is cyclic.

(ii) The inclusion $\supseteq$ is trivial. Conversely, let S be a p-second submodule of M. Then there exists $j \in I$ such that $S \not\subseteq (0) \oplus (\bigoplus_{j\neq i\in I}(M_i))$. By [7, Lemma 2.2 (b)], $\frac{S+((0)\oplus(\bigoplus_{j\neq i\in I}(M_i)))}{(0)\oplus(\bigoplus_{j\neq i\in I}(M_i))} \in Spec^s_p(M_j)$. This implies that $S = S_j \oplus (\bigoplus_{j\neq i\in I}(0))$, where $S_j \in Spec^s(M_j)$.

Definition 3.14. A family $(M_i)_{i\in I}$ of R-modules is said to be *second-compatible* if for all $i \neq j$ in I, there doesn't exist a prime ideal p in R with $Spec^s_p(M_i)$ and $Spec^s_p(M_j)$ both nonempty.

Theorem 3.15. *Let $(M_i)_{i\in I}$ be a family of R-modules and let $M = \bigoplus_{i\in I} M_i$. Then M is an X^s-injective (resp. a cotop) R-module if and only if $(M_i)_{i\in I}$ is a family of second-compatible X^s-injective (resp. cotop) R-modules.*

Proof. We consider the proof for two cases X^s-injective and cotop modules.

Case(I). X^s-injective modules.
$(\Rightarrow)$. Since every submodule of an X^s-injective module is X^s-injective, M_i's are X^s-injective. Let $i \neq j$ and let $S \in Spec^s_p(M_i)$ and $K \in Spec^s_p(M_j)$. Then $S \oplus (\bigoplus_{i\neq k\in I}(0)), K \oplus (\bigoplus_{j\neq k\in I}(0)) \in Spec^s_p(M)$. Since M is X^s-injective, $S \oplus (\bigoplus_{i\neq j\in I}(0)) = K \oplus (\bigoplus_{j\neq i\in I}(0))$, a contradiction. Hence M_i's are second-compatible.
$(\Leftarrow)$. Let S be a second submodule of M and let $p = Ann_R(S)$. Since $S \neq 0$, it follows that there exists $j \in I$ with $S \nsubseteq (\bigoplus_{j\neq i\in I} M_i) \oplus (0)$ and so $\frac{S+((\bigoplus_{j\neq i\in I} M_i)\oplus(0))}{(\bigoplus_{j\neq i\in I} M_i)\oplus(0)} \in Spec^s_p(M_j)$. By hypothesis, $Spec^s_p(M_i)$ is empty for all $j \neq i \in I$ and hence $S \subseteq (\bigoplus_{i\neq k\in I} M_k) \oplus (0)$. This implies that

$$S \subseteq \bigcap_{j\neq i\in I}\left(\left(\bigoplus_{i\neq k\in I} M_k\right) \oplus (0)\right) = M_j \oplus \left(\bigoplus_{j\neq i\in I}(0)\right).$$

Thus there exist $S_j \in Spec^s_p(M_j)$ such that $S = S_j \oplus (\bigoplus_{j\neq i\in I}(0))$. Now let $S, K \in Spec^s_p(M)$. By the above arguments, there exist $S_i \in Spec^s_p(M_i)$ and $S_j \in Spec^s_p(M_j)$ such that $S = S_i \oplus (\bigoplus_{i\neq j\in I}(0))$ and $K = S_j \oplus (\bigoplus_{j\neq i\in I}(0))$. This implies that $i = j$ because M_i's are second compatible. Hence $K = S$, i.e., M is X^s-injective.

Case(II). Cotop modules.
$(\Rightarrow)$. Since every submodule of a cotop module is cotop, M_i's are cotop. Let $i \neq j$ and let $S \in Spec^s_p(M_i)$ and $K \in Spec^s_p(M_j)$. Set $S_1 = S \oplus (\bigoplus_{i\neq k\in I}(0))$ and $K_1 = K \oplus (\bigoplus_{j\neq k\in I}(0))$. Thus $S_1 + K_1 \in Spec^s_p(M)$. Since M is cotop, $K_1 \subseteq S_1$ or $S_1 \subseteq K_1$, which is a contradiction. Therefore M_i's are second-compatible.
$(\Leftarrow)$. Let S be p-second and let S_1 and S_2 be socle submodules of M such that $S \subseteq S_1 + S_2$. Then by similar arguments in case(I), for each $i \in I$, there exist submodules S_{1i} and S_{2i} of M_i, such that $S_k = \bigoplus_{i\in I} S_{ki}$ $(k = 1, 2)$ and each submodule S_{ki} is either socle submodule or equals (0). Also there exists a second submodule S_j of M_j such that $S = S_j \oplus (\bigoplus_{j\neq i\in I}(0))$. Therefore $S_j \subseteq S_{1j} + S_{2j}$. Since M_j is cotop, $S_j \subseteq S_{1j}$ or $S_j \subseteq S_{2j}$. It follows that $S \subseteq S_1$ or $S \subseteq S_2$ and hence M is a cotop module.

An R-module M is said to have the *double annihilator conditions* if for each ideal I of R, we have $I = Ann_R(0 :_M I)$.

Corollary 3.16. (a) Let R be a domain with field of fractions Q and suppose M is an R-module such that $M/I_0(M)$ is finitely cogenerated. Then the R-module $Q \oplus M$ is a cotop (resp. an X^s-injective) module if and only if M is a cotorsion cotop (resp. X^s-injective) module.

(b) Let (R, m) be a local ring, I_λ $(\lambda \in \Lambda)$ a family of ideals of R, and $(0 :_M m) \neq 0$. If $M = \bigoplus_{\lambda\in\Lambda}(0 :_M I_\lambda)$ is a cotop (resp. an X^s-injective) R-module, then the ideals I_λ $(\lambda \in \Lambda)$ are comaximal.

(c) Let M be a weak comultiplication module and let $(I_\lambda)_{\lambda\in\Lambda}$ be a family of ideals of R. If $M = \bigoplus_{\lambda\in\Lambda}(0 :_M I_\lambda)$ and the ideals I_λ $(\lambda \in \Lambda)$ are comaximal, then M is a cotop (resp. an X^s-injective) R-module.

Proof. We just consider the proof for cotop modules. We have similar arguments when M is an X^s-injective module.

(a) $(\Leftarrow)$. We can see that $Spec^s(Q) = Spec^s_0(Q) = \{Q\}$. We show that R-module M and R-module Q are second-compatible. Let $S \in Spec^s_0(M)$. By [5, Theorem 2.10], $I^M_0(S) = S$. Since $I^M_0(S) \subseteq I^M_0(M)$ and M is cotorsion, we have $S = (0)$, a contradiction. Therefore $Q \oplus M$ is cotop by Theorem 3.15.
$(\Rightarrow)$. By Theorem 3.15, Q and M are cotop modules and second-compatible. If M is not cotorsion, then $I^M_0(M)$ belongs to $Spec^s_0(M)$ by [4, Corollary 2.10] which is a contradiction. Thus M is cotorsion.

(b) Let M be a cotop R-module, and $\lambda, \lambda' \in \Lambda$. If $I_\lambda + I_{\lambda'} \neq R$, then $I_\lambda + I_{\lambda'} \subseteq m$. Therefore $(0 :_M m) \subseteq (0 :_M I_\lambda + I_{\lambda'}) = (0 :_M I_\lambda) \cap (0 :_M I_{\lambda'})$. But $(0 :_M m)$ is an m-second submodule of M. It follows that $Spec^s_m((0 :_M I_\lambda))$ and $Spec^s_m((0 :_M I_{\lambda'}))$ are both nonempty sets which is a contradiction by Theorem 3.15.

(c) We show that for every $\lambda \in \Lambda$, $(0 :_M I_\lambda)$ is a cotop R-module. Let $S \in Spec^s_p((0 :_M I_\lambda))$ and S_1, S_2 are socle submodules of $(0 :_M I_\lambda)$ such that $S \subseteq S_1 + S_2$. Thus $Ann_R(S_1+S_2) = Ann_R(S_1)\cap Ann_R(S_2) \subseteq Ann_R(S) \in Spec(R)$. Therefore $Ann_R(S_1) \subseteq p$ or $Ann_R(S_2) \subseteq p$. Since M is weak comultiplication, $S = (0 :_M p) \subseteq (0 :_M Ann_R(S_1)) = S_1$ or $S = (0 :_M p) \subseteq (0 :_M Ann_R(S_1)) = S_2$. We show that $(0 :_M I_\lambda)$'s are second-compatible. Let $\lambda_1, \lambda_2 \in \Lambda$ and $q \in Spec(R)$ and let $S_1 \in Spec^s_q((0 :_M I_{\lambda_1}))$ and $S_2 \in Spec^s_q((0 :_M I_{\lambda_2}))$. Since $S_i \subseteq (0 :_M I_{\lambda_i})$, we have $I_{\lambda_i} \subseteq Ann_R((0 :_M I_{\lambda_i})) \subseteq Ann_R(S_i) = q$ $(i = 1, 2)$. This implies that $R = I_{\lambda_1} + I_{\lambda_2} \subseteq q$, a contradiction by hypothesis.

Theorem 3.17. *Let M be a non-zero X^s-injective Artinian R-module.*

(a) $Spec^s(M) = \{I^M_p((0 :_M p)) \mid p \in V(Ann_R(M)), I^M_p(0 :_M p) \neq (0)\}$,

$$Min(M) = \{(0 :_M p) \mid p \in Max(R), (0 :_M p) \neq (0)\}.$$

(b) If M is secondful, then

$$Spec^s(M) = \{I^M_p((0 :_M p)) \mid p \in V(Ann_R(M))\},$$

$$Min(M) = \{(0 :_M p) \mid p \in V(Ann_R(M)) \cap Max(R)\}.$$

(c) If R is PID and M is faithful secondful, then

$$Spec^s(M) = Min(M) \cup \{I_0^M(M)\}, \text{ where}$$

$$Min(M) = \{(0 :_M p) \mid p \in Max(R)\}.$$

Proof.

(a) Put $T = \{I_p^M((0 :_M p)) \mid p \in V(Ann_R(M)), I_p^M((0 :_M p)) \neq (0)\}$. Then $T \subseteq Spec^s(M)$ by [4, Lemma 2.9]. Let $S \in Spec^s(M)$. Then $Spec^s(M) \neq \emptyset$ for $p = Ann_R(S)$. Thus $I_p^M((0 :_M p)) \in Spec_p^s(M)$ by [4, Lemma 2.9] as $I_p^M((0 :_M p)) \neq (0)$. Since M is X^s-injective, $S = I_p^M((0 :_M p))$. This implies that $S \in T$. To prove the second assertion, put $\Omega = \{(0 :_M p) \mid p \in Max(R), (0 :_M p) \neq (0)\}$ and let $(0 :_M p) \in \Omega$. Then $(0 :_M p) \in Spec^s(M)$ by [30, Proposition 1.4]. Moreover, $(0 :_M p) \in Min(M)$ because if $K \subseteq (0 :_M p)$ for some non-zero submodule K of M, then $Ann_R((0 :_M p)) = Ann_R(K) = p$ and hence $(0 :_M p) = K$ because M is X^s-injective. Thus $\Omega \subseteq Min(M)$. Conversely, let $S \in Min(M)$. Then $S \in Spec^s(M)$ and we have $p = Ann_R(S) \in Max(R)$. Whence $p = Ann_R(S) = Ann_R((0 :_M p))$. It follows that $S = (0 :_M p) \neq (0)$. Thus $S \in \Omega$, and we can conclude that $Min(M) = \Omega$.

(b) Let $p \in V(Ann_R(M))$. Since M is secondful, there exists $S \in Spec^s(M)$ such that $Ann_R(S) = p$. Thus $I_p^M((0 :_M p)) \neq (0)$ by [4, Corollary 2.10]. Now the proof follows from part (a).

(c) Use part (b), the fact that $Ann_R(M) = (0)$, and that if M is secondful, then for every maximal ideal p of R, $I_p^M((0 :_M p)) = (0 :_M p)$.

Let M be an R-module. M said to be a *weak comultiplication module* if M does not have any second submodule or for every second submodule S of M, $S = (0 :_M I)$ for some ideal I of R (see [5]).

Assume that $V^s(sec(N)) = V^{s*}(sec(N))$ for every $N \leq M$. Then clearly, $\tau_M^{s*} \subseteq \tau_M^s$ and hence M is a cotop module.

Theorem 3.18. *(a) Let R be a perfect ring. Then the three classes of cotop, weak comultiplication, and X^s-injective modules are all equal.*

(b) Let R be a one-dimensional integral domain and let M be an Artinian R-module. Then M is cotorsion or cotorsion-free X^s-injective if and only if M is weak comultiplication.

(c) Let M be a secondful X^s-injective R-module. If $V^s(sec(N)) = V^{s*}(sec(N))$ for every submodule N of M, then $(Spec^s(M), \tau^{s*})$ is homeomorphic to $Spec(\overline{R})$. Therefore, $(Spec^s(M), \tau^{s*})$ is a spectral space.

(d) *Let M be an X^s-injective Artinian R-module. If for any $p \in V(Ann_R(M))$ and every family $\{p_i\}_{i\in I}$, where $p_i \in V(Ann_R(M))$, $\bigcap_{i\in I} p_i \subseteq p$ implies that $I_p^M((0 :_M p)) \subseteq \sum_{i\in I} I_{p_i}^M((0 :_M p_i))$, then we have $V^s(sec(N)) = V^{s*}(sec(N))$.*

Proof.

(a) First we assume that M is a cotop R-module and show that it is weak comultiplication. To see this, let $S \in Spec^s(M)$. Clearly, $S \subseteq (0 :_M Ann_R(S))$. Since R is perfect, $(0 :_M Ann_R(S))$ a simple $R/Ann_R(S)$-module by [7, Corollary 2.6 (a) and (d)] and hence a simple R-module. This implies that $S = (0 :_M Ann_R(S))$, as desired. Also, it is clear every weak comultiplication module is an X^s-injective module. To complete the proof, we assume that M is an X^s-injective module and show that M is a cotop module. To see this, let S be a second submodule and let N, L are socle submodules of M such that $S \subseteq N + L$. Put $N = \sum_{\alpha\in I} N_\alpha$ and $L = \sum_{\beta\in J} L_\beta$, where N_α and L_β are second submodules of M for each $\alpha \in I$ and $\beta \in J$. We have $Ann_R(N + L) = Ann_R(N) \cap Ann_R(L) \subseteq Ann_R(S)$ and so $Ann_R(N) \subseteq Ann_R(S)$ or $Ann_R(L) \subseteq Ann_R(S)$. Thus $\bigcap_{\alpha\in I} Ann_R(N_\alpha) \subseteq Ann_R(S)$ or $\bigcap_{\beta\in J} Ann_R(L_\beta) \subseteq Ann_R(S)$. Now since R is a perfect ring, both I and J are finite index sets. Consequently, there exists $\alpha_0 \in I$ or $\beta_0 \in J$ such that $Ann_R(N_{\alpha_0}) = Ann_R(S)$ or $Ann_R(L_{\beta_0}) = Ann_R(S)$. As M is an X^s-injective module, we have $N_{\alpha_0} = S$ or $L_{\beta_0} = S$. This implies that $S \subseteq N$ or $S \subseteq L$. Hence M is a cotop module.

(b) Let M be a cotorsion-free X^s-injective, and let S be a second submodule of M. Then we have $S = I_p^M(S) = I_p^M(0 :_M p)$ for $p = Ann_R(S)$ by [4, Corollary 2.10]. If $p = (0)$, then $S = I_0^M(S) = I_0^M(M) = M = (0 :_M 0)$. If $p \neq (0)$, then p is a maximal ideal so that $S = I_p^M(0 :_M p) = (0 :_M p)$ by [30, Proposition 1.4]. Now let M be a cotorsion X^s-injective R-module and S a p-second submodule of M. If $p = (0)$, then $S = I_0^M(S) = I_0^M(M) = (0)$, a contradiction. This implies that $p \neq (0)$ and hence $S = I_p^M(S) = I_p^M(0 :_M p) = (0 :_M p)$. Thus M is a weak comultiplication module. The reverse implication is clear.

(c) Let $\psi : Spec^s(M) \to Spec(\overline{R})$ be the natural map of $Spec^s(M)$. As M is a secondful X^s-injective R-module, ψ is a bijective map. Now let $\overline{I}$ be an ideal of $\overline{R}$. Then by [7, Lemma 3.3 (c) and Proposition 3.6], $(\psi)^{-1}(V(\overline{I})) = V^s(0 :_M I) = V^{s*}(0 :_M I)$, so ψ is continuous. Now let N be a submodule of M. Then we have

$$\psi(V^{s*}(N)) = \psi(V^{s*}(sec(N))) = \psi(V^s(sec(N))) = V(\overline{Ann_R(sec(N))}).$$

It follows that ψ is a closed map and the proof is completed.

(d) If $M = (0)$, there is nothing to prove. Hence we assume that $M \neq (0)$. Let N be a submodule of M. Clearly, $V^{s*}(sec(N)) \subseteq V^s(sec(N))$. So we assume that $S \in V^s(sec(N))$. Then by Theorem 3.17 (a), $S = I_p^M((0 :_M p))$, where $p = Ann_R(S) \in V(Ann_R(M))$. If set

$$I = \{q \in V(Ann_R(M)) \mid (0) \neq I_q^M((0 :_M q)) \subseteq N\},$$

then $sec(N) = \sum_{q\in I} I_q^M((0:_M q))$. It follows that

$$Ann_R(sec(N)) = Ann_R(\sum_{q\in I} I_q^M((0:_M q))) = \bigcap_{q\in I} q \subseteq p.$$

Now by using the assumption, we have

$$S = I_p^M((0:_M p)) \subseteq \sum_{q\in I} I_q^M((0:_M q)) = sec(N).$$

So $S \in V^{s*}(sec(N))$, as desired.

Example 3.19. Set $M = \bigoplus_{i=1}^n \mathbb{Z}_{p_i}$, where p_i are distinct positive prime integers. Then by using Proposition 3.13 (ii) and Theorem 3.15, we see that M is a secondful X^s-injective $\mathbb{Z}$- module and we have

$$Spec^s(M) = \{\mathbb{Z}_{p_j} \oplus (\bigoplus_{1\leq i\neq j\leq n} (0)) \mid 1 \leq j \leq n\}.$$

Moreover, $V^s(sec(N)) = V^{s*}(sec(N))$ for every submodule N of M. Hence $(Spec^s(M), \tau^{s*})$ is a spectral space by Theorem 3.18. This example shows that for each $n > 1$, when n is square free, $(Spec^s(\mathbb{Z}_n), \tau^{s*})$ is a spectral space.

Remark 3.20. (a) Let $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_q$, where p and q are distinct positive prime integers. Then M is an Artinian X^s-injective over the one dimensional integral domain $\mathbb{Z}$; but it is not a weak comultiplication $\mathbb{Z}$-module. This shows that the condition "M is cotorsion or cotorsion-free" in part (b) of Theorem 3.18 is a necessary condition and can not be omitted.

(b) Set $M = (\bigoplus_p \mathbb{Z}_p) \oplus \mathbb{Q}$, where p runs over all distinct positive prime integers. Then by using Proposition 3.13 (ii) and Theorem 3.15, M is a secondful X^s-injective $\mathbb{Z}$-module. Moreover, we see that $(Spec^s(M), \tau^{s*})$ is not a quasi-compact space and hence not a spectral space by Hochster's characterizations. This shows that the condition "$V^s(sec(N)) = V^{s*}(sec(N))$ for every submodule N of M" in Theorem 3.18 (d) is a necessary condition and can not be omitted.

We end this section with the following question.

Question 3.21. Is every cotop R-module an X^s-injective R-module?

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