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Sufficient conditions for starlikeness of reciprocal order

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Abstract. The object of the present paper is to derive certain sufficient conditions for starlikeness of reciprocal order of analytic functions in the open unit disk.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike of order α if it satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{2}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathcal{U}$. Clearly $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) = \mathcal{S}^*$, where $\mathcal{S}^*$is the class of functions that are starlike in $\mathcal{U}$.

A function $f \in \mathcal{A}$ is said to be starlike of reciprocal order α if

$$\Re\left\{\frac{f(z)}{zf'(z)}\right\} > \alpha \qquad (z \in \mathcal{U}) \tag{3}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote the class of such functions by $\mathcal{S}^{-1}{}^*(\alpha)$ (see, [1, 4, 8]).

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In view of the fact that

$$\Re p(z) > 0 \Rightarrow \Re \frac{1}{p(z)} = \Re \frac{p(z)}{|p(z)|^2} > 0,$$

it follows that a starlike function of reciprocal order 0 is same as a starlike function. In particular, every starlike function of reciprocal order $\alpha \geq 0$ is starlike and hence univalent (cf. [10, Example 1]).

Example 1. *The function* $f(z) = ze^{(1-\alpha)z}$ *is a starlike function of reciprocal order* $1/(2-\alpha)$ *[10, Example 2].*

Sufficient conditions were studied by various authors for starlikeness [e.g., see [2–7, 9–12]). The object of the present paper is to derive certain sufficient conditions for starlikeness of reciprocal order α by using the same techniques as in [9].

In order to establish our main results, we require the following lemma due to Nunokawa et al. [9].

Lemma 1. *Let* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *be analytic in* $\mathcal{U}$ *and suppose that there exists a point* $z_0 \in \mathcal{U}$ *such that*

$$\Re\{p(z)\} > 0 \text{ for } |z| < |z_0| \tag{4}$$

and

$$\Re\{p(z_0)\} = 0. \tag{5}$$

Then we have

$$z_0 p^{'}(z_0) \leq -\frac{1}{2}(1 + |p(z_0)|^2), \tag{6}$$

where $z_0 p^{'}(z_0)$ *is a negative real number.*

2. Sufficient conditions for starlikeness of reciprocal order

Our first result is contained in the following.

Theorem 2. *Let* $f(z) \in \mathcal{A}$ *satisfies* $f(z)\ f^{'}(z) \neq 0$ *in* $0 < |z| < 1$ *and*

$$\Re\left\{\frac{f(z)}{zf'(z)}\left(1 - \alpha\frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{\alpha}{2}\left(3 + \left|\frac{f(z)}{zf'(z)}\right|^2\right) \qquad (z \in \mathcal{U}; \alpha \geqslant 0). \tag{7}$$

Then $f(z)$ *is starlike of reciprocal order* 0 *in* $\mathcal{U}$ *and thus,* $f(z)$ *is starlike in* $\mathcal{U}$*.*

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{f(z)}{zf'(z)}. \tag{8}$$

Then $p(z)$ is analytic in $\mathcal{U}$ and $p(0) = 1$. Differentiating (8) logarithmically we obtain

$$\frac{f(z)}{zf'(z)}\left(1 - \alpha\frac{zf''(z)}{f'(z)}\right) = \alpha zp'(z) + (\alpha+1)p(z) - \alpha. \tag{9}$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\Re\{p(z)\} > 0 \text{ for } |z| < |z_0|$$

and

$$\Re\{p(z_0)\} = 0,$$

then from Lemma 1, we have,

$$z_0p'(z_0) \le -\frac{1}{2}(1 + |p(z_0)|^2).$$

Therefore from (9),we have

$$\begin{aligned}\Re\left\{\frac{f(z_0)}{z_0f'(z_0)}\left(1 - \alpha\frac{z_0f''(z_0)}{f'(z_0)}\right)\right\} &= \Re\left\{\alpha z_0p'(z_0) + (\alpha+1)p(z_0) - \alpha\right\}. \\ &\le -\frac{\alpha}{2}\left(1 + |p(z_0)|^2\right) - \alpha \\ &\le -\frac{\alpha}{2}\left(3 + \left|\frac{f(z_0)}{z_0f'(z_0)}\right|^2\right).\end{aligned}$$

which contradicts our condition (6) of Theorem 2. Thus we complete the proof of Theorem 2.

Next, we derive the following.

Theorem 3. *Let $f(z) \in \mathcal{A}$ satisfies $f(z)\ f'(z) \ne 0$ in $0 < |z| < 1$ and*

$$\Re\left\{\frac{f(z)}{zf'(z)}\left(-1 - \frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{5}{4} - \frac{1}{4}\left|\frac{2f(z)}{zf'(z)} - 1\right|^2 \qquad (z \in \mathcal{U}).$$

Then $f(z)$ is starlike of reciprocal order $\frac{1}{2}$ in $\mathcal{U}$.

Proof. Putting

$$p(z) = 2\left(\frac{f(z)}{zf'(z)} - \frac{1}{2}\right), \tag{10}$$

then we have $p(0) = 1$. Suppose that there exists a point $z_0 \in \mathcal{U}$ satisfies the conditions (4) and (5) of Lemma 1, from (10) we have

$$\Re\left\{\frac{f(z_0)}{z_0 f'(z_0)}\left(-1 - \frac{zf''(z_0)}{f'(z_0)}\right)\right\} = \Re\left\{\frac{1}{2}z_0 p'(z_0) - 1\right\}. \tag{11}$$

Using (6) of Lemma 1 in (11), it follows that

$$\begin{aligned}\Re\left\{\frac{f(z_0)}{z_0 f'(z_0)}\left(-1 - \frac{z_0 f''(z_0)}{f'(z_0)}\right)\right\} &\leq -\frac{1}{4}\left(1 + |p(z_0)|^2\right) - 1 \\ &\leq -\frac{5}{4} - \frac{1}{4}|p(z_0)|^2 \\ &\leq -\frac{5}{4} - \frac{1}{4}\left|\frac{2f(z_0)}{z_0 f'(z_0)} - 1\right|^2.\end{aligned}$$

which contradicts the hypothesis of Theorem 3 and therefore, we have

$$\Re\{p(z)\} > 0 \qquad (z \in \mathcal{U})$$

or

$$\Re\left\{\frac{f(z)}{zf'(z)}\right\} > \frac{1}{2} \qquad (z \in \mathcal{U}).$$

Finally, we discuss the following theorem.

Theorem 4. *Let $f(z) \in \mathcal{A}$ satisfies*

$$\Re\left\{\frac{f(z)}{zf'(z)}\left(1 - \alpha\frac{zf''(z)}{f'(z)}\right)\right\} > -\frac{\alpha}{(2-\alpha)}\left|\frac{f(z)}{zf'(z)} - \frac{\alpha}{2}\right|^2 + \frac{\alpha}{4}(3\alpha - 4) \quad (z \in \mathcal{U}; 0 \leq \alpha < 2). \tag{12}$$

Then $f(z)$ is starlike of reciprocal order $\frac{\alpha}{2}$ in $\mathcal{U}$.

Proof. Let the function $p(z)$ be defined by

$$\frac{f(z)}{zf'(z)} = \left(1 - \frac{\alpha}{2}\right)p(z) + \frac{\alpha}{2},\ p(0) = 1. \tag{13}$$

Suppose that there exists a point $z_0 \in \mathcal{U}$ satisfies the conditions (4) and (5) of Lemma 1, from (13) we have

$$\begin{aligned}&\Re\left\{\frac{f(z_0)}{z_0 f'(z_0)}\left(1 - \alpha\frac{z_0 f''(z_0)}{f'(z_0)}\right)\right\} \\ =\ &\Re\left\{\alpha\left(1 - \frac{\alpha}{2}\right)z_0 p'(z_0) + (1+\alpha)\left(1 - \frac{\alpha}{2}\right)p(z_0) + \frac{\alpha}{2}(\alpha - 1)\right\}.\end{aligned} \tag{14}$$

Thus, by using (5) and (6) of Lemma 1 in (14), it follows that

$$\begin{aligned}\Re\left\{\frac{f(z_0)}{z_0 f'(z_0)}\left(1-\alpha\frac{z_0 f''(z_0)}{f'(z_0)}\right)\right\} &\leq -\frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)\left(1+|p(z_0)|^2\right)+\frac{\alpha}{2}(\alpha-1)\\ &\leq -\frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)|p(z_0)|^2+\frac{\alpha}{4}(3\alpha-4)\\ &\leq -\frac{\alpha}{(2-\alpha)}\left|\frac{f(z_0)}{z_0 f'(z_0)}-\frac{\alpha}{2}\right|^2+\frac{\alpha}{4}(3\alpha-4)\end{aligned}$$

which contradicts the hypothesis (12). It follows that

$$\Re\left\{\frac{f(z)}{zf'(z)}\right\} > \frac{\alpha}{2} \qquad (z \in \mathcal{U}).$$

Thus proof of the Theorem 4 is completed.

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