Sufficient conditions for starlikeness of reciprocal order

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Abstract. The object of the present paper is to derive certain sufficient conditions for starlikeness of reciprocal order of analytic functions in the open unit disk.

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1. Introduction and definitions

Let \( \mathcal{A} \) denote the class of functions \( f(z) \) defined by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic and univalent in the open unit disk \( \mathcal{U} = \{ z : |z| < 1 \} \). A function \( f \in \mathcal{A} \) is said to be starlike of order \( \alpha \) if it satisfies

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U})
\]

for some \( 0 \leq \alpha < 1 \). We denote by \( \mathcal{S}^* (\alpha) \) the subclass of \( \mathcal{A} \) consisting of functions which are starlike of order \( \alpha \) in \( \mathcal{U} \). Clearly \( \mathcal{S}^* (\alpha) \subseteq \mathcal{S}^* (0) = \mathcal{S}^* \), where \( \mathcal{S}^* \) is the class of functions that are starlike in \( \mathcal{U} \).

A function \( f \in \mathcal{A} \) is said to be starlike of reciprocal order \( \alpha \) if

\[
\Re \left\{ \frac{f(z)}{zf'(z)} \right\} > \alpha \quad (z \in \mathcal{U})
\]

for some \( 0 \leq \alpha < 1 \). We denote the class of such functions by \( \mathcal{S}^{-1}* (\alpha) \) (see, [1, 4, 8]).

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In view of the fact that 
\[ R_p(z) > 0 \Rightarrow R_{1/p(z)} = R_{p(z)/|p(z)|^2} > 0, \]

it follows that a starlike function of reciprocal order 0 is same as a starlike function. In particular, every starlike function of reciprocal order \( \alpha \geq 0 \) is starlike and hence univalent (cf. [10, Example 1]).

**Example 1.** The function \( f(z) = z e^{(1-\alpha)z} \) is a starlike function of reciprocal order \( 1/(2-\alpha) \) [10, Example 2].

Sufficient conditions were studied by various authors for starlikeness [e.g., see [2–7, 9–12]]. The object of the present paper is to derive certain sufficient conditions for starlikeness of reciprocal order \( \alpha \) by using the same techniques as in [9].

In order to establish our main results, we require the following lemma due to Nunokawa et al. [9].

**Lemma 1.** Let \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) be analytic in \( U \) and suppose that there exists a point \( z_0 \in U \) such that
\[ R\{p(z)\} > 0 \text{ for } |z| < |z_0| \] (4)
and
\[ R\{p(z_0)\} = 0. \] (5)

Then we have
\[ z_0p'(z_0) \leq -\frac{1}{2}(1 + |p(z_0)|^2), \] (6)
where \( z_0p'(z_0) \) is a negative real number.

**2. Sufficient conditions for starlikeness of reciprocal order**

Our first result is contained in the following.

**Theorem 2.** Let \( f(z) \in A \) satisfies \( f(z) \neq 0 \) in \( 0 < |z| < 1 \) and
\[ R\left\{ \frac{f(z)}{zf'(z)} \left(1 - \alpha \frac{zf''(z)}{f'(z)} \right) \right\} > -\frac{\alpha}{2} \left[ 3 + \left| \frac{f(z)}{zf'(z)} \right|^2 \right] \] (z \in U; \alpha \geq 0). (7)

Then \( f(z) \) is starlike of reciprocal order \( 0 \) in \( U \) and thus, \( f(z) \) is starlike in \( U \).
Proof. Let us define the function \( p(z) \) by

\[
p(z) = \frac{f(z)}{zf'(z)}.
\]

(8)

Then \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). Differentiating (8) logarithmically we obtain

\[
\frac{f(z)}{zf'(z)} \left( 1 - \alpha \frac{zf''(z)}{f'(z)} \right) = \alpha zp'(z) + (\alpha + 1)p(z) - \alpha.
\]

(9)

Suppose that there exists a point \( z_0 \in U \) such that

\[
\Re \{p(z)\} > 0 \text{ for } |z| < |z_0|
\]

and

\[
\Re \{p(z_0)\} = 0,
\]

then from Lemma 1, we have,

\[
z_0 p'(z_0) \leq -\frac{1}{2}(1 + |p(z_0)|^2).
\]

Therefore from (9), we have

\[
\Re \left\{ \frac{f(z_0)}{z_0 f'(z_0)} \left( 1 - \alpha \frac{z_0 f''(z_0)}{f'(z_0)} \right) \right\} = \Re \left\{ \alpha z_0 p'(z_0) + (\alpha + 1)p(z_0) - \alpha \right\}.
\]

\[
\leq -\frac{\alpha}{2} \left( 1 + |p(z_0)|^2 \right) - \alpha
\]

\[
\leq -\frac{\alpha}{2} \left( 3 + \left| \frac{f(z_0)}{z_0 f'(z_0)} \right|^2 \right).
\]

which contradicts our condition (6) of Theorem 2. Thus we complete the proof of Theorem 2.

Next, we derive the following.

**Theorem 3.** Let \( f(z) \in A \) satisfies \( f(z) f'(z) \neq 0 \) in \( 0 < |z| < 1 \) and

\[
\Re \left\{ \frac{f(z)}{zf'(z)} \left( -1 - \frac{zf''(z)}{f'(z)} \right) \right\} > \frac{5}{4} - \frac{1}{4} \left| \frac{2f(z)}{zf'(z)} - 1 \right|^2 \quad (z \in U).
\]

Then \( f(z) \) is starlike of reciprocal order \( \frac{1}{2} \) in \( U \).

Proof. Putting

\[
p(z) = 2 \left( \frac{f(z)}{zf'(z)} - \frac{1}{2} \right),
\]

(10)
then we have $p(0) = 1$. Suppose that there exists a point $z_0 \in \mathcal{U}$ satisfies the conditions (4) and (5) of Lemma 1, from (10) we have

$$ \Re \left\{ \frac{f(z_0)}{z_0 f'(z_0)} \left( -1 - z_0 f''(z_0) \right) / f'(z_0) \right\} = \Re \left\{ \frac{1}{2} z_0 p'(z_0) - 1 \right\}. \quad (11) $$

Using (6) of Lemma 1 in (11), it follows that

$$ \Re \left\{ \frac{f(z_0)}{z_0 f'(z_0)} \left( -1 - z_0 f''(z_0) \right) / f'(z_0) \right\} \leq -\frac{1}{4} (1 + |p(z_0)|^2) - 1 \leq -\frac{5}{4} \left( \frac{3}{2} \right) |2f(z_0)/z_0 f'(z_0) - 1|^2. $$

which contradicts the hypothesis of Theorem 3 and therefore, we have

$$ \Re \{p(z)\} > 0 \quad (z \in \mathcal{U}) $$

or

$$ \Re \left\{ \frac{f(z)}{zf'(z)} \right\} > \frac{1}{2} \quad (z \in \mathcal{U}). $$

Finally, we discuss the following theorem.

**Theorem 4.** Let $f(z) \in \mathcal{A}$ satisfies

$$ \Re \left\{ \frac{f(z)}{zf'(z)} \left( 1 - \alpha zf''(z) / f'(z) \right) \right\} > -\frac{\alpha}{(2 - \alpha)} \left| \frac{f(z)}{zf'(z)} - \frac{\alpha}{2} \right|^2 + \frac{\alpha}{4} (3\alpha - 4) \quad (z \in \mathcal{U}; 0 \leq \alpha < 2). \quad (12) $$

Then $f(z)$ is starlike of reciprocal order $\frac{\alpha}{2}$ in $\mathcal{U}$.

**Proof.** Let the function $p(z)$ be defined by

$$ \frac{f(z)}{zf'(z)} = \left( 1 - \frac{\alpha}{2} \right) p(z) + \frac{\alpha}{2}, \quad p(0) = 1. \quad (13) $$

Suppose that there exists a point $z_0 \in \mathcal{U}$ satisfies the conditions (4) and (5) of Lemma 1, from (13) we have

$$ \Re \left\{ \frac{f(z_0)}{z_0 f'(z_0)} \left( 1 - \alpha z_0 f''(z_0) / f'(z_0) \right) \right\} = \Re \left\{ \alpha \left( 1 - \frac{\alpha}{2} \right) z_0 p'(z_0) + \left( 1 + \alpha \right) \left( 1 - \frac{\alpha}{2} \right) p(z_0) + \frac{\alpha}{2} (\alpha - 1) \right\}. \quad (14) $$
Thus, by using (5) and (6) of Lemma 1 in (14), it follows that

\[
\Re \left\{ \frac{f(z_0)}{z_0 f'(z_0)} \left( 1 - \frac{z_0 f''(z_0)}{f'(z_0)} \right) \right\} \leq -\frac{\alpha}{2} \left( 1 - \frac{\alpha}{2} \right) \left( 1 + |p(z_0)|^2 \right) + \frac{\alpha}{2} (\alpha - 1) \\
\leq -\frac{\alpha}{2} \left( 1 - \frac{\alpha}{2} \right) |p(z_0)|^2 + \frac{\alpha}{4} (3\alpha - 4) \\
\leq -\frac{\alpha}{2} \left( 1 - \frac{\alpha}{2} \right) \frac{f(z_0)}{z_0 f'(z_0)} - \frac{\alpha}{2} + \frac{\alpha}{4} (3\alpha - 4)
\]

which contradicts the hypothesis (12). It follows that

\[
\Re \left\{ \frac{f(z)}{z f'(z)} \right\} > \frac{\alpha}{2} \quad (z \in U).
\]

Thus proof of the Theorem 4 is completed.

References


