Modules that Have a $\delta$-supplement in Every Extension

Esra Öztürk Sözen$^{1,*}$, Şenol Eren$^1$

$^1$ Ondokuz Mayıs University, Faculty of Science and Arts
Department of Mathematics, Turkey

Abstract. Let $R$ be a ring and $M$ be a left $R$-module. In this paper, we define modules with the properties $(\delta-E)$ and $(\delta-EE)$, which are generalized version of Zöschinger’s modules with the properties $(E)$ and $(EE)$, and provide various properties of these modules. We prove that the class of modules with the property $(\delta-E)$ is closed under direct summands and finite direct sums. It is shown that a module $M$ has the property $(\delta-EE)$ if and only if every submodule of $M$ has the property $(\delta-E)$. It is a known fact that a ring $R$ is perfect if and only if every left $R$-module has the property $(E)$. As a generalization of this, we prove that if $R$ is a $\delta$-perfect ring then every left $R$-module has the property $(\delta-E)$. Moreover, the converse is also true on $\delta$-semiperfect rings.

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1. Introduction

In this paper $R$ is an associative ring with identity and all modules are unital left $R$-modules. Let $M$ be a module $X \leq M$ means that $X$ is a submodule of $M$ or $M$ is an extension of $X$. Recall that a submodule $N \leq M$ is called small, denoted by $N \ll M$, if $N + L \neq M$, for all proper submodules $L$ of $M$. We call $T$ a supplement of $N$ in $M$ if $M = T + N$ and $T \cap N$ is small in $T$. A module $M$ is called supplemented if every submodule of $M$ has a supplement in $M$ [14]. $L \leq M$ is said to be essential in $M$, denoted by $L \leq M$, if $L \cap K \neq 0$ for each nonzero submodule $K \leq M$. The singular submodule of a module $M$ (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid Ix = 0 \text{ for some ideal } I \leq R\}$. A module $M$ is called singular if $Z(M) = M$. Every submodule and every factor module of a singular module is singular. We refer to [6] for the further properties of singular modules.

In [15], Zhou introduced the concept of $\delta$-small submodules as a generalization of small submodules. A submodule $N$ of $M$ is said to be $\delta$-small in $M$ (denoted by $N \ll_{\delta} M$) if whenever $M = N + K$ and $K$ is singular, we have $M = K$. And we denote the sum of all $\delta$-small submodules of $M$ by $\delta(M)$. A submodule $L$ of $M$ is called a $\delta$-supplement of...

*Corresponding author.

Email addresses: esraozturk55@hotmail.com (E. Ö. Sözen), seren@omu.edu.tr (Ș. Eren)
$N$ in $M$ if $M = N + L$ and $N \cap L \ll \delta L$ and $M$ is called \textit{$\delta$-supplemented} in case every submodule of $M$ has a \textit{$\delta$-supplement} in $M$ \cite{7}.

For a module $M$ consider the following conditions:

- $(E) : M$ has a supplement in every extension.
- $(EE) : M$ has ample supplements in every extension.

The concept of these modules with these properties was first introduced by Zöschinger \cite{16}. Adapting his concept in \cite{4}, Çalışçı and Türkmen introduced modules with the properties $(CE)$ and $(CEE)$ as a generalization of the properties $(E)$ and $(EE)$. In addition, in \cite{9} the authors worked on modules that have a weak supplement in every extension and in \cite{5} Eryılmaz introduced modules that have a \textit{$\delta$-supplement} in every torsion extension.

In this paper we investigate the structure of modules with the properties $(\delta$-$E)$ and $(\delta$-$EE)$ as a generalization of Zöschinger’s modules with the properties $(E)$ and $(EE)$. We prove that a module has the property $(\delta$-$EE)$ if and only if every submodule has the property $(\delta$-$E)$. We show that every direct summand and $\delta$-small cover of $M$ with the property $(\delta$-$E)$ has the property $(\delta$-$E)$. Using the property $(\delta$-$E)$, we present a relation between $\delta$-perfect rings and modules with the property $(\delta$-$E)$, which are a generalization of perfect rings, that is, $R$ is a $\delta$-perfect ring, then every left $R$-module has the property $(\delta$-$E)$. Moreover we obtain that if every left $R$-module has the property $(\delta$-$E)$, then $R$ is a $\delta$-semiperfect ring.

2. Preliminaries

In this section, we begin by stating the following lemmas and theorems for the completeness.

2.1. $\delta$-Small Submodules

\textbf{Lemma 1. (\cite[Lemma 1.2]{15})}. Let $N$ be a submodule of $M$. The following are equivalent:

1. $N \ll \delta M$.
2. If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule $Y$ with \[ \begin{array}{c}
Y \subseteq N.
\end{array} \]
3. If $X + N = M$ with \[ \begin{array}{c}
\frac{M}{X} \text{ Goldie torsion, then } X = M.
\end{array} \]

\textbf{Lemma 2. (\cite[Lemma 1.3]{15})}. Let $M$ be a module.

1. For submodules $N$, $K$, $L$ of $M$ with $K \subseteq N$, we have

   (a) $N \ll \delta M$ if and only if $K \ll \delta M$ and \[ \frac{N}{K} \ll \delta \frac{M}{K}. \]
   (b) $N + L \ll \delta M$ if and only if $N \ll \delta M$ and $L \ll \delta M$.

2. If $K \ll \delta M$ and $f : M \rightarrow N$ is a homomorphism, then $f(K) \ll \delta N$.

In particular, if $K \ll \delta M \subseteq N$, then $K \ll \delta N$. 

3. Let \( K_1 \subseteq M_1 \subseteq M, K_2 \subseteq M_2 \subseteq M \) and \( M = M_1 \oplus M_2 \). Then \( K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2 \) if and only if \( K_1 \ll_\delta M_1 \) and \( K_2 \ll_\delta M_2 \).

2.2. \( \delta \)-Supplemented Modules

Lemma 3. ([7, Prop.2.7]). Let \( U \) and \( V \) be submodules of a module \( M \). Assume that \( V \) is a \( \delta \)-supplement of \( U \) in \( M \). Then

1. If \( W + V = M \) for some \( W \subseteq U \), then \( V \) is a \( \delta \)-supplement of \( W \) in \( M \).
2. If \( K \ll_\delta M \), then \( V \) is a \( \delta \)-supplement of \( U + K \) in \( M \).
3. For \( K \ll_\delta M \) we have \( K \cap V \ll_\delta V \) and so \( \delta(V) = V \cap \delta(M) \).
4. For \( L \subseteq U, \frac{V + L}{L} \) is a \( \delta \)-supplement of \( \frac{U}{L} \) in \( \frac{M}{L} \).
5. If \( \delta(M) \ll_\delta M \), or \( \delta(M) \subseteq U \) and if \( p : M \rightarrow \frac{M}{\delta(M)} \) is the canonical projection, then \( \frac{M}{\delta(M)} = p(U) \oplus p(V) \).

In [7], a projective module \( P \) is called a projective \( \delta \)-cover of a module \( M \) if there exists an epimorphism \( f : P \rightarrow M \) with \( \Ker(f) \ll_\delta M \), and a ring \( R \) is called \( \delta \)-perfect (resp., \( \delta \)-semiperfect) if every \( R \)-module (resp., every simple \( R \)-module) has a projective \( \delta \)-cover. In addition, a module \( M \) is called \( \delta \)-lifting if for any \( N \leq M \), there exists a decomposition \( M = A \oplus B \) such that \( A \leq N \) and \( N \cap B \) is \( \delta \)-small in \( B \) since \( B \) is a direct summand of \( M \).

Theorem 4. [7, Theorem3.3]. The following are equivalent for a ring \( R \):

1. \( R \) is \( \delta \)-semiperfect.
2. Every finitely generated module is \( \delta \)-supplemented.
3. Every finitely generated projective module is \( \delta \)-supplemented.
4. Every finitely generated projective module is \( \delta \)-lifting.
5. Every left ideal of \( R \) has a \( \delta \)-supplement in \( R \).

Theorem 5. [7, Theorem 3.4]. The following statements are equivalent for a ring \( R \):

1. \( R \) is \( \delta \)-perfect.
2. Every module is \( \delta \)-supplemented.
3. Every projective module is \( \delta \)-supplemented.
4. Every projective module is \( \delta \)-lifting.
3. Modules with the Properties \((\delta\text{-}E)\) and \((\delta\text{-}EE)\)

In this section, we define the concept of modules with the properties \((\delta\text{-}E)\) and \((\delta\text{-}EE)\).

**Definition 1.** A module \(M\) has the property \((\delta\text{-}E)\) if it has a \(\delta\)-supplement in each module in which it is contained as a submodule.

**Definition 2.** A module \(M\) has the property \((\delta\text{-}EE)\) if it has ample \(\delta\)-supplements in each module in which it is contained as a submodule, where \(U \leq M\) has ample \(\delta\)-supplements in \(M\) if for every \(V \leq M\) with \(U + V = M\), there is a \(\delta\)-supplement \(V'\) of \(U\) with \(V' \leq V\).

It is clear that every module with the property \((E)\) has the property \((\delta\text{-}E)\). Also there exists the same relation between modules with the properties \((EE)\) and \((\delta\text{-}EE)\). At the end of this section, we shall give an example of a module which has the property \((\delta\text{-}E)\) but not \((E)\).

Zöschinger proved in [16] that a module has the property \((EE)\) if and only if every submodule has the property \((E)\). We give an analogous characterization of our modules with the following proposition.

**Proposition 1.** A module \(M\) has the property \((\delta\text{-}EE)\) if and only if every submodule of \(M\) has the property \((\delta\text{-}E)\).

**Proof.** Let \(M\) be a module and \(N\) be any extension of \(M\). Suppose that for a submodule \(X \leq N\), \(X + M = N\). By hypothesis, the submodule \(X \cap M\) of \(M\) has a \(\delta\)-supplement \(V\) in \(X\), that is, \((X \cap M) + V = X\) and \((X \cap M) \cap V \ll_{\delta} V\). Then, \(N = M + X = M + [(X \cap M) + V] = M + V\) and \(M \cap V = M \cap (V \cap X) = (X \cap M) \cap V \ll_{\delta} V\). Hence, \(V\) is a \(\delta\)-supplement of \(M\) in \(N\) such that \(V \leq X\).

Conversely, let \(U\) be a submodule of \(M\) and \(N\) be any module containing \(U\). Then we can draw the following pushout:

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & F \\
\downarrow{i_1} & & \uparrow{\beta} \\
U & \xleftarrow{i_2} & N
\end{array}
\]

\(i_1\) and \(i_2\) are inclusion homomorphisms in this diagram. Additionally \(\alpha : M \rightarrow F\) and \(\beta : N \rightarrow F\) are monomorphisms by the properties of push out (see, for example, [11, Exercise 5.10]). Let \(\alpha(M) = M' \subseteq F\) and \(\beta(N) = N' \subseteq F\). Then it can be easily shown
that $F = M' + N'$. So by using hypothesis, $M' \cong M$ has a $\delta$-supplement $V$ in $F$ such that $V \leq N'$, that is, $M' + V = F$ and $M' \cap V \ll_\delta V$. Hence,

\[(M' \cap N') + V = (N' \cap M') + V = N' \cap (M' + V) = N' \cap F = N', \quad \text{and} \]

\[(M' \cap N') \cap V = M' \cap (N' \cap V) = M' \cap V \ll_\delta V.\]

So $V$ is a $\delta$-supplement of $M' \cap N'$ in $N'$. Now we will show that $\beta^{-1}(V)$ is a $\delta$-supplement of $U$ in $N$. We have an isomorphism $\tilde{\beta} : N \rightarrow N'$ defined as $\beta(x) = \beta(x)$ for all $x \in N$, since $\beta$ is a monomorphism. Using this, we obtain $\beta^{-1}(V)$ is a $\delta$-supplement of $\beta^{-1}(M' \cap N')$ in $\beta^{-1}(N')$ since $V$ is a $\delta$-supplement of $M' \cap N'$ in $N'$. It can be seen that $\beta^{-1}(V) = \beta^{-1}(V)$, $\beta^{-1}(N') = N$ and $\beta^{-1}(M' \cap N') = U$. Thus $\beta^{-1}(V)$ is a $\delta$-supplement of $U$ in $N$.

**Corollary 1.** A module with the property $(\delta$-$EE)$ has the property $(\delta$-$E)$ and it is also $\delta$-supplemented.

Recall that $R$ is a (right) $\delta$-$V$ ring if for any right $R$-module $M$, $\delta(M) = 0$ (see, [13]).

**Proposition 2.** Let $R$ be $\delta$-$V$ ring and $M$ be an $R$-module. Then the following statements are equivalent:

1. $M$ has the property $(\delta$-$E)$.
2. $M$ is injective.

**Proof.** (1) $\implies$ (2) : Suppose that $M$ has the property $(\delta$-$E)$. Let $N$ be any extension of $M$. So, there exists a $\delta$-supplement $V$ of $M$ in $N$, that is, $M + V = N$ and $M \cap V \ll_\delta V$ and so $M \cap V \leq \delta(V)$. Since $R$ is a $\delta$-$V$ ring, $\delta(V) = 0$. So, $N = M \oplus V$. Therefore, $M$ is injective.

(2) $\implies$ (1) : is clear.

Now we show that the property $(\delta$-$E)$ is preserved by direct summands in the following proposition:

**Proposition 3.** Every direct summand of any module with the property $(\delta$-$E)$ has the property $(\delta$-$E)$.

**Proof.** Let $M$ be a module with the property $(\delta$-$E)$, $U$ be a direct summand of $M$ and $N$ be any extension of $U$. Then there exists a submodule $A$ of $M$ such that $M = U \oplus A$. By hypothesis, $M$ has a $\delta$-supplement $V$ in $A \oplus N$ such that $(A \oplus U) + V = A \oplus N$ and $(A \oplus U) \cap V \ll_\delta V$. Let $g : A \oplus N \rightarrow N$ be the projection onto $N$. Then

\[N = g(A \oplus N) = g((A \oplus U) + V) = g(A \oplus U) + g(V) = U + g(V), \quad \text{and} \]

\[g((A \oplus U) \cap V) = U \cap g(V) \ll_\delta g(V).\]

Hence, $g(V)$ is a $\delta$-supplement of $U$ in $N$. 
Corollary 2. If $M_1$ and $M_2$ have the property $(\delta\cdot E)$, so does $M_1 \oplus M_2$.

Proof. Let $0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$ be a short exact sequence. Result follows by Proposition 4.

Proposition 5. Let $0 \to K \to M \to L \to 0$ be a short exact sequence. If $K$ and $L$ have the property $(\delta\cdot E)$, so does $M$. If the sequence splits the converse is also true.

Proof. Let $N$ be any extension of $M$. So $\frac{N}{K}$ is an extension of $\frac{M}{K}$ and is is a well known fact that $\frac{M}{K} \cong L$. Then there exists a $\delta$-supplement $\frac{V}{K}$ for $\frac{M}{K}$ in $\frac{N}{K}$, that means $\frac{M}{K} + \frac{V}{K} = \frac{N}{K}$ and $\frac{M}{K} \cap \frac{V}{K} \cong \frac{V}{K}$ for some $\frac{V}{K} \le \frac{N}{K}$. Since $K \le V$ and $K$ has the property $(\delta\cdot E)$, $V = K$ and by the modular law $(K + K') \cap M + T = V$. Following this, $V \cap M + T = V$ is obtained. It can be easily seen written that $\frac{V \cap M}{K} + \frac{T + K}{K} = \frac{V}{K}$, additionally, $\frac{V}{T + K}$ is singular since,

$$\frac{V}{T + K} = \frac{K + K'}{T + K} = \frac{T + K'}{T + K} \cong \frac{K'}{T + (K \cap K')} \le \frac{K'}{T}$$

and $\frac{M}{K} \cap \frac{V}{K} \cong \frac{V}{K}$ and of course $T + K = V$. $(T + K') \cap M + T = V$ can be seen and by the modular law, $(T + K') \cap K' = K'$ is obtained. This provides $T = K'$ since $K \cap K' \cong K'$ and $\frac{K'}{T}$ is singular. Moreover, suppose that the sequence splits, then $K$ and $L$ have the property $(\delta\cdot E)$ by corollary 2.

Corollary 3. Let $M_i (i = 1, 2, ..., n)$ be any finite collection of modules and $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. Then $M$ has the property $(\delta\cdot E)$ if and only if $M_i$ has the property $(\delta\cdot E)$ for each $i = 1, 2, ..., n$.

Proof. It can be proved easily for $n = 2$ by using the previous theorem and can be generalized on $n$.

We give the following known lemma for the completeness.
Lemma 6. Every simple submodule $S$ of a module $M$ is either a direct summand of $M$ or small in $M$ (see in [10])

Proposition 6. Every simple module has the property $(\delta\cdot E)$.

Proof. Let $S$ be a simple module and $N$ be any extension of $S$. Then by Lemma 4, $S \ll N$ and so $S \ll_\delta N$ or $S \oplus S' = N$ for a submodule $S' \leq N$. If $S \ll_\delta N$, then $N$ is a $\delta$-supplement of $S$ in $N$ or if $S$ is a direct summand of $N$ then $S'$ is a $\delta$-supplement of $S$ in $N$. So in each case $S$ has a $\delta$-supplement in $N$. This means that $S$ has the property $(\delta\cdot E)$.

Theorem 7. Every module with composition series has the property $(\delta\cdot E)$.

Proof. Let $0 = M_0 \leq M_1 \leq \ldots \leq M_{n-1} \leq M_n = M$ be any composition series of a module $M$. We shall prove the theorem by induction on $n \in \mathbb{N}$. If $n = 1$, then $M = M_1$ is simple, and so $M$ has the property $(\delta\cdot E)$ by Proposition 6. Assume that this is true for each $k \leq n - 1$. Then $M_{n-1}$ has the property $(\delta\cdot E)$. Since $M_{n-1}$ has the property $(\delta\cdot E)$ as a simple module, $M$ has the property $(\delta\cdot E)$ by Proposition 4.

Corollary 4. A finitely generated semisimple module has the property $(\delta\cdot E)$.

In the following proposition we will prove that modules with the property $(\delta\cdot E)$ are closed under factor modules, under a special condition.

Proposition 7. Let $A \leq B \leq C$ with $C_A$ injective. If $B$ has the property $(\delta\cdot E)$, so does $B_A$.

Proof. Let $N$ be any extension of $B_A$. So we have the following commutative diagram with exact rows since $C_A$ is injective, (see in [10]).

Since $h$ is monic and $B$ has the property $(\delta\cdot E)$, $B \cong h(B)$ has a $\delta$-supplement $V$ in $P$, that is, $h(B) + V = P$ and $h(B) \cap V \ll_\delta V$. We claim that $g(V)$ is a $\delta$-supplement of $B_A$ in $N$.

$$
\frac{B}{A} + g(V) = (f\sigma)(B) + g(V) = g(h(B)) + g(V) = g(P) = N, \text{ and } \frac{B}{A} \cap g(V) = f(\sigma(B)) \cap g(V) = g[h(B) \cap N] \ll_\delta g(V)
$$
since \( h(B) \cap V \ll_\delta V \) and \( g \) is a homomorphism.

A ring \( R \) is left perfect if and only if every left \( R \)-module has the property \((E)\) (see [16]). Now we show only one side of this fact is valid for \( \delta \)-perfect rings.

**Proposition 8.** If \( R \) is a \( \delta \)-perfect ring, then every left \( R \)-module has the property \((\delta-E)\).

**Proof.** Suppose that a ring \( R \) is \( \delta \)-perfect. Let \( M \) be an \( R \)-module and \( N \) be any extension of \( M \). \( N \) is \( \delta \)-supplemented since \( R \) is \( \delta \)-perfect. So \( M \) has a \( \delta \)-supplemented in \( N \) as a submodule of \( N \). Hence, \( M \) has the property \((\delta-E)\).

**Proposition 9.** Let \( R \) be a ring. If every left \( R \)-module has the property \((\delta-E)\), then \( R \) is a \( \delta \)-semiperfect ring.

**Proof.** Since every left \( R \)-module has the property \((\delta-E)\), every ideal of \( R \) also has the property \((\delta-E)\) as a submodule of \( R \). So every ideal of \( R \) has a \( \delta \)-supplement in \( R \). Hence \( R \) is \( \delta \)-semiperfect by [6, Theorem 3.3].

**Example 1.** Let \( F \) be a field, \[
I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}, \quad R = \{(x_1, \ldots, x_n, x, x, \ldots) \mid n \in \mathbb{N}, \ x_i \in M_2(F), \ x \in I\}
\]
with component-wise operations, \( R \) is a ring. By Example 4.3 in [15], \( R \) is a \( \delta \)-perfect ring that is not perfect. And so \( R \) is an example of a module that has the property \((\delta-E)\) but not have the property \((E)\).

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**References**


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