Contra $\delta gb$-Continuous Functions in Topological Spaces

S.S. Benchalli$^{1,2}$, P.G. Patil$^{2,*}$, J.B. Toranagatti$^3$, S.R. Vighnesri$^4$

$^1,^2$ Department of Mathematics, Karnatak University, Dharwad, India
$^3$ Department of Mathematics, Karnatak University’s Karnatak College, Dharwad, India
$^4$ Department of Mathematics, R.L.S College, Dharwad, India

Abstract. In this paper the notion of $\delta gb$-open sets in topological spaces is applied to study a new class of functions called contra $\delta gb$-continuous functions as a new generalization of contra continuity and obtain their characterizations and properties.

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1. Introduction and Preliminaries


Throughout this paper, $(X, \tau), (Y, \sigma)$ and $(Z, \eta)$ (or simply $X, Y$ and $Z$ ) represent topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a space $X$, the closure of $A$, interior of $A$ and complement of $A$ are denoted by $\text{cl}(A)$, $\text{int}(A)$ and $A^c$ respectively.

Definition 1. A subset $A$ of a topological space $X$ is called a

(i) pre-closed [9] if $\text{cl}(\text{int}(A)) \subseteq A$

(ii) b-closed [2] if $\text{cl}(\text{int}(A)) \cap \text{int}(\text{cl}(A)) \subseteq A$

(iii) regular-closed [14] if $A = \text{cl}(\text{int}(A))$

(iv) $\delta$-closed [17] if $A = cl_\delta(A)$ where $cl_\delta(A) = \{x \in X: \text{int}(cl(U)) \cap A \neq \emptyset, U \in \tau \text{ and } x \in U\}$

*Corresponding author.

Email addresses: benchalliss@gmail.com (S.S. Benchalli), pgpatil01@gmail.com (P.G. Patil), jagadeeshbt2000@gmail.com (J.B. Toranagatti), vighneshirs@gmail.com (S.R. Vignesri)

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(v) delta generalized b-closed (briefly, δgb-closed) \([4]\) if \(\text{bcl}(A) \subseteq G\) whenever \(A \subseteq G\) and \(G\) is \(\delta\)-open in \(X\).

The complements of the above mentioned closed sets are their respective open sets.

The \(b\)-closure of a subset \(A\) of \(X\) is the intersection of all \(b\)-closed sets containing \(A\) and is denoted by \(\text{bcl}(A)\).

**Definition 2.** A function \(f:X \to Y\) from a topological space \(X\) into a topological space \(Y\) is called a,

(i) contra continuous \([6]\) if \(f^{-1}(G)\) is closed in \(X\) for every open set \(G\) of \(Y\).

(ii) contra \(b\)-continuous \([10]\) if \(f^{-1}(G)\) is \(b\)-closed in \(X\) for every open set \(G\) of \(Y\).

(iii) contra rgb-continuous \([13]\) if \(f^{-1}(G)\) is rgb-closed in \(X\) for every open set \(G\) of \(Y\).

(iv) \(\delta gb\)-continuous \([5]\) if \(f^{-1}(G)\) is \(\delta gb\)-open in \(X\) for every open set \(G\) of \(Y\).

(v) completely-continuous \([3]\) if \(f^{-1}(G)\) is regular-open in \(X\) for every open set \(G\) of \(Y\).

(vi) perfectly-continuous \([12]\) if \(f^{-1}(G)\) is clopen in \(X\) for every open set \(G\) of \(Y\).

(vii) \(\delta^*\)-continuous if \(f^{-1}(G)\) is \(\delta\)-open in \(X\) for every open set \(G\) of \(Y\).

(viii) contra gb-continuous \([1]\) if \(f^{-1}(G)\) is gb-closed in \(X\) for every open set \(G\) of \(Y\).

(ix) pre-closed \([7]\) if for every closed subset \(A\) of \(X\) \(f(A)\) is pre-closed in \(Y\).

**Definition 3.** \([5]\) A topological space \(X\) is said to be a,

(i) \(T_{\delta gb}\)-space if every \(\delta gb\)-closed subset of \(X\) is closed.

(ii) \(\delta gbT_{\frac{1}{2}}\)-space if every \(\delta gb\)-closed subset of \(X\) is \(b\)-closed.

### 2. Contra \(\delta gb\)-Continuous Functions.

**Definition 4.** A function \(f:X \to Y\) is called contra \(\delta gb\)-continuous if \(f^{-1}(V)\) is \(\delta gb\)-closed in \(X\) for each open set \(V\) of \(Y\).

Clearly, \(f:X \to Y\) is contra \(\delta gb\)-continuous if and only if \(f^{-1}(G)\) is \(\delta gb\)-open in \(X\) for every closed set \(G\) in \(Y\).

**Theorem 1.** If \(f:X \to Y\) is contra gb-continuous then it is contra \(\delta gb\)-continuous.

**Proof:** Follows from the fact that every gb-closed set is \(\delta gb\)-closed.

**Theorem 2.** If \(f:X \to Y\) is contra \(b\)-continuous then it is contra \(\delta gb\)-continuous.

**Proof:** Follows from the fact that every contra \(b\)-continuous function is contra gb-continuous and Theorem 1.

**Remark 1.** The converse of Theorem 1 and Theorem 2 need not be true as seen from the following example.
Example 1. Let $X=Y=\{a,b,c\}$. Let $\tau=\{X,\phi,\{a\}\}$ and $\sigma=\{X,\phi,\{a\},\{b\}\}$ be topologies on $X$ and $Y$ respectively. Then the identity function $f:X\to Y$ is contra $\delta gb$-continuous but neither contra $b$-continuous and nor contra $gb$-continuous, since $\{a\}$ is open in $Y$ but $f^{-1}(\{a\})=\{a\}$ is not $gb$-closed in $X$ and hence not $b$-closed in $X$.

Theorem 3. If $f:X\to Y$ is contra $\delta gb$-continuous then it is contra $rgb$-continuous.

Proof: Follows from the fact that every $\delta gb$-closed set is $rgb$-closed.

Remark 2. The converse of Theorem 3 need not be true as seen from the following example.

Example 2. Let $X=Y=\{a,b,c\}$. Let $\tau=\{X,\phi,\{a\},\{b\}\}$ and $\sigma=\{X,\phi,\{a\}\}$ be topologies on $X$ and $Y$ respectively. Let $f:X\to Y$ be a function defined by $f(a)=a=f(b)$ and $f(c)=c$. Then $f$ is contra $rgb$-continuous but not contra $\delta gb$-continuous, since $\{a\}$ is open in $Y$ but $f^{-1}(\{a\})=\{a,b\}$ is not $\delta gb$-closed in $X$.

Theorem 4. Let $f:X\to Y$ be a function.

(i) If $X$ is $T_{\delta gb}$-space then $f$ is contra $\delta gb$-continuous if and only if it is contra continuous.

(ii) If $X$ is $\delta gbT_{1/2}$-space then $f$ is contra $\delta gb$-continuous if and only if it is contra $b$-continuous.

Proof: (i) Suppose $X$ is $T_{\delta gb}$-space and $f$ is contra $\delta gb$-continuous. Let $G$ be an open set in $Y$. Then by hypothesis $f^{-1}(G)$ is $\delta gb$-closed in $X$ and hence $f^{-1}(G)$ is closed in $X$. Therefore $f$ is contra continuous.

Converse is obvious.

(ii) Suppose $X$ is $\delta gbT_{1/2}$-space and $f$ is contra $\delta gb$-continuous. Let $G$ be an open set in $Y$ then $f^{-1}(G)$ is $\delta gb$-closed in $X$ and hence $f^{-1}(G)$ is $b$-closed in $X$. Therefore $f$ is contra $b$-continuous.

Converse is follows from the Theorem 2.

Theorem 5. [5] Let $A\subseteq X$. Then $x\in \delta gbcl(A)$ if and only if $U\cap A\neq \phi$, for every $\delta gb$-open set $U$ containing $x$.

Lemma 1. [8] The following properties are hold for subsets $A$ and $B$ of a space $X$:

(i) $x\in ker(A)$ if and only if $A\cap F=\phi$ for any closed set $F$ of $X$ containing $x$.

(ii) $A\subseteq ker(A)$ and $A=ker(A)$ if $A$ is open in $X$.

(iii) If $A\subseteq B$ then $ker(A)\subseteq ker(B)$.

Theorem 6. Suppose that $\delta GBC(X)$ is closed under arbitrary intersections. Then the following are equivalent for a function $f:X\to Y$:

(i) $f$ is contra $\delta gb$-continuous
For each $x \in X$ and each closed set $B$ of $Y$ containing $f(x)$ there exists an $\delta_{gb}$-open set $A$ of $X$ containing $x$ such that $f(A) \subseteq B$.

For each $x \in X$ and each open set $G$ of $Y$ not containing $f(x)$ there exists an $\delta_{gb}$-closed set $H$ in $X$ not containing $x$ such that $f^{-1}(G) \subseteq H$.

(iv) $f(\delta_{gb}cl(A)) \subseteq \ker(f(A))$ for every subset $A$ of $X$.

(v) $\delta_{gb}cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$ for every subset $B$ of $Y$.

Proof:

(i) $\rightarrow$ (ii) Let $B$ be a closed set in $Y$ containing $f(x)$ then $x \in f^{-1}(B)$. By (i), $f^{-1}(B)$ is $\delta_{gb}$-open set in $X$ containing $x$. Let $A = f^{-1}(F)$ then $f(A) = f(f^{-1}(B)) \subseteq B$.

(ii) $\rightarrow$ (i) Let $F$ be a closed set in $Y$ containing $f(x)$ then $x \in f^{-1}(F)$. From (ii), there exists a $\delta_{gb}$-open set $G_x$ in $X$ containing $x$ such that $f(G_x) \subseteq f^{-1}(F)$. Thus $f^{-1}(F) = \bigcup \{U_x : x \in f^{-1}(F)\}$ which is $\delta_{gb}$-open. Hence $f^{-1}(F)$ is $\delta_{gb}$-open set in $X$.

(iii) $\rightarrow$ (ii) Let $G$ be an open set in $Y$ not containing $f(x)$. Then $Y-G$ is a closed set in $Y$ containing $f(x)$. From (ii), there exists a $\delta_{gb}$-open set $F$ in $X$ containing $x$ such that $f(F) \subseteq Y-G$. This implies $F \subseteq f^{-1}(Y-G) = X-f^{-1}(G)$. Hence $f^{-1}(G) \subseteq X-F$. Set $H = X-F$, then $H$ is $\delta_{gb}$-closed set not containing $x$ in $X$ such that $f^{-1}(G) \subseteq H$.

(iv) $\rightarrow$ (v) Let $B \subseteq Y$ then $f^{-1}(B) \subseteq X$. By (iv), $f(\delta_{gb}cl(f^{-1}(B))) \subseteq \ker(f(f^{-1}(B))) \subseteq \ker(B)$. Thus $\delta_{gb}cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$.

(v) $\rightarrow$ (i) Let $V$ be any open subset of $Y$. Then by (v) and Lemma 1, $\delta_{gb}cl(f^{-1}(V)) \subseteq f^{-1}(\ker(V)) = f^{-1}(V)$ and $\delta_{gb}cl(f^{-1}(V)) = f^{-1}(V)$. Therefore $f^{-1}(V)$ is $\delta_{gb}$-closed set in $X$.

Lemma 2. [16] For a subset $A$ of a space $X$, the following are equivalent:

(i) $A$ is open and $\delta_{gb}$-closed

(ii) $A$ is regular open.

Theorem 7. [4] If $A \subseteq X$ is both $\delta$-open and $\delta_{gb}$-closed then it is $b$-closed.

Theorem 8. If $A \subseteq X$ is regular open then it is $b$-closed.

Lemma 3. For a subset $A$ of a space $X$ the following are equivalent:

(i) $A$ is $\delta$-open and $\delta_{gb}$-closed

(ii) $A$ is regular open
(iii) A is open and b-closed.

Proof: (i)→(ii): Let A be an \( \delta \)-open and \( \delta gb \)-closed set. Then by Theorem 7, A is b-closed that is \( bcl(A) \subseteq A \) and so \( int(cl(A)) \subseteq A \). Since A is \( \delta \)-open then A is pre-open and thus \( A \subseteq int(cl(A)) \). Hence A is regular open.

(ii)→(i): Follows from the fact that every regular open set is \( \delta \)-open and by Theorem 8.

(ii)→(iii): Follows from the fact that every regular open set is open and Theorem 8.

(iii)→(ii): Let A be an open and b-closed set then \( bcl(A) \subseteq A \) and so \( int(cl(A)) \subseteq A \). Since A is open, then A is pre-open and thus \( A \subseteq int(cl(A)) \), which implies A = \( int(cl(A)) \).

As a consequence of the above lemma, we have the following result:

**Theorem 9.** The following statements are equivalent for a function \( f:X \to Y \):

(i) \( f \) is completely continuous

(ii) \( f \) is contra \( \delta gb \)-continuous and \( \delta^* \)-continuous

(iii) \( f \) is contra b-continuous and continuous.

**Definition 5.** [16] A subset A of X is said to be Q-set if \( int(cl(A))=cl(int(A)) \).

**Definition 6.** [16] A function \( f:X \to Y \) is Q-continuous if \( f^{-1}(V) \) is Q-set in X for every open set V of Y.

**Theorem 10.** For a subset A of a space X the following are equivalent:

(i) A is clopen

(ii) A is \( \delta \)-open and \( \delta \)-closed

(iii) A is regular-open and regular-closed.

**Theorem 11.** For a subset A of a space X the following are equivalent:

(i) A is clopen

(ii) A is \( \delta \)-open, Q-set and \( \delta gb \)-closed

(iii) A is open, Q-set and b-closed.

Proof: (i)→(ii): Let A be clopen then by Theorem 10 we have A = \( int(cl(A))=cl(int(A)) \). Hence A is Q-set. Again by Theorem 10, A is \( \delta \)-open and \( \delta \)-closed. Since every \( \delta \)-closed set is \( \delta gb \)-closed. Therefore (ii) holds.

(ii)→(iii): Follows from the Theorem 7.

(iii)→(i): Let A be an open, Q-set and b-closed set then by Lemma 3, A is regular open. Since A is Q-set, then A = \( int(cl(A))=cl(int(A)) \) which implies A is regular closed. Hence by Theorem 10, A is clopen.
Theorem 12. The following statements are equivalent for a function \( f: X \rightarrow Y \):

(i) \( f \) is perfectly continuous
(ii) \( f \) is \( \delta^* \)-continuous, Q-continuous and contra \( \delta gb \)-continuous
(iii) \( f \) is continuous, Q-continuous and contra \( b \)-continuous.

Definition 7. A space \( X \) is called locally \( \delta gb \)-indiscrete if every \( \delta gb \)-open set is closed in \( X \).

Theorem 13. If \( f:X \rightarrow Y \) is a contra \( \delta gb \)-continuous and \( X \) is locally \( \delta gb \)-indiscrete space then \( f \) is continuous.

Proof: Let \( G \) be a closed set in \( Y \). Since \( f \) is contra \( \delta gb \)-continuous and \( X \) is locally \( \delta gb \)-indiscrete space then \( f^{-1}(G) \) is a closed set in \( X \). Hence \( f \) is continuous.

Definition 8. [11] A space \( X \) is called locally indiscrete if every open set is closed in \( X \).

Theorem 14. If \( f:X \rightarrow Y \) is a contra \( \delta gb \)-continuous preclosed surjection and \( X \) is \( T_{\delta gb} \)-space then \( Y \) is locally indiscrete.

Proof: Let \( V \) be an open set in \( Y \). Since \( f \) is contra \( \delta gb \)-continuous and \( X \) is \( T_{\delta gb} \)-space then \( f^{-1}(G) \) is closed in \( X \). Also \( f \) is preclosed then \( V \) is preclosed in \( Y \). Now we have \( \text{cl}(V)=\text{cl}(\text{int}(V)) \subseteq V \). This means \( V \) is closed in \( Y \) and hence \( Y \) is indiscrete.

Theorem 15. Suppose that \( \delta GBC(X) \) is closed under arbitrary intersections. If \( f:X \rightarrow Y \) is contra \( \delta gb \)-continuous and \( Y \) is Urysohn then \( G(f) \) is contra \( \delta gb \)-closed in \( X \times Y \).

Proof: Let \( (x,y) \in \text{(X×Y)}-G(f) \), then \( y \neq f(x) \) and there exist open sets \( A \) and \( B \) such that \( f(x) \in A, y \in B \) and \( \text{cl}(A) \cap \text{cl}(B)=\phi \). Since \( f \) is contra \( \delta gb \)-continuous then there exists \( U \in \delta gbO(X,x) \) such that \( f(U) \subseteq \text{cl}(A) \). Therefore we obtain \( f(U) \cap \text{cl}(B)=\phi \). This shows that \( G(f) \) is contra \( \delta gb \)-closed.
Theorem 18. If $f:X \to Y$ is $\delta gb$-continuous and $Y$ is $T_1$ then $G(f)$ is contra $\delta gb$-closed in $X \times Y$.

Proof: Let $(x,y) \in (X \times Y)-G(f)$ then $y \neq f(x)$ and there exists open set $U$ such that $f(x) \in U$ and $y \notin U$. Since $f$ is $\delta gb$-continuous, then there exists $V \in gbO(X,x)$ such that $f(V) \subseteq U$. Therefore we obtain $f(V) \cap (Y-U) = \phi$ and $Y-U \in C(Y,y)$. This shows that $G(f)$ is contra $\delta gb$-closed.

Theorem 19. Let $f:X \to Y$ be a function and $g:X \to X \times Y$ be the graph function of $f$ defined by $g(x) = (x,f(x))$ for each $x \in X$. If $g$ is contra $\delta gb$-continuous then $f$ is contra $\delta gb$-continuous.

Proof: Let $U$ be an open set in $Y$ then $X \times U$ is an open set in $X \times Y$. Since $g$ is contra $\delta gb$-continuous. It follows that $f^{-1}(U) = g^{-1}(X \times U)$ is $\delta gb$-closed in $X$. Thus $f$ is contra $\delta gb$-continuous.

Theorem 20. If $f:X \to Y$ is contra $\delta gb$-continuous then for each $x \in X$ and for each closed set $V$ in $Y$ with $f(x) \in V$ there exists a $\delta gb$-open set $U$ containing $x$ such that $f(U) \subseteq V$.

Proof: Let $x \in X-K$. Then $f(x) \neq g(x)$. Since $Y$ is Urysohn there exist open sets $U$ and $V$ such that $f(x) \in U, g(x) \in V$ and $cl(U) \cap cl(V) = \phi$. Since $f$ and $g$ are contra $\delta gb$-continuous, $f^{-1}(cl(U))$ and $g^{-1}(cl(V))$ are $\delta gb$-open sets in $X$. Let $A=f^{-1}(cl(U))$ and $B=g^{-1}(cl(V))$. Then $A$ and $B$ are $\delta gb$-open sets containing $x$. Set $C=A \cap B$, then $C$ is $\delta gb$-open set in $X$. Hence $f(C) \cap g(C) = f(A \cap B) \cap g(A \cap B) \subseteq f(A) \cap g(B) = cl(U) \cap cl(V) = \phi$. Therefore $C \cap K = \phi$.

By Theorem 5, $x \notin gbcl(K)$. Hence $K$ is $\delta gb$-closed in $X$.

Definition 11. A space $X$ is called $\delta gb$-connected provided that $X$ is not the union of two disjoint nonempty $\delta gb$-open sets.

Theorem 24. If $f:X \to Y$ is a contra $\delta gb$-continuous function from a $\delta gb$-connected space $X$ onto any space $Y$ then $Y$ is not a discrete space.
Proof: Since \( f \) is contra \( \delta_{gb} \)-continuous and \( X \) is \( \delta_{gb} \)-connected space. Suppose \( Y \) is a discrete space. Let \( V \) be a proper non empty open and closed subset of \( Y \). Then \( f^{-1}(V) \) is proper nonempty \( \delta_{gb} \)-open and \( \delta_{gb} \)-closed subset of \( X \), which contradicts the fact that \( X \) is \( \delta_{gb} \)-connected space. Hence \( Y \) is not a discrete space.

**Theorem 25.** If \( f:X \to Y \) is a contra \( \delta_{gb} \)-continuous surjection and \( X \) is \( \delta_{gb} \)-connected space then \( Y \) is connected.

**Proof:** Let \( f:X \to Y \) is a contra \( \delta_{gb} \)-continuous and \( X \) is \( \delta_{gb} \)-connected space. Suppose \( Y \) is not connected. Then there exist disjoint open sets \( U \) and \( V \) in \( Y \) such that \( Y=U\cup V \). Therefore \( U \) and \( V \) are clopen in \( Y \). Since \( f \) is contra \( \delta_{gb} \)-continuous \( f^{-1}(U) \) and \( f^{-1}(V) \) are \( \delta_{gb} \)-open sets in \( X \). Further \( f \) is surjective implies, \( f^{-1}(U) \) and \( f^{-1}(V) \) are non empty disjoint and \( X = f^{-1}(U) \cup f^{-1}(V) \). This contradicts the fact that \( X \) is \( \delta_{gb} \)-connected space. Therefore \( Y \) is connected.

**Theorem 26.** Let \( X \) be a \( \delta_{gb} \)-connected and \( Y \) be \( T_1 \)-space. If \( f:X \to Y \) is contra \( \delta_{gb} \)-continuous then \( f \) is constant.

**Proof:** Since \( Y \) is \( T_1 \)-space, \( U=\{f^{-1}(y): y \in Y \} \) is a disjoint \( \delta_{gb} \)-open partition of \( X \). If \( |U| \geq 2 \) then \( X \) is the union of two nonempty \( \delta_{gb} \)-open sets. This contradicts the fact that \( X \) is \( \delta_{gb} \)-connected. Therefore \( |U|=1 \) and hence \( f \) is constant.

**Definition 12.** [4] A topological space \( X \) is said to be \( \delta_{gb} \)-\( T_2 \) space if for any pair of distinct points \( x \) and \( y \) there exist disjoint \( \delta_{gb} \)-open sets \( G \) and \( H \) such that \( x \in G \) and \( y \in H \).

**Theorem 27.** Let \( X \) and \( Y \) be topological spaces. If

(i) for each pair of distinct points \( x \) and \( y \) in \( X \) there exists a function \( f:X \to Y \) such that \( f(x) \neq f(y) \),

(ii) \( Y \) is Urysohn space and

(iii) \( f \) is contra \( \delta_{gb} \)-continuous at \( x \) and \( y \). Then \( X \) is \( \delta_{gb} \)-\( T_2 \).

**Proof:** Let \( x \) and \( y \) be any distinct points in \( X \) and \( f \) is a function such that \( f(x) \neq f(y) \). Let \( a=f(x) \) and \( b=f(y) \) then \( a \neq b \). Since \( Y \) is an Urysohn space there exist open sets \( V \) and \( W \) in \( Y \) containing \( a \) and \( b \) respectively such that \( cl(V) \cap cl(W) = \emptyset \). Since \( f \) is contra \( \delta_{gb} \)-continuous at \( x \) and \( y \) then there exist \( \delta_{gb} \)-open sets \( A \) and \( B \) in \( X \) containing \( x \) and \( y \) respectively such that \( f(A) \subseteq cl(V) \) and \( f(B) \subseteq cl(W) \). We have \( A \cap B \subseteq f^{-1}(cl(V)) \cap f^{-1}(cl(W)) = f^{-1}(\emptyset) = \emptyset \). Hence \( X \) is \( \delta_{gb} \)-\( T_2 \).

**Corollary 2.** Let \( f:X \to Y \) be a contra \( \delta_{gb} \)-continuous injective function from a space \( X \) into Urysohn space \( Y \) then \( X \) is \( \delta_{gb} \)-\( T_2 \).

**Definition 13.** [14] A topological space \( X \) is called Ultra Hausdorff space if for every pair of distinct points \( x \) and \( y \) in \( X \) there exist disjoint clopen sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \) respectively.
Theorem 28. If \( f: X \to Y \) be contra \( \delta gb \)-continuous injective function from space \( X \) into a Ultra Hausdorff space \( Y \) then \( X \) is \( \delta gb-T_2 \).

Proof: Let \( x \) and \( y \) be any two distinct points in \( X \). Since \( f \) is injective \( f(x) \neq f(y) \) and \( Y \) is Ultra Hausdorff space implies there exist disjoint clopen sets \( U \) and \( V \) of \( Y \) containing \( f(x) \) and \( f(y) \) respectively. Then \( x \in f^{-1}(U) \) and \( y \in f^{-1}(V) \) where \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint \( \delta gb \)-open sets in \( X \). Therefore \( X \) is \( \delta gb-T_2 \).

Definition 14. [14] A space \( X \) is called Ultra normal space if each pair of disjoint closed sets can be separated by disjoint clopen sets.

Definition 15. [4] A topological space \( X \) is said to be \( \delta gb \)-normal if each pair of disjoint closed sets can be separated by disjoint \( \delta gb \)-open sets.

Theorem 29. If \( f: X \to Y \) be contra \( \delta gb \)-continuous closed injection and \( Y \) is ultra normal then \( X \) is \( \delta gb \)-normal.

Proof: Let \( E \) and \( F \) be disjoint closed subsets of \( X \). Since \( f \) is closed and injective \( f(E) \) and \( f(F) \) are disjoint closed sets in \( Y \). Since \( Y \) is ultra normal there exists disjoint clopen sets \( U \) and \( V \) in \( Y \) such that \( f(E) \subseteq U \) and \( f(F) \subseteq V \). This implies \( E \subseteq f^{-1}(U) \) and \( F \subseteq f^{-1}(V) \). Since \( f \) is contra \( \delta gb \)-continuous injection, \( f^{-1}(U) \) and \( f^{-1}(V) \) are disjoint \( \delta gb \)-open sets in \( X \). This shows \( X \) is \( \delta gb \)-normal.

Remark 3. The composition of two contra-\( \delta gb \)-continuous functions need not be contra-\( \delta gb \)-continuous as seen from the following example.

Example 3. Let \( X=Y=Z=\{a,b,c\}, \tau=\{X,\phi,\{a\},\{b\},\{a,b\}\}, \sigma=\{Y,\phi,\{a\}\} \) and \( \eta=\{Z,\phi,\{b,c\}\} \) be topologies on \( X, Y \) and \( Z \) respectively. Then the identity function \( f: X \to Y \) and a function \( g: Y \to Z \) defined by \( g(a)=b, g(b)=c \) and \( g(c)=a \) are contra \( \delta gb \)-continuous but \( g \circ f: X \to Z \) is not contra \( \delta gb \)-continuous, since there exists a open set \( \{b,c\} \) in \( Z \) such that \( (g \circ f)^{-1}(b,c)=\{a,b\} \) is not \( \delta gb \)-closed in \( X \).

Theorem 30. Let \( f: X \to Y \) and \( g: Y \to Z \) be any two functions.

(i) If \( f \) is contra \( \delta gb \)-continuous and \( g \) is continuous then \( g \circ f \) is contra \( \delta gb \)-continuous.

(ii) If \( f \) is contra \( \delta gb \)-continuous and \( g \) is contra continuous then \( g \circ f \) is \( \delta gb \)-continuous.

(iii) If \( f \) is \( \delta gb \)-continuous and \( g \) is contra continuous then \( g \circ f \) is contra \( \delta gb \)-continuous.

(iv) If \( f \) is \( \delta gb \)-irresolute and \( g \) is contra \( \delta gb \)-continuous then \( g \circ f \) is contra \( \delta gb \)-continuous.

Proof: (i) Let \( h=g \circ f \) and \( V \) be an open set in \( Z \). Since \( g \) is continuous, \( g^{-1}(V) \) is open in \( Y \). Therefore \( f^{-1}(g^{-1}(V))=h^{-1}(V) \) is \( \delta gb \)-closed in \( X \) because \( f \) is contra \( \delta gb \)-continuous.

Hence \( g \circ f \) is contra \( \delta gb \)-continuous.

The proofs of (ii),(iii) and (iv) are similar to (i).

Theorem 31. Let \( f: X \to Y \) be contra \( \delta gb \)-continuous and \( g: Y \to Z \) be \( \delta gb \)-continuous. If \( Y \) is \( T_{\delta gb} \)-space, then \( g \circ f: X \to Z \) is contra \( \delta gb \)-continuous.

Proof: Let \( V \) be any open set in \( Z \). Since \( g \) is \( \delta gb \)-continuous \( g^{-1}(V) \) is \( \delta gb \)-open in \( Y \) and since \( Y \) is \( T_{\delta gb} \)-space, \( g^{-1}(V) \) open in \( Y \). Since \( f \) is contra \( \delta gb \)-continuous, then \( f^{-1}(g^{-1}(V))=(g \circ f)^{-1}(V) \) is \( \delta gb \)-closed set in \( X \). Therefore \( g \circ f \) is contra \( \delta gb \)-continuous.
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