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Existence, nonexistence and decay estimate of global solutions for a viscoelastic wave equation with nonlinear boundary damping and internal source terms

Huafei Di[1], Yadong Shang[1,*]

[1] *School of Mathematics and Information Science, Guangzhou University, Guangdong, P R China*

Abstract. In this paper, we consider the initial boundary value problem for a viscoelastic wave equation with nonlinear boundary damping and internal source terms. We first prove the existence of global weak solutions by the combination of Galerkin approximation, potential well and monotonicity-compactness methods. Then, we give an explicit decay rate estimate of the energy by making use of the perturbed energy method. Finally, the finite time blow up result of the solutions is investigated under certain assumptions on the relaxation function g and initial data.

2010 Mathematics Subject Classifications: 35L35, 35L75, 35R15

Key Words and Phrases: Viscoelastic equation, Nonlinear boundary damping, Internal source, Blow up, Decay rate estimate, Perturbed energy method

1. Introduction

We are concerned with the following initial boundary value problem of the viscoelastic wave equation with nonlinear boundary damping and internal source terms

$$\begin{cases} |u_t|^{\rho-1}u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds = |u|^{p-1}u, & in\ \Omega \times (0,\infty), \\ u(x,t) = 0, & on\ \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + |u_t|^{q-1}u_t = 0, & on\ \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\rho, p, q \geq 1$, and Ω is a bounded domain of R^n with a smooth boundary Γ. Let $\{\Gamma_0, \Gamma_1\}$ be a partition of its boundary Γ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and $\mathrm{meas}(\Gamma_0) > 0$. Here, ν is the unit outward normal to Γ, and g represents the kernel of memory term, namely the relaxation function, satisfying certain conditions to be specified later.

*Corresponding author.

Email addresses: dihuafei@yeah.net (H.F. Di), gzydshang@126.com (Y.D. Shang)

It is well known that viscoelastic materials present nature damping, which is due to some special properties of these materials to keep memory of their past trace. From the mathematical point of view, these damping effects are modeled by integro-differential operators. This type of equations with viscoelastic term describes a variety of important physical processes, such as the analysis of heat conduction in viscoelastic materials, electric signals in nonlinear telegraph line with nonlinear damping, viscous flow in viscoelastic materials [1], vibration of nonlinear elastic rod with viscosity [2], nonlinear bidirectional shallow water waves [3], and the velocity evolution of ion-acoustic waves in a collision less plasma when a ion viscosity is invoked [4] and so on.

For the nonlinear viscoelastic wave equations with homogeneous Dirichlet boundary condition, many authors have given attention to them for quite a long time. There are extensive literature on the existence or nonexistence of global solutions, blow up results in finite time, and the asymptotic behavior of the solutions for this type of problems. Berrimi and Messaoudi [5] considered the following nonlinear viscoelastic wave equation

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds = u|u|^{p-2}, \quad in\ \Omega \times (0,\infty), \tag{1.2}$$

in a bounded domain and $p \geq 2$. They established a local existence result and showed that the local solution is global and decays uniformly if the initial data are small enough. Kim and Han [6] proved that any weak solution with negative initial energy blows up in finite time under suitable conditions on the relaxation function g for the equation (1.2).

In [7], Wang et al. studied the following nonlinear viscoelastic wave equation

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds + u_t = u|u|^{p-2}, \quad in\ \Omega \times (0,\infty). \tag{1.3}$$

Under some appropriate assumptions on g, by introducing potential wells they obtained the existence of global solution and the explicit exponential energy decay estimates. Later, Wang [8] proved that solution with arbitrary positive initial energy blows up in finite time under some appropriate assumptions on the relaxation function g and the initial data. Messaoudi [9] changed the linear damping term u_t into the nonlinear damping term $au_t|u_t|^{m-2}$. Under suitable conditions on g, he proved that the solution with negative initial energy blows up in finite time. This blow up result was extended by the same author [10] to certain solution with positive initial energy.

Song and Zhong [11] considered the nonlinear viscoelastic wave equation with strong damping term

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \triangle u_t = u|u|^{p-2}, \quad in\ \Omega \times (0,\infty), \tag{1.4}$$

with homogeneous Dirichlet boundary condition. They proved that the solution with positive initial energy blows up in finite time.

In [12], Han and Wang studied the general decay of energy for the following nonlinear viscoelastic equation without source term

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \triangle u_{tt} + u_t|u_t|^{m-2} = 0, \quad in\ \Omega \times (0,\infty). \tag{1.5}$$

More recently, Xu, Yang and Liu [13] investigated the following strongly damped viscoelastic wave equation

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \triangle u_t - \triangle u_{tt} + u_t = u|u|^{p-1},\ in\ \Omega \times (0,\infty). \quad (1.6)$$

They proved the existence and nonexistence of global weak solution with low initial energy by introducing a family of potential wells. Then, they established a blow up result for certain solutions with arbitrary positive initial energy.

Messaoudi and Tatar [14] considered the following nonlinear viscoelastic equation

$$|u_t|^\rho u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \triangle u_{tt} = bu|u|^{p-1},\quad in\ \Omega \times (0,\infty), \quad (1.7)$$

with Dirichlet boundary condition. By using the potential well method, they proved that the viscoelastic term is enough to ensure the global existence and uniform decay of solutions provided that the initial data are in same stable set. Liu [15] proved that for certain class of relaxation function g and certain initial data in the unstable set, there are the solutions with positive initial energy that blow up in finite time.

Cavalcanti et al. [16] considered the following nonlinear viscoelastic equation without source and weak damping terms

$$|u_t|^\rho u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds - \gamma\triangle u_t - \triangle u_{tt} = 0,\quad in\ \Omega \times (0,\infty). \quad (1.8)$$

They obtained the global existence of weak solution and uniform decay rates of the energy by assuming that the relaxation g has a exponential decay.

In [17], Wu studied the following viscoelastic equation with nonlinear source and weak damping terms

$$|u_t|^\rho u_{tt} - \triangle u - \triangle u_{tt} + \int_0^t g(t-s)\triangle u(s)ds + |u_t|^m u_t = |u|^{p-1}u,\ in\ \Omega \times (0,\infty). \quad (1.9)$$

He discussed the general uniform decay estimate of solution energy under suitable conditions on the relaxation function g, the initial data and the parameters ρ, m, p.

We also note that the potential well method is a very popular and important way to study the global existence and finite time blow up of solutions for nonlinear evolution equations. This method was first introduced by Sattinger [26] to study the global existence of solutions for nonlinear hyperbolic equations. And it also plays a very vital role in deriving the threshold results between the global existence and nonexistence of solutions. Hence, the potential well method has been widely used and extended by many authors to study different kinds of evolution equations, we refer the reader to see [25,27-29] and the papers cited therein.

For the viscoelastic equation with nonlinear boundary condition, there are also some results about the well-posedness for this type of problems. We refer readers to see [18]-[22] and the papers cited therein. In [18]-[20], the initial boundary value problem of the viscoelastic equation with a nonlinear boundary damping term

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds = 0, \quad in\ \Omega \times (0,\infty), \\ u(x,t) = 0, \quad on\ \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + h(u_t) = 0, \quad on\ \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.10}$$

was studied. Cavalcanti et al. [18] obtained a global existence result for strong and weak solutions under the classical assumptions on g. Some uniform decay rate results were established under quite restrictive assumptions on both the damping term h and the relaxation function g. Later, Cavalcanti et al. [19] studied the problem (1.10) under weaker conditions on the relaxation function g and without imposing a growth assumption on the function h. They obtained the decay rate estimates of the energy depending on the behavior of h near zero and on the behavior of the relaxation g at infinity. For a wider class of relaxation function g and without imposing any restrictive growth assumptions on the damping term h, Messaoudi and Mustafa [20] also established an explicit and general decay rate result for the problem (1.10).

Lu et al. [21] considered the following initial boundary value problem of the viscoelastic wave equation with nonlinear boundary damping and source terms

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds = 0, \quad in\ \Omega \times (0,\infty), \\ u(x,t) = 0, \quad on\ \Gamma_0 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + u_t|u_t|^{m-2} = u|u|^{p-2}, \quad on\ \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega. \end{cases} \tag{1.11}$$

They obtained the global existence of solution and a general decay of the energy under some appropriate assumptions on the function g and certain initial data. In [22], Liu and Yu first extended the decay result obtained by Lu et al. Then, they established two blow up results: one is for certain solutions with nonpositive initial energy as well as positive initial energy in the case $m \geq 2$, the other is for certain solutions with arbitrary positive initial energy in the case $m = 2$.

Motivated by the above researches, in the present work we consider the viscoelastic wave equation with internal nonlinear terms $|u_t|^{\rho-1}u_{tt}$, $|u|^{p-1}u$ and boundary nonlinear damping term $|u_t|^{q-1}u_t$. First of all, we prove the existence of global weak solutions by the combination of Galerkin approximation, potential well and monotonicity-compactness methods. Then, we give an explicit decay rate estimate of the energy by making use of the perturbed energy method introduced by Cavalcanti et al.[16,18,23], Messaoudi and Tatar [14,24] and Liu [22] coupled with some technical Lemmas. Finally, the finite time blow up result of the solutions is investigated under certain assumptions on the relaxation function g and initial data.

The rest of this paper is organized as follows: In Section 2, we give some preliminaries and state our main results. The proof of the existence of global weak solutions and an exponential decay result will be given in Sections 3 and 4. In the last Section, we investigate the finite time blow up result of solutions under certain conditions.

2. Preliminaries

In order to state our results precisely, we first give some notations, basic definitions and important Lemmas which will be needed in the course of this paper.

Let Ω be a bounded open domain of R^n with a smooth boundary Γ. We consider $m(x) = x - x_0$ (x_0 is a fixed point of R^n), and introduce a partition of the boundary Γ such that

$$\Gamma_0 = \{x \in \Gamma : \ m(x) \cdot \nu(x) \leq 0\}, \quad \Gamma_1 = \{x \in \Gamma : \ m(x) \cdot \nu(x) > 0\}.$$

We define some inner products and norms

$$(u, v) = \int_\Omega u(x)v(x)dx, \quad (u, v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)d\Gamma,$$

$$\|u\|_p^p = \int_\Omega |u(x)|^p dx, \quad \|u\|_{\Gamma_1,p}^p = \int_{\Gamma_1} |u(x)|^p d\Gamma, \quad \|u\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|$$

and the Hilbert space

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) \middle| \ u = 0 \ \text{ on } \ \Gamma_0\right\}.$$

Since Γ_0 has positive $(n-1)$ dimensional Lebesgue measure, by Poincaré inequality, we can endow $H^1_{\Gamma_0}(\Omega)$ with the equivalent norm $\|u\|_{H^1_{\Gamma_0}} = \|\nabla u\|_2$ (see [25] for details).

Now, we state the general hypotheses.

(A1) The relaxation function g: $[0, \infty) \to (0, \infty)$ is a C^1 function satisfying

$$g'(t) \leq 0, \quad b = 1 - \int_0^\infty g(s)ds \leq 1 - \int_0^t g(s)ds = l(t).$$

(A2) There exists a positive differentiable function $\xi(t)$ such that

$$g'(t) \leq -\xi(t)g(t), \quad t \geq 0,$$

and for some positive constant k, $\xi(t)$ satisfies

$$\left|\frac{\xi'(t)}{\xi(t)}\right| \leq k, \quad \xi'(t) \leq 0, \quad \forall \ t > 0.$$

(A3) We also assume that

$$1 < p < \infty \ \text{ if } \ n \leq 2, \quad 1 < p \leq \frac{n+2}{n-2} \ \text{ if } \ n \geq 3,$$

$$1 < q < \infty \ \text{ if } \ n \leq 2, \quad 1 < q \leq \frac{n}{n-2} \ \text{ if } \ n \geq 3.$$

Next, we shall define some functionals and study their some basic properties which are related with potential well.

Firstly, let us consider the functionals

$$E(t)=\frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1}+\frac{1}{2}\big(1-\int_0^t g(s)ds\big)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{2.1}$$

$$J(u)=\frac{1}{2}\big(1-\int_0^t g(s)ds\big)\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{2.2}$$

$$I(u)=\big(1-\int_0^t g(s)ds\big)\|\nabla u\|_2^2+(g\circ\nabla u)(t)-\|u\|_{p+1}^{p+1}, \tag{2.3}$$

where $(g\circ\nabla u)(t)=\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2ds$, $\forall\ u\in H^1_{\Gamma_0}(\Omega)$.

Lemma 1. *Let the assumptions* $(A1)$, $(A3)$ *hold, then for any* $u\in H^1_{\Gamma_0}(\Omega)$, $\|u\|_{H^1_{\Gamma_0}}\neq 0$, *it follows that*
(1) $\lim_{\lambda\to 0^+}J(\lambda u)=0,\quad \lim_{\lambda\to+\infty}J(\lambda u)=-\infty$*;*
(2) *On the interval* $0<\lambda<\infty$, *there exists a unique* $\lambda^*=\lambda^*(u)$ *such that*

$$\frac{d}{d\lambda}J(\lambda u)\big|_{\lambda=\lambda^*}=0;$$

(3) $J(\lambda u)$ *is increasing on* $0\le\lambda\le\lambda^*$, *decreasing on* $\lambda^*\le\lambda<\infty$, *and takes the maximum at* $\lambda=\lambda^*$*;*
(4) $I(\lambda u)>0$ *for* $0\le\lambda<\lambda^*$, $I(\lambda u)<0$ *for* $\lambda^*<\lambda<\infty$, *and* $I(\lambda^* u)=0$.

Proof. (1) From the definition of the functional (2.2), we have

$$J(\lambda u)=\frac{1}{2}l(t)\lambda^2\|\nabla u\|_2^2+\frac{1}{2}\lambda^2(g\circ\nabla u)(t)-\frac{\lambda^{p+1}}{p+1}\|u\|_{p+1}^{p+1}.$$

Hence, the conclusion holds.

(2) The conclusion follows from

$$\frac{d}{d\lambda}J(\lambda u)=\lambda l(t)\|\nabla u\|_2^2+\lambda(g\circ\nabla u)(t)-\lambda^p\|u\|_{p+1}^{p+1}=0. \tag{2.4}$$

(3) From the conclusion of (2), we can easily get

$$\frac{d}{d\lambda}J(\lambda u)\ge 0,\quad for\ 0\le\lambda\le\lambda^*,\qquad \frac{d}{d\lambda}J(\lambda u)\le 0,\quad for\ \lambda^*\le\lambda<\infty.$$

(4) The conclusion follows from

$$I(\lambda u)=\lambda^2 l(t)\|\nabla u\|_2^2+\lambda^2(g\circ\nabla u)(t)-\lambda^p\|u\|_{p+1}^{p+1}=\lambda\frac{d}{d\lambda}J(\lambda u). \tag{2.5}$$

Then, for $t \geq 0$, we define

$$d(t) = \inf_{u \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}} \big\{ \sup_{\lambda > 0} J(\lambda u) \big\}. \tag{2.6}$$

In fact (see [26,27] for details), d(t) is positive and equal to

$$\inf_{I(u)=0, u \neq 0} J(u). \tag{2.7}$$

Lemma 2. *Let the assumptions* (*A*1), (*A*3) *hold, then for all* $t \in [0, \infty)$, *we have*

$$0 < \tilde{d} \leq d(t) \leq \tilde{\tilde{d}}(u) = \sup_{\lambda > 0} J(\lambda u), \tag{2.8}$$

where $\tilde{d} = \frac{p-1}{2(p+1)} (\frac{b}{B^2_{p+1}})^{\frac{p+1}{p-1}}$, *and* B_{p+1} *is the optimal constant satisfying the Sobolev inequality* $\|u\|_{p+1} \leq B_{p+1} \|\nabla u\|_2$.

Proof. From the definition of $d(t)$, we get $d(t) \leq \tilde{\tilde{d}} = \sup_{\lambda > 0} J(\lambda u)$. By the Sobolev inequality, it follows that

$$\begin{aligned} J(\lambda u) &= \frac{1}{2} l(t) \|\nabla \lambda u\|_2^2 + \frac{1}{2} (g \circ \nabla \lambda u)(t) - \frac{1}{p+1} \|\lambda u\|_{p+1}^{p+1} \\ &\geq \frac{1}{2} b \|\nabla \lambda u\|_2^2 - \frac{1}{p+1} B^{p+1}_{p+1} \|\nabla \lambda u\|_2^{p+1}. \end{aligned} \tag{2.9}$$

Here, we define the function $h(\lambda) = \frac{1}{2} b \lambda^2 - \frac{1}{p+1} B^{p+1}_{p+1} \lambda^{p+1}$, $\lambda > 0$. By the direct computation, we deduce that h is increasing for $0 < \lambda < \lambda_1$, decreasing for $\lambda > \lambda_1$ and $\lambda_1 = (\frac{b}{B^{p+1}_{p+1}})^{\frac{1}{p-1}}$ is the absolute maximum point of h such that

$$\tilde{d} = h(\lambda_1) = \frac{p-1}{2(p+1)} (\frac{b}{B^2_{p+1}})^{\frac{p+1}{p-1}}.$$

By the combination of (2.9) and the definition of $\tilde{d}$, it follows that $\tilde{d} = h(\lambda_1) \leq J(\lambda u)$. Moreover, from the definition of $d(t)$, we have $\tilde{d} \leq \inf_{u \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}} \big\{ \sup_{\lambda > 0} J(\lambda u) \big\} = d(t)$. The proof is completed.

To obtain the results of this paper, we will construct the potential wells associated with the functionals J(u) and I(u). Next, let us introduce the stable and unstable sets:

$$W = \big\{ u \in H^1_{\Gamma_0}(\Omega) \big| \ I(u) > 0, \ J(u) < \tilde{d} \big\} \cup \{0\}, \tag{2.10}$$

and

$$V = \big\{ u \in H^1_{\Gamma_0}(\Omega) \big| \ I(u) < 0, \ J(u) < \tilde{d} \big\}. \tag{2.11}$$

For simplicity, we define the weak solutions of (1.1) over the interval $\Omega \times [0, T)$, but it is to be understood that T is either infinity or the limit of the existence interval.

Definition 1. *We say that $u(x,t)$ is called a weak solution of the problem* (1.1) *on the interval* $\Omega\times[0,T)$. *If* $u\in L^\infty(0,T;H^1_{\Gamma_0}(\Omega))$ *with* $u_t\in L^\infty(0,T;L^{\rho+1}(\Omega))\cap L^{q+1}(0,T;L^{q+1}(\Gamma_1))$ *satisfy the following conditions*
(*i*) *For any* $v\in H^1_{\Gamma_0}(\Omega)\cap L^{q+1}(\Gamma_1)\cap L^{\rho+1}(\Omega)$ *and a.e* $0\le t\le T$, *such that*

$$\begin{aligned}&\frac{1}{\rho}(|u_t|^{\rho-1}u_t,v)+\int_0^t b_1(u,v)ds+\int_0^t b_2(u,v)ds\\ -&\int_0^t(|u|^{p-1}u,v)ds+\int_0^t(|u_t|^{q-1}u_t,v)_{\Gamma_1}ds=\frac{1}{\rho}(|u_1|^{\rho-1}u_1,v),\end{aligned} \tag{2.12}$$

where

$$b_1(u,v)=(\nabla u,\nabla v),$$

$$b_2(u,v)=-(\int_0^s g(s-\tau)\nabla u(\tau)d\tau,\nabla v);$$

(*ii*) $u(x,0)=u_0(x)$ *in* $H^1_{\Gamma_0}(\Omega)$, $u_t(x,0)=u_1(x)$ *in* $L^{\rho+1}(\Omega)\cap L^{q+1}(\Gamma_1)$.
(*iii*) *The following energy inequality holds*

$$E(t)\le E(0), \tag{2.13}$$

for any $0\le t<T$.

The following Lemma is similar to the Lemmas of [28,29] with slight modification.

Lemma 3. *Let the assumptions* (*A*1), (*A*3) *hold and u be a solution of problem* (1.1). *Further assume that* $u_0(x)\in H^1_{\Gamma_0}(\Omega)$, $u_1(x)\in L^{\rho+1}(\Omega)\cap L^{q+1}(\Gamma_1)$, *we have*
(1) *If* $E(0)<\tilde d$, $I(u_0)>0$ *or* $\|u_0\|_{H^1_{\Gamma_0}}=0$, *then the solution* $u(t)\in W$ *for all* $t\in[0,T)$;
(2) *If* $E(0)<\tilde d$, $I(u_0)<0$, *then the solution* $u(t)\in V$ *for all* $t\in[0,T)$.

Proof. (1) Let u be any solution of problem (1.1) with $E(0)<\tilde d$ and $I(u_0)>0$ or $\|u_0\|_{H^1_{\Gamma_0}}=0$. If $\|u_0\|_{H^1_{\Gamma_0}}=0$, then $u_0(x)\in W$. If $I(u_0)>0$, from the inequality

$$\frac{1}{\rho+1}\|u_1\|^{\rho+1}_{\rho+1}+J(u_0)=E(0)<\tilde d, \tag{2.14}$$

we have $u_0(x)\in W$.

We prove $u(t)\in W$ for $0<t<T$. Arguing by contradiction and considering the time continuity of $I(u)$, we suppose that there exists a time $t_0\in(0,T)$ such that $u(t_0)\in\partial W$, which means that $I(u(t_0))=0$, $\|u(t_0)\|_{H^1_{\Gamma_0}}\ne 0$ or $J(u(t_0))=\tilde d$. From (2.13), it follows that

$$\frac{1}{\rho+1}\|u_t\|^{\rho+1}_{\rho+1}+J(u)\le E(0)<\tilde d,\ \ 0<t<T. \tag{2.15}$$

Thus, we see that $J(u(t_0)) \neq \tilde{d}$. If $I(u(t_0)) = 0$, $\|u(t_0)\|_{H^1_{\Gamma_0}} \neq 0$, then by the definition of d we have $J(u(t_0)) \geq d$ which contradicts (2.15). The proof of (1) is completed.

(2) Let $u(t)$ be any solution of problem (1.1) with $E(0) < \tilde{d}$ and $I(u_0) < 0$. From (2.11) we get that $u_0(x) \in V$. We prove $u(t) \in V$ for $0 < t < T$. Arguing by contradiction, we suppose that there exists a time $t_0 \in (0, T)$ such that $u(t_0) \in \partial V$ which means that $I(u(t_0)) = 0$ or $J(u(t_0)) = \tilde{d}$. Again (2.15) shows that $J(u(t_0)) \neq \tilde{d}$. If $I(u(t_0)) = 0$, then by the definition of d we have $J(u(t_0)) \geq d$ which contradicts (2.15).

Lemma 4. *Let the assumptions (A1), (A3) hold. For any fixed positive number $\beta < 1$, assume that $I(u_0) < 0$, $E(0) < \beta\tilde{d}$, then we have $I(u(t)) < 0$ for all $t \in [0, T)$ and*

$$\tilde{d} < \frac{p-1}{2(p+1)}[l(t)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)] < \frac{p-1}{2(p+1)}\|u\|_{p+1}^{p+1}. \tag{2.16}$$

Proof. Arguing by contradiction, we can get $I(u(t)) < 0$ for all $t \in [0, T)$. In fact, suppose this is not true, then there exist some $t_0 \in [0, T)$ such that $I(u(t_0)) = 0$ and $I(u(t)) < 0$ for $0 \leq t < t_0$. Hence,

$$l(t)\|\nabla u\|_2^2 + (g \circ \nabla u)(t) < \|u\|_{p+1}^{p+1}, \quad 0 \leq t < t_0. \tag{2.17}$$

From the definition of $\tilde{d}$, it follows that

$$\begin{aligned}
\tilde{d} &= \frac{p-1}{2(p+1)}\left(\frac{b}{B_{p+1}^2}\right)^{\frac{p+1}{p-1}} \\
&\leq \frac{p-1}{2(p+1)}\left\{\frac{l(t)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)}{\|u\|_{p+1}^2}\right\}^{\frac{p+1}{p-1}} \\
&< \frac{p-1}{2(p+1)}\left\{\frac{l(t)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)}{(l(t)\|\nabla u\|_2^2 + (g \circ \nabla u)(t))^{\frac{2}{p+1}}}\right\}^{\frac{p+1}{p-1}} \\
&= \frac{p-1}{2(p+1)}[l(t)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)], \quad 0 \leq t < t_0.
\end{aligned} \tag{2.18}$$

We deduce from (2.17) and (2.18) that

$$\|u\|_{p+1}^{p+1} > \frac{2(p+1)}{p-1}\tilde{d} > 0, \quad 0 \leq t < t_0. \tag{2.19}$$

Since $t \to \|u(t)\|_{p+1}^{p+1}$ is continuous, from (2.19) we have

$$\tilde{d} \leq \frac{p-1}{2(p+1)}\|u(t_0)\|_{p+1}^{p+1} = J(u(t_0)). \tag{2.20}$$

This is impossible since $J(u(t_0)) \leq E(t_0) \leq E(0) < \tilde{d}$. Hence, we obtain $I(u(t)) < 0$ for all $t \in [0, T)$. Furthermore, from (2.18) again, we have

$$\tilde{d} < \frac{p-1}{2(p+1)}[l(t)\|\nabla u\|_2^2 + (g \circ \nabla u)(t)]$$

$$< \frac{(p-1)}{2(p+1)}\|u\|_{p+1}^{p+1}, \quad 0 \le t < T. \tag{2.21}$$

Thus, the proof is completed.

3. Existence of global weak solutions

In this section, we are going to obtain the existence of global weak solutions for the problem (1.1) with the initial conditions $E(0) < \tilde{d}$ and $I(u_0) > 0$ or $\|u_0\|_{H^1_{\Gamma_0}} = 0$ by the combination of Galerkin approximation, potential well and monotonicity-compactness methods.

Theorem 5. *Let the assumptions (A1), (A3) hold, $u_0(x) \in H^1_{\Gamma_0}(\Omega)$, $u_1(x) \in L^{\rho+1}(\Omega) \cap L^{q+1}(\Gamma_1)$. Further assume that $E(0) < \tilde{d}$ and $I(u_0) > 0$ or $\|u_0\|_{H^1_{\Gamma_0}} = 0$, then the problem (1.1) admits a global weak solution satisfying*

$$u \in L^\infty(0,\infty; H^1_{\Gamma_0}(\Omega)), \quad u_t \in L^\infty(0,\infty; L^{\rho+1}(\Omega)) \cap L^{q+1}(0,\infty; L^{q+1}(\Gamma_1)), \quad u(t) \in W$$

for $0 \le t < \infty$, and the energy identity

$$E(t) + \int_0^t \|u_t(s)\|_{\Gamma_1,q+1}^{q+1} ds - \frac{1}{2}\int_0^t (g' \circ \nabla u)(s) ds + \frac{1}{2}\int_0^t g(s)\|\nabla u(s)\|_2^2 ds = E(0), \tag{3.1}$$

holds for $0 \le t < \infty$.

Remark 1. From (3.1), we can easily obtain

$$E'(t) = -\|u_t(t)\|_{\Gamma_1,q+1}^{q+1} + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \le 0. \tag{3.2}$$

Proof. Let $\{w_j(x)\}$ be a complete orthogonal system in $H^1_{\Gamma_0}(\Omega) \cap L^{q+1}(\Gamma_1) \cap L^{\rho+1}(\Omega)$. We suppose that the approximate weak solution u_m of the problem (1.1) can be written

$$u_m(t) = \sum_{j=1}^m d_{mj}(t) w_j(x), \quad m = 1, 2, \cdots\cdots. \tag{3.3}$$

According to Galerkin's method, these coefficients d_{mj} need to satisfy the following initial value problem of nonlinear ordinary integro-differential equations

$$\begin{aligned} &\frac{1}{\rho}(|u_m'|^{\rho-1}u_m', w_j) + \int_0^t b_1(u_m, w_j) ds + \int_0^t b_2(u_m, w_j) ds \\ &\quad - \int_0^t (|u_m|^{p-1}u_m, w_j) ds + \int_0^t (|u_m'|^{q-1}u_m', w_j)_{\Gamma_1} ds \\ &= \frac{1}{\rho}(|u_m'(0)|^{\rho-1}u_m'(0), w_j), \quad j = 1, 2, \cdots, m, \end{aligned} \tag{3.4}$$

$$u_m(x,0)=\sum_{j=1}^{m}d_{mj}(0)\omega_j(x)\rightarrow u_0(x),\quad in\ \ H^1_{\Gamma_0}(\Omega), \tag{3.5}$$

$$u'_m(x,0)=\sum_{j=1}^{m}d'_{mj}(0)\omega_j(x)\rightarrow u_1(x),\quad in\ \ L^{\rho+1}(\Omega)\cap L^{q+1}(\Gamma_1), \tag{3.6}$$

where

$$b_1(u_m,w_j)=(\nabla u_m,\nabla w_j),$$

$$b_2(u_m,w_j)=-(\int_0^s g(s-\tau)\nabla u_m(\tau)d\tau,\nabla w_j).$$

We will prove that the initial value problem (3.4)-(3.6) of the nonlinear integro-differential equations have global weak solutions in the interval $[0,\infty)$. Furthermore, we show that the solutions of the problem (1.1) can be approximated by the functions u_m.

Now, differentiating (3.4) with respect to t, and multiplying the obtained equation by $d'_{mj}(t)$, summing for $j=1,\cdots,m$, then we have

$$(|u'_m|^{\rho-1}u''_m,u'_m)+b_1(u_m,u'_m)+b_2(u_m,u'_m)+(|u'_m|^{q-1}u'_m,u'_m)_{\Gamma_1}=(|u_m|^{p-1}u_m,u'_m). \tag{3.7}$$

By a direct calculation, it follows that

$$(|u'_m|^{\rho-1}u''_m,u'_m)=\frac{1}{\rho+1}\frac{d}{dt}\|u'_m\|^{\rho+1}_{\rho+1}, \tag{3.8}$$

$$b_1(u_m,u'_m)=(\nabla u_m,\nabla {u'}_m)=\frac{1}{2}\frac{d}{dt}\|\nabla u_m\|_2^2, \tag{3.9}$$

$$(|u_m|^{p-1}u_m,u'_m)=\frac{1}{p+1}\frac{d}{dt}\|u_m\|^{p+1}_{p+1}, \tag{3.10}$$

and

$$\begin{aligned}
b_2(u_m,u'_m)=&-\int_\Omega\int_0^t g(t-s)\nabla u_m(s)\nabla {u'}_m(t)dsdx\\
&=-\int_\Omega\int_0^t g(t-s)[\nabla u_m(s)-\nabla u_m(t)]\nabla {u'}_m(t)dsdx\\
&-\int_\Omega\int_0^t g(t-s)\nabla u_m(t)\nabla {u'}_m(t)dsdx\\
&=\frac{1}{2}\int_\Omega\int_0^t g(t-s)\frac{d}{dt}[\nabla u_m(s)-\nabla u_m(t)]^2dsdx
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\int_\Omega\int_0^t g(t-s)\frac{d}{dt}[\nabla u_m(t)]^2dsdx\\
=&\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^t g(t-s)[\nabla u_m(s)-\nabla u_m(t)]^2dsdx\\
&-\frac{1}{2}\int_\Omega\int_0^t g'(t-s)[\nabla u_m(s)-\nabla u_m(t)]^2dsdx\\
&-\frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\|\nabla u_m(t)\|_2^2+\frac{1}{2}g(t)\|\nabla u_m(t)\|_2^2. \qquad (3.11)
\end{aligned}
$$

Inserting (3.8)-(3.11) into (3.7), we have

$$
\begin{aligned}
&\frac{1}{\rho+1}\frac{d}{dt}\|u_m'\|_{\rho+1}^{\rho+1}+\frac{1}{2}\frac{d}{dt}[(1-\int_0^t g(s)ds)\|\nabla u_m(t)\|_2^2]\\
&+\frac{1}{2}\frac{d}{dt}(g\circ\nabla u_m)(t)-\frac{1}{p+1}\frac{d}{dt}\|u_m\|_{p+1}^{p+1}\\
=&-\|u_m'\|_{\Gamma_1,q+1}^{q+1}+\frac{1}{2}(g'\circ\nabla u_m)(t)-\frac{1}{2}g(t)\|\nabla u_m(t)\|_2^2\le 0, \qquad (3.12)
\end{aligned}
$$

which implies that

$$
E_m'(t)=-\|u_m'\|_{\Gamma_1,q+1}^{q+1}+\frac{1}{2}(g'\circ\nabla u_m)(t)-\frac{1}{2}g(t)\|\nabla u_m(t)\|_2^2\le 0, \qquad (3.13)
$$

where

$$
\begin{aligned}
E_m(t)=&\frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1}+\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u_m(t)\|_2^2\\
&+\frac{1}{2}(g\circ\nabla u_m)(t)-\frac{1}{p+1}\|u_m\|_{p+1}^{p+1}\\
=&\frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1}+J(u_m),\quad 0\le t<\infty. \qquad (3.14)
\end{aligned}
$$

From $E(0)<\tilde{d}$ and $I(u_0)>0$ or $\|u_0\|_{H_{\Gamma_0}^1}=0$, it follows that $u_0(x)\in W$. Hence, we obtain from (3.5) and (3.6) that $E_m(0)<\tilde{d}$, $I(u_m(0))>0$ and $u_m(0)\in W$ for sufficiently large m. In what follows, from the (3.14) and the arguments in the proof of Lemma 3 (1), we can obtain $u_m(t)\in W$ for sufficiently large m and $0\le t<\infty$ such that

$$
\begin{aligned}
J(u_m)=&\frac{1}{2}l(t)\|\nabla u_m\|_2^2+\frac{1}{2}(g\circ\nabla u_m)(t)-\frac{1}{p+1}\|u_m\|_{p+1}^{p+1}\\
=&\frac{p-1}{2(p+1)}[l(t)\|\nabla u_m\|_2^2+(g\circ\nabla u_m)(t)]+\frac{1}{p+1}I(u_m)\\
\ge&\frac{p-1}{2(p+1)}[l(t)\|\nabla u_m\|_2^2+(g\circ\nabla u_m)(t)]. \qquad (3.15)
\end{aligned}
$$

By the combination of (3.13) and (3.15), we get

$$
\frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1}+\frac{p-1}{2(p+1)}[l(t)\|\nabla u_m\|_2^2+(g\circ\nabla u_m)(t)]\le E_m(t)\le E_m(0)<\tilde{d}, \qquad (3.16)
$$

for sufficiently large m and $0 \leq t < \infty$.
Integrating (3.13) with respect to t, then we have

$$\begin{aligned} &E_m(t) + \int_0^t \|u_m'\|_{\Gamma_1,q+1}^{q+1} ds \\ &= \frac{1}{2}\int_0^t (g' \circ \nabla u_m)(s)ds - \frac{1}{2}\int_0^t g(s)\|\nabla u_m(s)\|_2^2 ds + E_m(0). \end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\frac{1}{\rho+1}\|u_m'\|_{\rho+1}^{\rho+1} + \int_0^t \|u_m'\|_{\Gamma_1,q+1}^{q+1} ds + \frac{p-1}{2(p+1)}[l(t)\|\nabla u_m\|_2^2 + (g \circ \nabla u_m)] \leq E_m(0) < \tilde{d}, \tag{3.18}$$

for sufficiently large m and $0 \leq t < \infty$.
From (3.18), we have

$$l(t)\|\nabla u_m\|_2^2 + (g \circ \nabla u_m)(t) < \frac{2(p+1)}{p-1}\tilde{d}, \quad 0 \leq t < \infty, \tag{3.19}$$

$$\|u_m'\|_{\rho+1}^{\rho+1} < (\rho+1)\tilde{d}, \quad 0 \leq t < \infty, \tag{3.20}$$

$$\int_0^t \|u_m'\|_{\Gamma_1,q+1}^{q+1} ds < \tilde{d}, \quad 0 \leq t < \infty. \tag{3.21}$$

Using the Sobolev inequality and (3.19), it follows that

$$\|u_m\|_{p+1}^2 \leq B_{p+1}^2\|\nabla u_m\|_2^2 < B_{p+1}^2\frac{2(p+1)}{(p-1)b}\tilde{d}, \quad 0 \leq t < \infty. \tag{3.22}$$

Furthermore, by (3.20) and (3.22), we get

$$|(|u_m'|^{\rho-1}u_m', u_m')| \leq \|u_m'\|_{\rho+1}^{\rho+1} < (\rho+1)\tilde{d}, \quad 0 \leq t < \infty, \tag{3.23}$$

$$|(|u_m|^{p-1}u_m, u_m)| \leq \|u_m\|_{p+1}^{p+1} < B_{p+1}^{p+1}\left(\frac{2(p+1)}{(p-1)b}\tilde{d}\right)^{\frac{p+1}{2}}, \quad 0 \leq t < \infty. \tag{3.24}$$

The estimates (3.19)-(3.24) permit us to obtain a subsequences of $\{u_m\}$ which from now on will be also denoted by $\{u_m\}$ and functions u, χ_1, χ_2, χ_3 such that

$$u_m \to u \ in \ L^\infty(0,\infty; H_{\Gamma_0}^1(\Omega)) \ weakly \ star, \quad m \longrightarrow \infty, \tag{3.25}$$

$$u_m' \to u' \ in \ L^\infty(0,\infty; L^{\rho+1}(\Omega)) \ weakly \ star, \quad m \longrightarrow \infty, \tag{3.26}$$

$$|u_m'|^{q-1}u_m' \to \chi_1 \ in \ L^{\frac{q+1}{q}}(0,\infty; L^{\frac{q+1}{q}}(\Gamma_1)) \ weakly, \quad m \longrightarrow \infty, \tag{3.27}$$

$$|u_m|^{p-1}u_m \to \chi_2 \; in \; L^\infty(0,\infty; L^{\frac{p+1}{p}}(\Omega)) \; weakly \; star, \quad m \longrightarrow \infty, \tag{3.28}$$

$$|u'_m|^{\rho-1}u'_m \to \chi_3 \; in \; L^\infty(0,\infty; L^{\frac{\rho+1}{\rho}}(\Omega)) \; weakly \; star, \quad m \longrightarrow \infty. \tag{3.29}$$

Since $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^2(\Omega)$ are compact(see [25] for details), we have, thanks to Aubin-Lions theorem, that

$$u_m \to u \; in \; L^2(0,\infty; L^2(\Omega)) \; strongly, \; m \longrightarrow \infty, \tag{3.30}$$

and consequently, making use of the Lemma 1.3 in [30], we deduce

$$|u_m|^{p-1}u_m \to \chi_2 = |u|^{p-1}u \; in \; L^\infty(0,\infty; L^{\frac{p+1}{p}}(\Omega)) \; weakly \; star, \quad m \longrightarrow \infty. \tag{3.31}$$

From the trace Theorem and (3.25), we deduce that $\frac{\partial u_m}{\partial \nu} \in L^\infty(0,\infty; H^{-\frac{1}{2}}_{\Gamma_0}(\Omega))$, which implies that

$$\begin{aligned} |u'_m|^{q-1}u'_m &= -\frac{\partial u_m}{\partial \nu} + \int_0^t g(t-s)\frac{\partial u_m}{\partial \nu}(s)ds \to -\frac{\partial u}{\partial \nu} + \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds \\ &= |u'|^{q-1}u' \quad in \; L^\infty(0,\infty; H^{-\frac{1}{2}}_{\Gamma_0}(\Omega)) \;\; weakly \; star, \quad m \longrightarrow \infty. \end{aligned} \tag{3.32}$$

Combining (3.27) and the above convergence, we have

$$|u'_m|^{q-1}u'_m \to \chi_1 = |u'|^{q-1}u' \; in \; L^{\frac{q+1}{q}}(0,\infty; L^{\frac{q+1}{q}}(\Gamma_1)) \; weakly, \quad m \longrightarrow \infty, \tag{3.33}$$

Passing to the limit in (3.4) and making use of (3.25)-(3.27), (3.29), (3.31) and (3.33), we obtain

$$\begin{aligned} &\frac{1}{\rho}(\chi_3, w_j) + \int_0^t b_1(u, w_j)ds + \int_0^t b_2(u, w_j)ds \\ &+ \int_0^t (|u'|^{q-1}u', w_j)_{\Gamma_1}ds - \int_0^t (|u|^{p-1}u, w_j)ds = \frac{1}{\rho}(|u_1|^{\rho-1}u_1, w_j). \end{aligned} \tag{3.34}$$

Since $\{w_j(x)\}$ is a basic of $H^1_{\Gamma_0}(\Omega) \cap L^{q+1}(\Gamma_1) \cap L^{\rho+1}(\Omega)$, then for all $t > 0$, multiplying (3.34) by $d'_{mj}(t)$, and summing for $j = 1, \cdots\cdots$, then we have

$$\begin{aligned} &\frac{1}{\rho}(\chi_3, u') + \int_0^t b_1(u, u')ds + \int_0^t b_2(u, u')ds \\ &+ \int_0^t (|u'|^{q-1}u', u')_{\Gamma_1}ds - \int_0^t (|u|^{p-1}u, u')ds = \frac{1}{\rho}(|u_1|^{\rho-1}u_1, u'). \end{aligned} \tag{3.35}$$

In what follows, multiplying (3.4) by $d'_{mj}(t)$, and summing for $j = 1, \cdots, m$, then we obtain

$$\frac{1}{\rho}(|u'_m|^{\rho-1}u'_m, u'_m) + \int_0^t b_1(u_m, u'_m)ds + \int_0^t (|u'_m|^{q-1}u'_m, u'_m)_{\Gamma_1}ds$$

$$+\int_0^t b_2(u_m,u_m')ds-\int_0^t(|u_m|^{p-1}u_m,u_m')ds=\frac{1}{\rho}(|u_m'(0)|^{\rho-1}u_m'(0),u_m'). \tag{3.36}$$

Taking $m\to\infty$ in (3.36), it follows that

$$\begin{aligned}&\frac{1}{\rho}\lim_{m\to\infty}(|u_m'|^{\rho-1}u_m',u_m')+\int_0^t b_1(u,u')ds+\int_0^t(|u'|^{q-1}u',u')_{\Gamma_1}ds\\&+\int_0^t b_2(u,u')ds-\int_0^t(|u|^{p-1}u,u')ds=\frac{1}{\rho}(|u_1|^{\rho-1}u_1,u').\end{aligned} \tag{3.37}$$

Combining (3.35) and (3.37), we deduce that

$$(\chi_3,u')=\lim_{m\to\infty}(|u_m'|^{\rho-1}u_m',u_m'). \tag{3.38}$$

On the other hand, utilizing the non-decreasing monotonicity of the function $|s|^{\rho-1}s$, $s\in \mathrm{R}$, we have

$$(|u_m'|^{\rho-1}u_m'-|\psi|^{\rho-1}\psi,u_m'-\psi)\geq 0, \tag{3.39}$$

for all $\psi\in L^{\rho+1}(\Omega)$. Thus, we get from the inequality (3.39) that

$$(|u_m'|^{\rho-1}u_m',\psi)+(|\psi|^{\rho-1}\psi,u_m'-\psi)\leq(|u_m'|^{\rho-1}u_m',u_m'). \tag{3.40}$$

Passing to the limit in (3.40) as $m\to\infty$, it follows that

$$(\chi_3-|\psi|^{\rho-1}\psi,u'-\psi)\geq 0. \tag{3.41}$$

In order to prove $\chi_3=|u'|^{\rho-1}u'$ from (3.41), we use the semi-continuity of the function $|s|^{\rho-1}s$, $s\in\mathrm{R}$ ([30], Chapter 2). Let $\psi=u'-\mu\phi$, $\mu\geq 0$ and $\forall\ \phi\in L^{\rho+1}(\Omega)$, then

$$(\chi_3-|u'-\mu\phi|^{\rho-1}(u'-\mu\phi),\phi)\geq 0. \tag{3.42}$$

Passing to the limit in (3.42) as $\mu\to 0$, we have

$$(\chi_3-|u'|^{\rho-1}u',\phi)\geq 0,\quad \forall\ \phi\in L^{\rho+1}(\Omega). \tag{3.43}$$

In a similar way, let $\psi=u'-\mu\phi$, $\mu\leq 0$ and $\forall\ \phi\in L^{\rho+1}(\Omega)$, then we obtain

$$(\chi_3-|u'|^{\rho-1}u',\phi)\leq 0,\quad \forall\ \phi\in L^{\rho+1}(\Omega). \tag{3.44}$$

From the combination of (3.43) and (3.44), we see that

$$\chi_3=|u'|^{\rho-1}u'. \tag{3.45}$$

Next, we shall prove that u satisfies (2.13). From the discussion above, we obtain for each fixed $t>0$ that

$$|(g\circ\nabla u)(t)-(g\circ\nabla u_m)(t)|$$

$$
\begin{aligned}
&= \Big|\int_0^t g(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dxds - \int_0^t g(t-s)\int_\Omega |\nabla u_m(s)-\nabla u_m(t)|^2 dxds\Big| \\
&\leq \int_0^t g(t-s)\|\nabla u(s)-\nabla u_m(s)\|_2\|\nabla u(s)+\nabla u_m(s)\|_2 ds \\
&\quad + \int_0^t g(t-s)\|\nabla u(s)-\nabla u_m(s)\|_2 ds\|\nabla u(t)+\nabla u_m(t)\|_2 \\
&\quad + \int_0^t g(t-s)\|\nabla u(s)+\nabla u_m(s)\|_2 ds\|\nabla u(t)-\nabla u_m(t)\|_2 \\
&\quad + \int_0^t g(s)ds\|\nabla u(t)+\nabla u_m(t)\|_2\|\nabla u(t)-\nabla u_m(t)\|_2 \\
&\leq C\int_0^t g(t-s)\|\nabla u(s)-\nabla u_m(s)\|_2 ds + C\int_0^t g(s)ds\|\nabla u(t)-\nabla u_m(t)\|_2 \to 0, \qquad (3.46)
\end{aligned}
$$

as $m \to \infty$. Taking into account the nonlinear term of the functional $J(u)$, we deduce

$$
\begin{aligned}
&\|u_m\|_{p+1}^{p+1} - \|u\|_{p+1}^{p+1} \\
&\leq (p+1)\Big|\int_\Omega |u+\theta_m u_m|^{p-1}(u+\theta_m u_m)(u_m-u)dx\Big| \\
&\leq (p+1)\|u+\theta_m u_m\|_{p+1}^{p}\|u_m-u\|_{p+1} \\
&\leq C\|u_m-u\|_{p+1} \to 0, \qquad (3.47)
\end{aligned}
$$

as $m \to \infty$, where $0<\theta_m<1$. Hence, we have

$$
\lim_{m\to\infty}(g\circ\nabla u_m)(t) = (g\circ\nabla u)(t), \quad \lim_{m\to\infty}\|u_m\|_{p+1}^{p+1} = \|u\|_{p+1}^{p+1}. \qquad (3.48)
$$

From (3.5),(3.6), it follows that $E_m(0) \to E(0)$ as $m \to \infty$. Therefore, making use of Fatou's Lemma and (3.14), we deduce

$$
\begin{aligned}
&\frac{1}{\rho+1}\|u'\|_{\rho+1}^{\rho+1} + \frac{1}{2}l\|\nabla u\|_2^2 \\
&\leq \liminf_{m\to\infty}\Big[\frac{1}{\rho+1}\|u_m\|_{\rho+1}^{\rho+1} + \frac{1}{2}l(t)\|\nabla u_m\|_2^2\Big] \\
&= \liminf_{m\to\infty}\Big[E_m(t) + \frac{1}{p+1}\|u_m\|_{p+1}^{p+1} - \frac{1}{2}(g\circ\nabla u_m)(t)\Big] \\
&\leq \lim_{m\to\infty}\Big[E_m(0) + \frac{1}{p+1}\|u_m\|_{p+1}^{p+1} - \frac{1}{2}(g\circ\nabla u_m)(t)\Big] \\
&= E(0) + \frac{1}{p+1}\|u\|_{p+1}^{p+1} - \frac{1}{2}(g\circ\nabla u)(t). \qquad (3.49)
\end{aligned}
$$

which yields (2.13). Thus, we obtain that u is a global weak solution of problem (1.1). Then, making use of Lemma 3 (1) again, we get $u(t) \in W$ for $0 \leq t < \infty$. Finally, taking $m \to \infty$ in (3.17), we deduce that the energy identity (3.1) also holds for $0 \leq t < \infty$.

4. Decay estimate

In this section, we shall prove the energy decay estimate of the global solutions obtained in the previous section by making use of the perturbed energy method introduced by Cavalcanti et al.[16,18,23], Messaoudi and Tatar [14,24] and Liu [22] coupled with some new technical Lemmas.

Theorem 6. *Let the assumptions* $(A1)-(A3)$ *hold,* $u_0(x) \in H^1_{\Gamma_0}(\Omega)$, $u_1(x) \in L^{\rho+1}(\Omega) \cap L^{q+1}(\Gamma_1)$. *Further assume that* $1 < \rho < \infty$ *if* $n \le 2$, $1 < \rho \le \frac{n+2}{n-2}$ *if* $n \ge 3$, $E(0) < \tilde{d}$ *and* $I(u_0) > 0$, *then for each* $t_0 > 0$, *there exist two positive constants* L *and* η *such that the solutions of the problem* (1.1) *satisfies*

$$E(t) \le Le^{-\eta \int_{t_0}^{t} \xi(s)ds}, \quad t \ge t_0.$$

For this purpose, we introduce the functional

$$F(t) = ME(t) + \varepsilon\Psi(t) + \Phi(t), \tag{4.1}$$

where ε, M are positive constants which shall be determined later, and

$$\Psi(t) = \frac{\xi(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t u dx, \tag{4.2}$$

$$\Phi(t) = -\frac{\xi(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t \int_0^t g(t-s)[u(t)-u(s)]dsdx. \tag{4.3}$$

Remark 2. This functional was first introduced in [14] but choose $\xi(t) \equiv 1$ and in [22,24] for $\xi(t) \not\equiv 1$. Here, we can choose ε sufficiently small and M sufficiently large (if needed) in (4.1) so that $F(t) \sim E(t)$.

Firstly, we state several Lemmas to prove the decay rate estimate of the energy.

Lemma 7. *Let* $u \in L^\infty(0,\infty; H^1_{\Gamma_0}(\Omega))$ *be the solution of* (1.1) *and* $E(0) < \tilde{d}$, $I(u_0) > 0$, *then we have*

$$\int_\Omega \left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^{\rho+1} dx \le B^{\rho+1}_{\rho+1}(1-b)^\rho \left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}} (g\circ\nabla u)(t), \tag{4.4}$$

where $B_{\rho+1}$ *is the optimal constant satisfying the Sobolev inequality* $\|u\|_{\rho+1} \le B_{\rho+1}\|\nabla u\|_2$.

Proof. From $E(0) < \tilde{d}$, $I(u_0) > 0$ and Lemma 3 (1), we can obtain $u(t) \in W$ for $0 \le t < \infty$. Thus we have

$$\begin{aligned}
&\frac{1}{\rho+1}\|u'\|^{\rho+1}_{\rho+1} + \frac{p-1}{2(p+1)}[l(t)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)] \\
\le& \frac{1}{\rho+1}\|u'\|^{\rho+1}_{\rho+1} + \frac{p-1}{2(p+1)}[l(t)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)] + \frac{1}{p+1}I(u)
\end{aligned}$$

$$= \frac{1}{\rho+1}\|u'\|_{\rho+1}^{\rho+1} + J(u)$$
$$= E(t) \le E(0) < \tilde{d}. \tag{4.5}$$

Taking the Hölder inequality and (4.5) into account, we have

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)[u(t)-u(s)]ds \right)^{\rho+1} dx \\
&= \int_\Omega \left(\int_0^t [g(t-s)]^{\frac{\rho}{\rho+1}} [g(t-s)]^{\frac{1}{\rho+1}} [u(t)-u(s)]ds \right)^{\rho+1} dx \\
&\le (\int_0^t g(s)ds)^\rho \int_0^t g(t-s) \int_\Omega |u(t)-u(s)|^{\rho+1} dxds \\
&\le (1-l(t))^\rho B_{\rho+1}^{\rho+1} \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|_2^{\rho+1} ds \\
&\le B_{\rho+1}^{\rho+1}(1-b)^\rho \left(\frac{4(p+1)E(0)}{(p-1)b} \right)^{\frac{\rho-1}{2}} (g \circ \nabla u)(t).
\end{aligned} \tag{4.6}$$

Lemma 8. *For $\varepsilon > 0$ is small enough and $M > 0$ is large enough, the inequality*

$$C_1 F(t) \le E(t) \le C_2 F(t) \tag{4.7}$$

holds for two positive constants C_1 and C_2.

Proof. By using Young inequality, Sobolev embedding theorem and (4.5), we deduce that

$$\begin{aligned}
&|\frac{1}{\rho} \int_\Omega |u_t|^{\rho-1} u_t u dx| \\
&\le \frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{1}{\rho(\rho+1)}\|u\|_{\rho+1}^{\rho+1} \\
&\le \frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}\|\nabla u\|_2^{\rho+1} \\
&\le \frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)} \left(\frac{2(p+1)E(0)}{(p-1)b} \right)^{\frac{\rho-1}{2}} \|\nabla u\|_2^2.
\end{aligned} \tag{4.8}$$

From the Young inequality, (4.5) and lemma 5, we get that

$$\begin{aligned}
&|\frac{1}{\rho} \int_\Omega |u_t|^{\rho-1} u_t \int_0^t g(t-s)[u(t)-u(s)]dsdx| \\
&\le \frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{1}{\rho(\rho+1)} \int_\Omega \left(\int_0^t g(t-s)[u(t)-u(s)]ds \right)^{\rho+1} dx
\end{aligned}$$

$$\leq \frac{1}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} + \frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}(1-b)^{\rho}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}(g\circ\nabla u)(t). \tag{4.9}$$

Considering the expressions of $F(t)$, $E(t)$, $\Psi(t)$, $\Phi(t)$ and the conditions $(A2)$, it follows that

$$\begin{aligned}
F(t) \leq{} & ME(t) + \left(\frac{1}{\rho+1} + \frac{\varepsilon}{\rho+1}\right)\xi(t)\|u_t\|_{\rho+1}^{\rho+1} + \varepsilon\frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(t)\|\nabla u\|_2^2 \\
& + \frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}(1-l(t))^{\rho}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(t)(g\circ\nabla u)(t) \\
\leq{} & ME(t) + \left(\frac{1}{\rho+1} + \frac{\varepsilon}{\rho+1}\right)\xi(0)\|u_t\|_{\rho+1}^{\rho+1} + \varepsilon\frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(0)\|\nabla u\|_2^2 \\
& + \frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}(1-b)^{\rho}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(0)(g\circ\nabla u)(t) \\
\leq{} & \frac{1}{C_1}E(t),
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
F(t) \geq{} & ME(t) - \left(\frac{1}{\rho+1} + \frac{\varepsilon}{\rho+1}\right)\xi(0)\|u_t\|_{\rho+1}^{\rho+1} - \varepsilon\frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(0)\|\nabla u\|_2^2 \\
& - \frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}(1-l(t))^{\rho}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(0)(g\circ\nabla u)(t) \\
\geq{} & \left[\frac{M}{\rho+1} - \left(\frac{1}{\rho+1} + \frac{\varepsilon}{\rho+1}\right)\xi(0)\right]\|u_t\|_{\rho+1}^{\rho+1} \\
& + \left[\frac{M(p-1)b}{2(p+1)} - \varepsilon\frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(0)\right]\|\nabla u\|_2^2 \\
& + \left[\frac{M(p-1)}{2(p+1)} - \frac{B_{\rho+1}^{\rho+1}}{\rho(\rho+1)}(1-b)^{\rho}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(0)\right](g\circ\nabla u)(t) \\
\geq{} & \frac{1}{C_2}E(t),
\end{aligned} \tag{4.11}$$

where $\varepsilon > 0$ is small enough and $M > 0$ is large enough.

Lemma 9. *Let the assumptions* $(A1)$*-*$(A3)$ *hold and* $1 < \rho < \infty$ *if* $n \leq 2$, $1 < \rho \leq \frac{n+2}{n-2}$ *if* $n \geq 3$. *Furthermore assume that* $E(0) < \tilde{d}$ *and* $I(u_0) > 0$, *then the functional* $\Psi(t) = \frac{\xi(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t u\,dx$ *satisfies the following inequality*

$$\Psi'(t) \leq \left[\frac{1}{\rho} + \frac{1}{(\rho+1)\rho\alpha}k\right]\xi(t)\|u_t\|_{\rho+1}^{\rho+1} - \left[\frac{b}{2} - \frac{\alpha}{\rho+1}kB_{\rho+1}^{\rho+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\right.$$

$$- \frac{q\alpha}{q+1} B_{q+1}^{q+1} \left(\frac{2(p+1)E(0)}{(p-1)b} \right)^{\frac{q-1}{2}} \Bigg] \xi(t) \|\nabla u\|_2^2 + \frac{1-b}{2b} \xi(t)(g \circ \nabla u)(t)$$
$$+ \xi(t)\|u\|_{p+1}^{p+1} + \frac{1}{(q+1)\alpha} \xi(t)\|u_t\|_{\Gamma_1,q+1}^{q+1}. \tag{4.12}$$

Proof. By using the equation of (1.1), we deduce that

$$\begin{aligned}\Psi'(t) &= \frac{\xi(t)}{\rho}\|u_t\|_{\rho+1}^{\rho+1} + \xi(t)\int_\Omega |u_t|^{\rho-1}u_{tt}udx + \frac{\xi'(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t u dx \\ &= \frac{\xi(t)}{\rho}\|u_t\|_{\rho+1}^{\rho+1} - \xi(t)\|\nabla u\|_2^2 + \xi(t)\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)dsdx \\ &\quad + \xi(t)\|u\|_{p+1}^{p+1} + \frac{\xi'(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t u dx - \xi(t)\int_{\Gamma_1} u|u_t|^{q-1}u_t d\Gamma. \end{aligned} \tag{4.13}$$

From the Young inequality and the fact that $\int_0^t g(s)ds \le \int_0^\infty g(s)ds = 1-b$, we have

$$\begin{aligned} &\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)dsdx \\ &\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)(|\nabla u(s)-\nabla u(t)| + |\nabla u(t)|)ds\right)^2 dx \\ &\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(1+\eta)\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)|ds\right)^2 dx \\ &\quad + \frac{1}{2}(1+\frac{1}{\eta})\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\right)^2 dx \\ &\le \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}(1+\eta)(1-b)^2\|\nabla u\|_2^2 + \frac{1}{2}(1+\frac{1}{\eta})(1-b)(g\circ\nabla u)(t) \end{aligned} \tag{4.14}$$

for any $\eta > 0$. We choose $\eta = \frac{b}{1-b}$, then (4.14) yields

$$\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)dsdx \le \frac{2-b}{2}\|\nabla u\|_2^2 + \frac{1-b}{2b}(g\circ\nabla u)(t). \tag{4.15}$$

Applying the Young inequality, Hölder inequality and (4.5), it is easy to see that

$$\begin{aligned} &\int_\Omega |u_t|^{\rho-1}u_t u dx \\ &\le \|u_t\|_{\rho+1}^{\rho}\|u\|_{\rho+1} \le \frac{\rho\alpha}{\rho+1}\|u\|_{\rho+1}^{\rho+1} + \frac{1}{(\rho+1)\alpha}\|u_t\|_{\rho+1}^{\rho+1} \\ &\le \frac{\rho\alpha}{\rho+1}B_{\rho+1}^{\rho+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\|\nabla u\|_2^2 + \frac{1}{(\rho+1)\alpha}\|u_t\|_{\rho+1}^{\rho+1}, \end{aligned} \tag{4.16}$$

for any $\alpha > 0$. By the Young inequality, trace theorem and (4.5), it follows that

$$\int_{\Gamma_1} |u_t|^{q-1}u_t u d\Gamma$$

$$\leq \frac{q\alpha}{q+1}\|u\|_{\Gamma_1,q+1}^{q+1} + \frac{1}{(q+1)\alpha}\|u_t\|_{\Gamma_1,q+1}^{q+1}$$
$$\leq \frac{q\alpha}{q+1}B_{q+1}^{q+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}}\|\nabla u\|_2^2 + \frac{1}{(q+1)\alpha}\|u_t\|_{\Gamma_1,q+1}^{q+1}. \quad (4.17)$$

where B_{q+1} is the optimal constant satisfying the inequality $\|u\|_{\Gamma_1,q+1} \leq B_{q+1}\|\nabla u\|_2$. Inserting (4.15)-(4.17) into (4.13) and applying the conditions (A2), we deduce that

$$\begin{aligned}\Psi'(t) \leq& \frac{\xi(t)}{\rho}\|u_t\|_{\rho+1}^{\rho+1} - \xi(t)\|\nabla u\|_2^2 + \frac{2-b}{2}\xi(t)\|\nabla u\|_2^2 + \frac{1-b}{2b}\xi(t)(g\circ\nabla u)(t) + \xi(t)\|u\|_{p+1}^{p+1}\\
&+ \frac{\alpha}{\rho+1}k\xi(t)B_{\rho+1}^{\rho+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\|\nabla u\|_2^2 + \frac{1}{(q+1)\alpha}\xi(t)\|u_t\|_{\Gamma_1,q+1}^{q+1}\\
&+ \frac{1}{(\rho+1)\rho\alpha}k\xi(t)\|u_t\|_{\rho+1}^{\rho+1} + \frac{q\alpha}{q+1}B_{q+1}^{q+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}}\xi(t)\|\nabla u\|_2^2\\
=& \left[\frac{1}{\rho} + \frac{1}{(\rho+1)\rho\alpha}k\right]\xi(t)\|u_t\|_{\rho+1}^{\rho+1} - \left[\frac{b}{2} - \frac{\alpha}{\rho+1}kB_{\rho+1}^{\rho+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\right.\\
&\left.- \frac{q\alpha}{q+1}B_{q+1}^{q+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}}\right]\xi(t)\|\nabla u\|_2^2 + \frac{1-b}{2b}\xi(t)(g\circ\nabla u)(t)\\
&+ \xi(t)\|u\|_{p+1}^{p+1} + \frac{1}{(q+1)\alpha}\xi(t)\|u_t\|_{\Gamma_1,q+1}^{q+1}.\end{aligned}$$

Lemma 10. *Let the assumptions (A1)-(A3) hold and* $1 < \rho < \infty$ *if* $n \leq 2$, $1 < \rho \leq \frac{n+2}{n-2}$ *if* $n \geq 3$. *Furthermore assume that* $E(0) < \tilde{d}$ *and* $I(u_0) > 0$, *then the functional* $\Phi(t) = -\frac{\xi(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx$ *satisfies the following inequality*

$$\begin{aligned}\Phi'(t) \leq& \delta\left[1 + 2(1-b)^2 + \frac{p}{p+1}B_{p+1}^{p+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{p-1}{2}}\right]\xi(t)\|\nabla u\|_2^2\\
&+ \left[\frac{\delta}{\rho+1} + \frac{k\delta}{\rho+1} - \frac{\int_0^t g(s)ds}{\rho}\right]\xi(t)\|u_t\|_{\rho+1}^{\rho+1} + \left[(2\delta + \frac{1}{2\delta})(1-b)\right.\\
&+ \frac{B_{p+1}^{p+1}(1-b)^p}{(p+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{p-1}{2}} + \frac{B_{q+1}^{q+1}(1-b)^q}{(q+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}}\\
&\left.+ \frac{kB_{\rho+1}^{\rho+1}(1-b)^\rho}{\rho(\rho+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\right]\xi(t)(g\circ\nabla u)(t) + \frac{q\delta}{q+1}\xi(t)\|u_t\|_{\Gamma_1,q+1}^{q+1}\\
&- \frac{g(0)^\rho B_{\rho+1}^{\rho+1}}{\rho(\rho+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(t)(g'\circ\nabla u)(t). \quad (4.18)\end{aligned}$$

Proof. Applying the equation of (1.1) and integrating by parts, we deduce that

$$\Phi'(t) = -\xi(t)\int_\Omega |u_t|^{\rho-1}u_{tt}\int_0^t g(t-s)[u(t)-u(s)]dsdx$$

$$
\begin{aligned}
&-\frac{\xi'(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx\\
&-\frac{\xi(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t\int_0^t g'(t-s)[u(t)-u(s)]dsdx\\
&-\frac{\xi(t)}{\rho}\int_0^t g(s)ds\int_\Omega |u_t|^{\rho+1}dx\\
=&\,\xi(t)\int_\Omega \nabla u(t)\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx\\
&-\xi(t)\int_\Omega\int_0^t g(t-s)\nabla u(s)ds\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx\\
&-\xi(t)\int_\Omega |u|^{p-1}u\int_0^t g(t-s)[u(t)-u(s)]dsdx\\
&+\xi(t)\int_{\Gamma_1} |u_t|^{q-1}u_t\int_0^t g(t-s)[u(t)-u(s)]dsd\Gamma\\
&-\frac{\xi'(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx\\
&-\frac{\xi(t)}{\rho}\int_\Omega |u_t|^{\rho-1}u_t\int_0^t g'(t-s)[u(t)-u(s)]dsdx\\
&-\frac{\xi(t)}{\rho}\int_0^t g(s)ds\int_\Omega |u_t|^{\rho+1}dx. \qquad (4.19)
\end{aligned}
$$

From the Young inequality and Hölder inequality, for any $\delta > 0$, we have

$$
\int_\Omega \nabla u(t)\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx \le \delta\|\nabla u\|_2^2+\frac{1-b}{4\delta}(g\circ\nabla u)(t). \qquad (4.20)
$$

By the calculation similar to (4.14), we get

$$
\begin{aligned}
&-\int_\Omega\int_0^t g(t-s)\nabla u(s)ds\int_0^t g(t-s)[\nabla u(t)-\nabla u(s)]dsdx\\
\le&\,\delta\int_\Omega\left(\int_0^t g(t-s)[|\nabla u(s)-\nabla u(t)|+|\nabla u(t)|]ds\right)^2dx\\
&+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2dx\\
\le&\,(2\delta+\frac{1}{4\delta})\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\right)^2dx+2\delta(1-b)^2\|\nabla u\|_2^2\\
\le&\,(2\delta+\frac{1}{4\delta})(1-b)(g\circ\nabla u)(t)+2\delta(1-b)^2\|\nabla u\|_2^2. \qquad (4.21)
\end{aligned}
$$

Using the Young inequality and Sobolev inequality, it follows that

$$
-\int_\Omega |u|^{p-1}u\int_0^t g(t-s)[u(t)-u(s)]dsdx
$$

$$\leq \frac{p\delta}{p+1}\|u\|_{p+1}^{p+1}+\frac{1}{(p+1)\delta}\int_\Omega\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^{p+1}dx$$
$$\leq \frac{p\delta}{p+1}\|u\|_{p+1}^{p+1}+\frac{B_{p+1}^{p+1}(1-b)^p}{(p+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{p-1}{2}}(g\circ\nabla u)(t). \tag{4.22}$$

Taking the Young inequality and trace theorem into account, we deduce

$$\int_{\Gamma_1}|u_t|^{q-1}u_t\int_0^t g(t-s)[u(t)-u(s)]dsd\Gamma$$
$$\leq \frac{q\delta}{q+1}\|u_t\|_{\Gamma_1,q+1}^{q+1}+\frac{1}{(q+1)\delta}\int_{\Gamma_1}\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^{q+1}d\Gamma$$
$$\leq \frac{q\delta}{q+1}\|u_t\|_{\Gamma_1,q+1}^{q+1}+\frac{B_{q+1}^{q+1}(1-b)^q}{(q+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}}(g\circ\nabla u)(t). \tag{4.23}$$

Making use of the Young inequality and lemma 5, we have

$$-\frac{1}{\rho}\int_\Omega|u_t|^{\rho-1}u_t\int_0^t g(t-s)[u(t)-u(s)]dsdx$$
$$\leq \frac{\delta}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1}+\frac{1}{\rho(\rho+1)\delta}\int_\Omega\left(\int_0^t g(t-s)[u(t)-u(s)]ds\right)^{\rho+1}dx$$
$$\leq \frac{\delta}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1}+\frac{B_{\rho+1}^{\rho+1}(1-b)^\rho}{\rho(\rho+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}(g\circ\nabla u)(t). \tag{4.24}$$

Furthermore, similar calculation give that

$$-\frac{1}{\rho}\int_\Omega|u_t|^{\rho-1}u_t\int_0^t g'(t-s)[u(t)-u(s)]dsdx$$
$$\leq \frac{\delta}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1}+\frac{1}{\rho(\rho+1)\delta}\int_\Omega\left(\int_0^t g'(t-s)[u(t)-u(s)]ds\right)^{\rho+1}dx$$
$$\leq \frac{\delta}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1}+\frac{g(0)^\rho B_{\rho+1}^{\rho+1}}{\rho(\rho+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}(-g'\circ\nabla u)(t). \tag{4.25}$$

Finally, Inserting (4.20)-(4.25) into (4.19) and applying the conditions (A2), we can obtain that the conclusion of Lemma holds.

Now, we are ready to give the proof of the Theorem 2.

Proof. Since the function g is positive, continuous and $g(0)>0$, for any $t_0>0$ we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0,\quad \forall\ t\geq t_0. \tag{4.26}$$

Combining (4.1),(4.12),(4.18) and lemma 4, by a series of computations, we have that

$$F'(t)\leq ME'(t)+\varepsilon\left[\frac{1}{\rho}+\frac{k}{(\rho+1)\rho\alpha}\right]\xi(t)\|u_t\|_{\rho+1}^{\rho+1}-\varepsilon\left[\frac{b}{2}-\frac{\alpha k B_{\rho+1}^{\rho+1}}{\rho+1}\right.$$

$$
\begin{aligned}
&\times \left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}} - \frac{q\alpha B_{q+1}^{q+1}}{q+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}}\Bigg]\xi(t)\|\nabla u\|_2^2 \\
&+\varepsilon\frac{1-b}{2b}\xi(t)(g\circ\nabla u)(t) + \varepsilon\xi(t)\|u\|_{p+1}^{p+1} + \varepsilon\frac{1}{(q+1)\alpha}\xi(t)\|u_t\|_{\Gamma_1,q+1}^{q+1} \\
&+\delta\left[1+2(1-b)^2 + \frac{p}{p+1}B_{p+1}^{p+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{p-1}{2}}\right]\xi(t)\|\nabla u\|_2^2 \\
&+\left[\frac{\delta}{\rho+1} + \frac{k\delta}{\rho+1} - \frac{\int_0^t g(s)ds}{\rho}\right]\xi(t)\|u_t\|_{\rho+1}^{\rho+1} + \Bigg[(2\delta+\frac{1}{2\delta})(1-b) \\
&+\frac{B_{p+1}^{p+1}(1-b)^p}{(p+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{p-1}{2}} + \frac{B_{q+1}^{q+1}(1-b)^q}{(q+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}} \\
&+\frac{k}{\rho(\rho+1)\delta}B_{\rho+1}^{\rho+1}(1-b)^\rho\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\Bigg]\xi(t)(g\circ\nabla u)(t) \\
&-\frac{g(0)^\rho B_{\rho+1}^{\rho+1}}{\rho(\rho+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(t)(g'\circ\nabla u)(t) + \frac{q\delta}{q+1}\xi(t)\|u_t\|_{\Gamma_1,q+1}^{q+1} \\
\le& -\Bigg\{\varepsilon\left[\frac{b}{2} - \frac{\alpha k B_{\rho+1}^{\rho+1}}{\rho+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}} - \frac{q\alpha B_{q+1}^{q+1}}{q+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}}\right] \\
&-\delta\left[1+2(1-b)^2+\frac{p}{p+1}B_{p+1}^{p+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{p-1}{2}}\right]\Bigg\}\xi(t)\|\nabla u\|_2^2 \\
&-\left\{\frac{M}{2} - \frac{1}{\rho(\rho+1)\delta}g(0)^\rho B_2^{\rho+1}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\xi(0)\right\}(-g'\circ\nabla u)(t) \\
&+\Bigg\{\varepsilon\frac{1-b}{2b} + \Bigg[(2\delta+\frac{1}{2\delta})(1-b) + \frac{B_{p+1}^{p+1}(1-b)^p}{(p+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{p-1}{2}} \\
&+\frac{B_{q+1}^{q+1}(1-b)^q}{(q+1)\delta}\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}} + \frac{kB_2^{\rho+1}(1-b)^\rho}{\rho(\rho+1)\delta} \\
&\times\left(\frac{4(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}}\Bigg]\Bigg\}\xi(t)(g\circ\nabla u)(t) + \varepsilon\xi(t)\|u\|_{p+1}^{p+1} \\
&-\left\{M - \varepsilon\frac{1}{(q+1)\alpha}\xi(0) - \frac{q\delta}{q+1}\xi(0)\right\}\|u_t\|_{\Gamma_1,q+1}^{q+1} \\
&-\left\{\frac{g_0}{\rho} - \varepsilon\left[\frac{1}{\rho}+\frac{k}{(\rho+1)\rho\alpha}\right] - \left[\frac{\delta}{\rho+1}+\frac{k\delta}{\rho+1}\right]\right\}\xi(t)\|u_t\|_{\rho+1}^{\rho+1},
\end{aligned}
\tag{4.27}
$$

for all $t \ge t_0$. At this point, we choose $\alpha > 0$ so small that

$$
\frac{b}{2} - \frac{\alpha k B_{\rho+1}^{\rho+1}}{\rho+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{\rho-1}{2}} - \frac{\alpha q B_{q+1}^{q+1}}{q+1}\left(\frac{2(p+1)E(0)}{(p-1)b}\right)^{\frac{q-1}{2}} > 0. \tag{4.28}
$$

When α is fixed, we choose $\varepsilon > 0$ small enough so that lemma 9 holds and that

$$\varepsilon < \frac{g_0(\rho+1)\alpha}{(\rho+1)\alpha+k}. \tag{4.29}$$

Once α and ε are fixed, we choose a positive constant δ small enough such that

$$\begin{aligned}\varepsilon\Big[\frac{b}{2} - \frac{\alpha k B_{\rho+1}^{\rho+1}}{\rho+1}\Big(\frac{2(p+1)E(0)}{(p-1)b}\Big)^{\frac{\rho-1}{2}} - \frac{\alpha q B_{q+1}^{q+1}}{q+1}\Big(\frac{2(p+1)E(0)}{(p-1)b}\Big)^{\frac{q-1}{2}}\Big] \\ - \delta\Big[1+2(1-b)^2+\frac{pB_{p+1}^{p+1}}{p+1}\Big(\frac{2(p+1)E(0)}{(p-1)b}\Big)^{\frac{p-1}{2}}\Big] > 0,\end{aligned} \tag{4.30}$$

and

$$\frac{g_0}{\rho} - \varepsilon\Big[\frac{1}{\rho}+\frac{k}{(\rho+1)\rho\alpha}\Big] - \Big[\frac{\delta}{\rho+1}+\frac{k\delta}{\rho+1}\Big] > 0. \tag{4.31}$$

Then, we pick M sufficiently large such that lemma 9 holds and that

$$\begin{aligned}&\Big\{\frac{M}{2} - \frac{g(0)^\rho B_{\rho+1}^{\rho+1}}{\rho(\rho+1)\delta}\Big(\frac{4(p+1)E(0)}{(p-1)b}\Big)^{\frac{\rho-1}{2}}\xi(0)\Big\} - \Big\{\varepsilon\frac{1-b}{2b} + \Big[(2\delta+\frac{1}{2\delta})(1-b) \\ &+ \frac{B_{p+1}^{p+1}(1-b)^p}{(p+1)\delta}\Big(\frac{4(p+1)E(0)}{(p-1)b}\Big)^{\frac{p-1}{2}} + \frac{B_{q+1}^{q+1}(1-b)^q}{(q+1)\delta}\Big(\frac{4(p+1)E(0)}{(p-1)b}\Big)^{\frac{q-1}{2}} \\ &+ \frac{kB_{\rho+1}^{\rho+1}(1-b)^\rho}{\rho(\rho+1)\delta}\Big(\frac{4(p+1)E(0)}{(p-1)b}\Big)^{\frac{\rho-1}{2}}\Big]\Big\} > 0,\end{aligned} \tag{4.32}$$

and

$$M - \frac{\varepsilon}{(q+1)\alpha}\xi(0) - \frac{q\delta}{q+1}\xi(0) > 0. \tag{4.33}$$

Therefore, from the conditions (A2), we obtain that there exists a positive constant $\beta_1 > 0$ such that

$$\begin{aligned}&\Big\{\frac{M}{2} - \frac{g(0)^\rho B_{\rho+1}^{\rho+1}}{\rho(\rho+1)\delta}\Big(\frac{4(p+1)E(0)}{(p-1)b}\Big)^{\frac{\rho-1}{2}}\xi(0)\Big\}(-g'\circ\nabla u)(t) - \Big\{\varepsilon\frac{1-b}{2b} + \Big[(2\delta+\frac{1}{2\delta})(1-b) \\ &+ \frac{B_{p+1}^{p+1}(1-b)^p}{(p+1)\delta}\Big(\frac{4(p+1)E(0)}{(p-1)b}\Big)^{\frac{p-1}{2}} + \frac{B_{q+1}^{q+1}(1-b)^q}{(q+1)\delta}\Big(\frac{4(p+1)E(0)}{(p-1)b}\Big)^{\frac{q-1}{2}} \\ &+ \frac{kB_{\rho+1}^{\rho+1}(1-b)^\rho}{\rho(\rho+1)\delta}\Big(\frac{4(p+1)E(0)}{(p-1)b}\Big)^{\frac{\rho-1}{2}}\Big]\Big\}\xi(t)(g\circ\nabla u)(t) \\ &> \beta_1\xi(t)(g\circ\nabla u)(t).\end{aligned} \tag{4.34}$$

Combining (4.27)-(4.34), the definition of $E(t)$ and lemma 8, we deduce that there exists a positive constant $\beta_2 > 0$ such that

$$F'(t) \le -\beta_2\xi(t)E(t) \le -C_1\beta_2\xi(t)F(t), \quad \forall\, t \ge t_0. \tag{4.35}$$

A simple integration of (4.35) over (t_0, t), it follows that

$$F(t) \leq F(t_0)e^{-C_1\beta_2 \int_{t_0}^{t} \xi(s)ds}, \quad \forall\, t \geq t_0. \tag{4.36}$$

Furthermore, by lemma 6 and (4.36), we obtain

$$E(t) \leq C_2 F(t_0)e^{-C_1\beta_2 \int_{t_0}^{t} \xi(s)ds} = Le^{-\eta \int_{t_0}^{t} \xi(s)ds}, \quad \forall\, t \geq t_0. \tag{4.37}$$

where $L = C_2F(t_0)$ and $\eta = C_1\beta_2$. This completes the proof.

5. Finite time blow up of the solutions

To prove the blow up result for certain solutions with nonpositive initial energy as well as positive initial energy, we modified and improved the methods of [9,21].

Theorem 11. *Let the assumptions* $(A1)$, $(A3)$ *hold. For any fixed positive number* $\beta < 1$, *assume that* $u_0(x) \in H^1_{\Gamma_0}(\Omega)$, $u_1(x) \in L^{\rho+1}(\Omega) \cap L^{q+1}(\Gamma_1)$, *and satisfy*

$$I(u_0) < 0, \quad E(0) < \beta\tilde{d}. \tag{5.1}$$

Further assume that $\rho < p$ *and the relaxation function* g *satisfies*

$$\int_0^\infty g(s)ds < \frac{[(p-1)(1-\beta)-\gamma]^2 + 2[(p-1)(1-\beta)-\gamma]}{[(p-1)(1-\beta)-\gamma+2]^2+1}, \tag{5.2}$$

where $0 < \gamma < (p-1)(1-\beta)$. *Then, the solutions of problem* (1.1) *blows up in finite time, that is, the maximum existence time* T_{max} *of* $u(t)$ *is finite and*

$$\lim_{t \to T_{max}} \left(\|u_t\|_{\rho+1}^{\rho+1} + \|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1}\right) = +\infty. \tag{5.3}$$

First of all, we introduce the following Lemma which will be needed in the course of this section.

Lemma 12. *Let the assumptions* $(A3)$ *hold. Then there exists a positive constant* $C > 1$ *depending on* Ω *only such that*

$$\|u\|_{p+1}^{s} \leq C(\|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1}) \tag{5.4}$$

for any $u \in H^1_{\Gamma_0}(\Omega)$ *and* $2 \leq s \leq p+1$.

Now, we are ready to prove blow up result of the solutions for the problem (1.1).

Proof. In Lemma 3 (2), we have proved that if $I(u_0) < 0$ then $I(u) < 0$ for any $t \in [0, T_{\max})$ in the case of $E(0) < \beta\tilde{d}$. By contradiction, we assume that the solution of problem (1.1) is global. Then, for any $T > 0$ we may consider functional $\theta : [0, T] \to R^+$ defined by

$$\theta(t) = \|u_t\|_{\rho+1}^{\rho+1} + \|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1}. \tag{5.5}$$

As $\theta(t)$ is continuous on $[0,T]$, there exist $\delta_1, \delta_2 > 0$ such that $\delta_1 \le \theta(t) \le \delta_2$. First, we set

$$N(t) = \beta\tilde{d} - E(t) \tag{5.6}$$

for all $t \in [0,T]$. Differentiating the identity (5.6) with respect to t, we have

$$\begin{aligned} N'(t) = -E'(t) &= \|u_t\|_{\Gamma_1,q+1}^{q+1} + \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ &- \frac{1}{2}\int_\Omega \int_0^t g'(t-s)[\nabla u(s) - \nabla u(t)]^2 ds dx \ge 0. \end{aligned} \tag{5.7}$$

Hence

$$N(t) \ge N(0) = \beta\tilde{d} - E(0) > 0. \tag{5.8}$$

From the Lemma 4 and (2.1), it follows that

$$N(t) \le \beta\tilde{d} + \frac{1}{p+1}\|u\|_{p+1}^{p+1} \le \big(\frac{\beta(p-1)}{2(p+1)} + \frac{1}{p+1}\big)\|u\|_{p+1}^{p+1}, \tag{5.9}$$

for all $t \in [0,T]$. Next, we define

$$G(t) = N^{1-\sigma}(t) + \frac{\varepsilon}{\rho}\int_\Omega u|u_t|^{\rho-1}u_t dx, \quad \forall\, t \in [0,T], \tag{5.10}$$

where $0 < \varepsilon \ll 1$ to be chosen later and

$$0 < \sigma < \min\left\{\frac{1}{\rho+1}, \frac{1}{q}\right\}, \tag{5.11}$$

which will be used later. Differentiating the identity (5.10) with respect to t and using equation (1.1), we obtain

$$\begin{aligned} G'(t) &= (1-\sigma)N^{-\sigma}(t)N'(t) + \frac{\varepsilon}{\rho}\|u_t\|_{\rho+1}^{\rho+1} + \varepsilon(|u_t|^{\rho-1}u_{tt}, u) \\ &= (1-\sigma)N^{-\sigma}(t)N'(t) + \frac{\varepsilon}{\rho}\|u_t\|_{\rho+1}^{\rho+1} - \varepsilon\|\nabla u\|_2^2 + \varepsilon\|u\|_{p+1}^{p+1} \\ &+ \varepsilon\int_\Omega \nabla u(t)\int_0^t g(t-s)\nabla u(s) ds dx - \varepsilon\int_{\Gamma_1}|u_t|^{q-1}u_t u d\Gamma. \end{aligned} \tag{5.12}$$

Considering the relation

$$\begin{aligned} &(p+1-\gamma)N(t) \\ &= (p+1-\gamma)\beta\tilde{d} - \frac{p+1-\gamma}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1} - \frac{p+1-\gamma}{2}l(t)\|\nabla u\|_2^2 \\ &- \frac{p+1-\gamma}{2}(g\circ\nabla u)(t) + \frac{(p+1-\gamma)}{p+1}\|u\|_{p+1}^{p+1} \end{aligned} \tag{5.13}$$

and Young inequality

$$\int_\Omega \nabla u(t)\int_0^t g(t-s)[\nabla u(s)-\nabla u(t)]dsdx$$
$$\leq \frac{1}{4\xi}\int_0^t g(s)ds\|\nabla u(t)\|_2^2+\xi\int_0^t g(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|^2 dsdx, \tag{5.14}$$

$$\int_{\Gamma_1} |u_t|^{q-1}u_t u d\Gamma \leq \frac{\mu^{q+1}}{q+1}\|u\|_{\Gamma_1,q+1}^{q+1}+\frac{q}{q+1}\mu^{-\frac{q+1}{q}}\|u_t\|_{\Gamma_1,q+1}^{q+1}, \tag{5.15}$$

where $\gamma,\xi,\mu>0$ to be determined later, we get from (5.12) that

$$\begin{aligned}
G'(t) &= (1-\sigma)N^{-\sigma}(t)N'(t)+\varepsilon(p+1-\gamma)N(t)-\varepsilon(p+1-\gamma)\beta\tilde{d}+\frac{\varepsilon}{\rho}\|u_t\|_{\rho+1}^{\rho+1}\\
&+\varepsilon\frac{p+1-\gamma}{\rho+1}\|u_t\|_{\rho+1}^{\rho+1}+\varepsilon\frac{p+1-\gamma}{2}l(t)\|\nabla u\|_2^2+\varepsilon\frac{p+1-\gamma}{2}(g\circ\nabla u)(t)\\
&-\varepsilon\frac{(p+1-\gamma)}{p+1}\|u\|_{p+1}^{p+1}-\varepsilon\|\nabla u\|_2^2+\varepsilon\int_\Omega\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx+\varepsilon\|u\|_{p+1}^{p+1}\\
&-\varepsilon\frac{\mu^{q+1}}{q+1}\|u\|_{\Gamma_1,q+1}^{q+1}-\varepsilon\frac{q}{q+1}\mu^{-\frac{q+1}{q}}\|u_t\|_{\Gamma_1,q+1}^{q+1}\\
&\geq\left[(1-\sigma)N^{-\sigma}(t)-\varepsilon\frac{q}{q+1}\mu^{-\frac{q+1}{q}}\right]\|u_t\|_{\Gamma_1,q+1}^{q+1}+\varepsilon(p+1-\gamma)N(t)\\
&+\varepsilon\left[(\frac{p+1-\gamma}{2}-1)-(\frac{p+1-\gamma}{2}+\frac{1}{4\xi})\int_0^t g(s)ds\right]\|\nabla u\|_2^2\\
&+\varepsilon\left[\frac{1}{\rho}+\frac{p+1-\gamma}{\rho+1}\right]\|u_t\|_{\rho+1}^{\rho+1}+\varepsilon\left[\frac{p+1-\gamma}{2}-\xi\right](g\circ\nabla u)(t)-\varepsilon(p+1-\gamma)\beta\tilde{d}\\
&+\varepsilon\frac{\gamma}{p+1}\|u\|_{p+1}^{p+1}-\varepsilon\frac{\mu^{q+1}}{q+1}\|u\|_{\Gamma_1,q+1}^{q+1}.
\end{aligned} \tag{5.16}$$

As in [9], we take $\mu^{-\frac{q+1}{q}}=DN^{-\sigma}(t)$ for some large D to be specified later and if this is substituted in (5.16), we have

$$\begin{aligned}
G'(t) &\geq\left[(1-\sigma)-\varepsilon\frac{q}{q+1}D\right]N^{-\sigma}(t)\|u_t\|_{\Gamma_1,q+1}^{q+1}+\varepsilon\left[\frac{1}{\rho}+\frac{p+1-\gamma}{\rho+1}\right]\|u_t\|_{\rho+1}^{\rho+1}\\
&+\varepsilon\left[(p+1-\gamma)N(t)-\frac{D^{-q}}{q+1}N^{\sigma q}(t)\|u\|_{\Gamma_1,q+1}^{q+1}\right]-\varepsilon(p+1-\gamma)\beta\tilde{d}\\
&+\varepsilon\left[(\frac{p+1-\gamma}{2}-1)-(\frac{p+1-\gamma}{2}+\frac{1}{4\xi})\int_0^t g(s)ds\right]\|\nabla u\|_2^2\\
&+\varepsilon\left[\frac{p+1-\gamma}{2}-\xi\right](g\circ\nabla u)(t)+\varepsilon\frac{\gamma}{p+1}\|u\|_{p+1}^{p+1}.
\end{aligned} \tag{5.17}$$

Taking into account the Lemma 4, we deduce

$$-\varepsilon(p+1-\gamma)\beta\tilde{d}\geq-\varepsilon(p+1)\beta(\frac{1}{2}-\frac{1}{p+1})[l(t)\|\nabla u\|_2^2+(g\circ\nabla u)(t)]$$

$$= \frac{-\varepsilon\beta(p-1)}{2}[l(t)\|\nabla u\|_2^2 + (g\circ\nabla u)(t)]. \tag{5.18}$$

From (5.5),(5.9) and trace Theorem, we get

$$\begin{aligned} N^{\sigma q}(t)\|u\|_{\Gamma_1,q+1}^{q+1} &\le \left(\frac{\beta(p-1)+2}{2(p+1)}\right)^{\sigma q}\|u\|_{p+1}^{(p+1)\sigma q}\|u\|_{\Gamma_1,q+1}^{q+1}\\ &\le \left(\frac{\beta(p-1)+2}{2(p+1)}\right)^{\sigma q}\|u\|_{p+1}^{(p+1)\sigma q}B_{q+1}^{q+1}\|\nabla u\|_2^{q+1}\\ &\le \left(\frac{\beta(p-1)+2}{2(p+1)}\right)^{\sigma q}\|u\|_{p+1}^{(p+1)\sigma q}B_{q+1}^{q+1}\delta_2^{\frac{q+1}{2}}. \end{aligned} \tag{5.19}$$

Using the following inequality

$$z^{\theta} \le z+1 \le (1+\frac{1}{\varsigma})(z+\varsigma),\ 0<\theta\le 1,\ \ \forall\ z>0,\ \ \varsigma>0,$$

and taking $\varsigma = N(0)$, then we have

$$N^{\sigma q}(t)\|u\|_{\Gamma_1,q+1}^{q+1} \le \left(\frac{\beta(p-1)+2}{2(p+1)}\right)^{\sigma q}B_{q+1}^{q+1}\delta_2^{\frac{q+1}{2}}k\left(\|u\|_{p+1}^{(p+1)} + N(t)\right), \tag{5.20}$$

where $k = 1+\frac{1}{N(0)}$. Inserting (5.18) and (5.20) into (5.17), we obtain

$$\begin{aligned} G'(t) \ge& \left[(1-\sigma) - \varepsilon\frac{q}{q+1}D\right]N^{-\sigma}(t)\|u_t\|_{\Gamma_1,q+1}^{q+1} + \varepsilon\left[\frac{1}{\rho}+\frac{p+1-\gamma}{\rho+1}\right]\|u_t\|_{\rho+1}^{\rho+1}\\ &+\varepsilon\left[(p+1-\gamma) - \frac{D^{-q}}{q+1}\left(\frac{\beta(p-1)+2}{2(p+1)}\right)^{\sigma q}B_{q+1}^{q+1}\delta_2^{\frac{q+1}{2}}k\right]N(t)\\ &+\varepsilon\left[\frac{p-1-\gamma}{2} - \frac{\beta(p-1)}{2} - (\frac{p+1-\gamma}{2} - \frac{\beta(p-1)}{2} + \frac{1}{4\xi})\int_0^t g(s)ds\right]\|\nabla u\|_2^2\\ &+\varepsilon\left[\frac{p+1-\gamma}{2} - \frac{\beta(p-1)}{2} - \xi\right](g\circ\nabla u)(t)\\ &+\varepsilon\left[\frac{\gamma}{p+1} - \frac{D^{-q}}{q+1}\left(\frac{\beta(p-1)+2}{2(p+1)}\right)^{\sigma q}B_{q+1}^{q+1}\delta_2^{\frac{q+1}{2}}k\right]\|u\|_{p+1}^{p+1}\\ =& K_1N^{-\sigma}(t)\|u_t\|_{\Gamma_1,q+1}^{q+1} + \varepsilon\left[\frac{1}{\rho}+\frac{p+1-\gamma}{\rho+1}\right]\|u_t\|_{\rho+1}^{\rho+1}\\ &+K_2N(t) + K_3\|u\|_{p+1}^{p+1} + K_4\|\nabla u\|_2^2 + K_5(g\circ\nabla u)(t). \end{aligned} \tag{5.21}$$

Utilizing (5.2) and taking $0<\xi\le\frac{(1-\beta)(p-1)}{2}+\frac{2-\gamma}{2}$, we have

$$K_4 = \frac{(p-1)(1-\beta)-\gamma}{2} - (\frac{(1-\beta)(p-1)}{2} + \frac{2-\gamma}{2} + \frac{1}{4\xi})\int_0^t g(s)ds > 0, \tag{5.22}$$

$$K_5 = \frac{(1-\beta)(p-1)}{2} + \frac{2-\gamma}{2} - \xi \ge 0, \tag{5.23}$$

where the positive constant γ satisfies $0 < \gamma < (p-1)(1-\beta)$.
At this point, we choose D large enough such that

$$K_2 = (p+1-\gamma) - \frac{D^{-q}}{q+1}\left(\frac{\beta(p-1)+2}{2(p+1)}\right)^{\sigma q} B_{q+1}^{q+1} \delta_2^{\frac{q+1}{2}} k > 0, \tag{5.24}$$

$$K_3 = \frac{\gamma}{p+1} - \frac{D^{-q}}{q+1}\left(\frac{\beta(p-1)+2}{2(p+1)}\right)^{\sigma q} B_{q+1}^{q+1} \delta_2^{\frac{q+1}{2}} k > 0. \tag{5.25}$$

Once D is fixed, we pick ε small enough so that

$$K_1 = (1-\sigma) - \varepsilon \frac{q}{q+1} D > 0, \tag{5.26}$$

and

$$G(0) = N^{1-\sigma}(0) + \frac{\varepsilon}{\rho} \int_\Omega u_0 |u_1|^{\rho-1} u_1 dx > 0. \tag{5.27}$$

Thus, we have

$$G'(t) \geq \varepsilon\eta\big[N(t) + \|u_t\|_{\rho+1}^{\rho+1} + \|\nabla u\|_2^2 + (g \circ \nabla u)(t) + \|u\|_{p+1}^{p+1}\big] \geq 0, \tag{5.28}$$

for some small number $\eta > 0$. Consequently, from (5.27) and (5.28) we obtain

$$G(t) \geq G(0) > 0, \quad t \geq 0. \tag{5.29}$$

Now, by the Hölder inequality and Sobolev inequality, we estimate

$$\begin{aligned} \Big| \int_\Omega u|u_t|^{\rho-1}u_t \Big|^{\frac{1}{1-\sigma}} &\leq \|u_t\|_{\rho+1}^{\frac{\rho}{1-\sigma}} \|u\|_{\rho+1}^{\frac{1}{1-\sigma}} \\ &\leq C\|u_t\|_{\rho+1}^{\frac{\rho}{1-\sigma}} \|u\|_{p+1}^{\frac{1}{1-\sigma}} \\ &\leq C(\|u_t\|_{\rho+1}^{\frac{\rho\varrho}{1-\sigma}} + \|u\|_{p+1}^{\frac{\varrho'}{1-\sigma}}), \end{aligned} \tag{5.30}$$

where $\frac{1}{\varrho} + \frac{1}{\varrho'} = 1$. We choose $\varrho = \frac{(\rho+1)(1-\sigma)}{\rho} (> 1)$, then

$$\frac{\varrho'}{1-\sigma} = \frac{\rho+1}{(\rho+1)(1-\sigma)-\rho}. \tag{5.31}$$

By using lemma 9 and (5.31), then (5.30) becomes

$$\Big| \int_\Omega u|u_t|^{\rho-1}u_t \Big|^{\frac{1}{1-\sigma}} \leq C\big(\|u_t\|_{\rho+1}^{\rho+1} + \|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1}\big). \tag{5.32}$$

Hence, combining (5.10) and (5.32) , we deduce that

$$G^{\frac{1}{1-\sigma}}(t) = \big(N^{1-\sigma}(t) + \frac{\varepsilon}{\rho}\int_\Omega |u_t|^{\rho-1}u_t u dx\big)^{\frac{1}{1-\sigma}}$$

$$\leq C\big(N(t) + \|u_t\|_{\rho+1}^{\rho+1} + \|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1}\big). \tag{5.33}$$

By the combination of (5.28) and (5.33), we obtain

$$G'(t) \geq QG^{\frac{1}{1-\sigma}}(t), \quad \forall\, t \in [0,T], \tag{5.34}$$

where Q is a positive constant depending only on C and $\varepsilon\eta$.
A simple integration of (5.34) over (0,t), it follows that

$$G^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{G^{-\sigma/(1-\sigma)}(0) - Q\sigma t/(1-\sigma)}, \qquad \forall\, t \in [0,T]. \tag{5.35}$$

This shows that $G(t)$ blows up in finite time

$$T^* \leq \frac{1-\sigma}{G^{\sigma/1-\sigma}(0)Q\sigma}. \tag{5.36}$$

Furthermore, we have from (5.33) that there exists a finite time $T^* \in (0,T)$ such that

$$\lim_{t\to T^{*-}} \big(\|u_t\|_{\rho+1}^{\rho+1} + \|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1}\big) = +\infty,$$

which contradicts $T_{\max} = \infty$. Hence, the solutions of the problem (1.1) blows up in finite time.

Remark 3. Noting that from

$$\frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + \frac{p-1}{2(p+1)}\|\nabla u_0\|_2^2 + \frac{1}{p+1}I(u_0) \leq \frac{1}{\rho+1}\|u_1\|_{\rho+1}^{\rho+1} + J(u_0) = E(0), \tag{5.37}$$

we see that if $E(0) < 0$, then $I(u_0) \geq 0$ is impossible. If $E(0) = 0$, then either $I(u_0) > 0$ or $I(u_0) = 0$ with $\|\nabla u_0\|_2^2 \neq 0$ is impossible. If $0 < E(0) < \beta\tilde{d}$ ($\beta\tilde{d} < \tilde{d}$), it follows from the definition of d that $I(u_0) = 0$ with $\|\nabla u_0\|_2^2 \neq 0$ is impossible. Otherwise, we have $J(u_0) \geq d$ which contradicts (5.37). Thus, all possible cases already have been considered in Theorems 1, 3.

From the discussion above in Sections 3, 5, a threshold result of global existence and nonexistence of solutions for problem (1.1) has been obtained as follows.

Corollary 1. *Let the assumptions $\rho < p$, (A1), (A3) and* (5.2) *hold. Further assume that $u_0(x) \in H_0^1(\Omega)$, $u_1(x) \in L^{\rho+1}(\Omega) \cap L^{q+1}(\Gamma_1)$ and $E(0) < \beta\tilde{d}$ ($\beta\tilde{d} < \tilde{d}$). Then problem* (1.1) *admits a global weak solution provided $I(u_0) \geq 0$ (includes $\|\nabla u_0\|_2^2 = 0$); Problem* (1.1) *dose not admit any global solutions provided $I(u_0) < 0$.*

Remark 4. In the above threshold result stated in Corollary 1, we see that the manifold $N = \big\{u \in H_{\Gamma_0}^1(\Omega) \big|\ I(u) = 0\big\}$ plays a key role as a borderline separating region of weak solutions with the initial energy $E(0) < \beta\tilde{d}$ ($\beta\tilde{d} < \tilde{d}$) into two parts: the global existence and finite time blow up of weak solutions for the problem (1.1).

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