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# Centre of Core Regular Double Stone Algebra

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**Abstract.** In literature there is an elegant characterization of factor congruences on a distributive lattice. In this paper, we make an attempt such type of characterization of factor congruences on a Core Regular Double Stone Algebra (CRDSA) and we identify that the factor congruences on a CRDSA $A$ with certain elements of $A$ and proved that set of all factor congruences forms a Boolean centre for $A$. Further Birkhoff centre is defined for CRDSA and finally it is shown that Birkhoff centre of CRDSA is isomorphic to its Boolean centre

**2010 Mathematics Subject Classifications**: 06D99, 03G10,06D15

**Key Words and Phrases**: Core Regular Double Stone Algebra, Boolean centre, Birkhoff centre

## 1. Introduction

The concept of a core regular double Stone algebra was introduced by Ravi Kumar etal and obtained a decomposition theorem for a complete atomic core regular double Stone algebra [5]. In [6], U.M. Swamy and G.S. Murti introduced the concept of the Boolean center of an universal Algebra. In this paper we make an attempt to characterize the Boolean centre of a CRDSA $A$ and the concept of Birkhoff's 'Centre' of a bounded poset is extended to CRDSA $A$ and referred to this, as 'Birkhoff centre' of $A$. It is also proved that Birkhoff centre $BC(A)$ is isomorphic to Boolean centre of $A$.

## 2. Preliminaries

In this section the concept of the isomorphism of RDSA is extended to CRDSA and a new characterization for centre of a CRDSA based on core element is done. We start with certain basic definitions and properties of RDSA.

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**Definition 1.** *A Regular double Stone algebra (RDSA)* $< A, \wedge, \vee, *, +, 0, 1 >$ *is an algebra of type* $< 2, 2, 1, 1, 0, 0 >$ *such that*

(i) $< A, \wedge, \vee, 0, 1 >$ *is a bounded distributive lattice.*

(ii) $*$ *is a pseudo complementation satisfying the Stone identity* $x^* \vee x^{**} = 1$

(iii) $+$ *is a dual pseudo complementation satisfying the dual Stone* $x^+ \wedge x^{++} = 0$

(iv) *For any* $x, y \in A, x^* = y^*$ *and* $x^+ = y^+$ *then* $x = y$

**Example 1.** *Consider the hasse diagrams of lattices* $L_1$ *and* $L_2$.

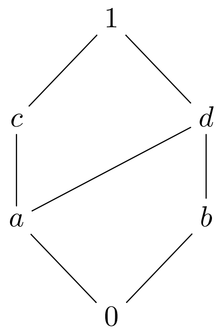


Figure 1: $L_1$

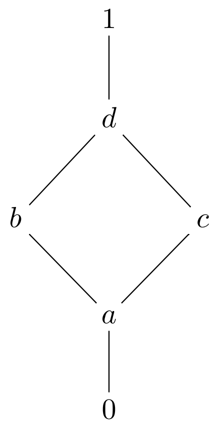


Figure 2: $L_2$

*Clearly* $L_1, L_2$ *are bounded distributive lattices, pseudo complemented and dual pseudo complemented , and in* $L_1, 0^* = 1, a^* = b, b^* = c, c^* = b, d^* = 0, 1^* = 0$ *and* $0^+ = 1, a^+ = 1, b^+ = c, c^+ = b, d^+ = c, 1^+ = 0$. *Clearly* $L_1$ *is a regular double Stone algebra where as in* $L_2, a^* = b^* = c^* = d^* = 1^* = 0, 0^* = 1$ *and* $a^+ = b^+ = c^+ = d^+ = 0^+ = 1, 1^+ = 0$. *Here* $a^* = b^*$ *and* $a^+ = b^+$ *but* $a \neq b$ *therefore* $L_2$ *is not a regular double Stone algebra.*

**Definition 2.** *Let A be a regular double Stone algebra. An element a of A is called a central element of A if* $a^* = a^+$. *The set of all central elements of A is called the centre of A and is denoted by* $C(A)$*; that is,* $C(A) = \{a \in A | a^* = a^+\}$

*Note that* $C(A)$ *can be described in various ways as follows;*

$$\begin{aligned} C(A) &= \{a \in A | a = a^{**}\} \\ &= \{a^* | a \in A\} \\ &= \{a \in A | a = a^{++}\} \end{aligned}$$

$$= \{a^+ | a \in A\}$$
$$= \{a \in A | a \vee a^* = 1\}$$
$$= \{a \in A \mid a \wedge a^+ = 0\}$$
$$= \{a \in A \mid a \wedge b = 0 \text{ and } a \vee b = 1 \text{ for some } b \in A\}$$

**Theorem 2.1.** *Let $A$ be a regular double Stone algebra.Then $C(A)$ is a Boolean sub algebra of $A$ with respect to the induced operations $\wedge, \vee$ and $*$.*

**Definition 3.** *Let $A$ be a regular double Stone algebra. The set $D(A) := \{a \in A \mid a^* = 0\}$ is called the dense set of $A$ and the elements of $D(A)$ are called dense elements of $A$. The dual of $D(A) := \{a \in A \mid a^+ = 1\}$ is called dual dense set of $A$ and denoted by $\overline{D(A)}$. The elements of $\overline{D(A)}$ are called dual dense elements of $A$.*

*Note that $D(A) = \{a \vee a^* \mid a \in A\}$ and $\overline{D(A)} = \{a \wedge a^+ \mid a \in A\}$.*

**Theorem 2.2.** *Let $A$ be a regular double Stone algebra. Then $D(A)$ is a filter of $A$ and $\overline{D(A)}$ is an ideal of $A$.*

**Definition 4.** *The core of a double Stone algebra $A$ is defined to be $K(A) = D(A) \cap \overline{D(A)}$*

*$K(A)$ is non empty if and only if $A$ does not have **2** $=\{0,1\}$ as a factor. When it is non empty the behavior of $K(A)$ in certain respects governs the behaviour of $A$. It is easy to prove that in any RDSA there exists at most one core element. We call a regular double Stone algebra with non empty core as **Core Regular Double Stone Algebra**(CRDSA).*

***Note:*** *In any CRDSA $A$, $|K(A)| = 1$.*

**Example 2.** *Every three element chain is CRDSA. We call it as a discrete CRDSA*

**Example 3.** *Consider the hasse diagrams of RDSAs $L_1 = (L_1, \wedge, \vee, *, +, 0, 1)$ and $L_3 = (L_3, \wedge, \vee, *, +, 0, 1)$.*

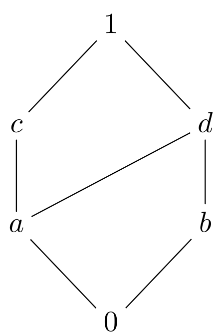

Figure 3: $L_1$

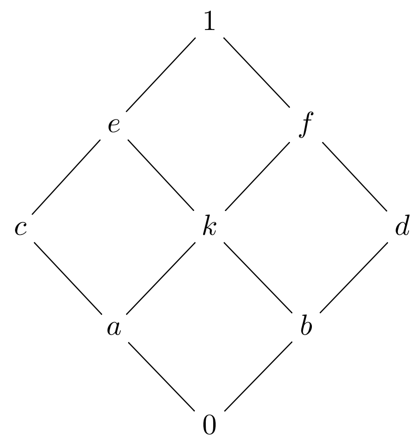

Figure 4: $L_3$

*Clearly $L_1, L_3$ are RDSAs, and it is seen that core of $L_1$ is empty where as $L_3$ has the core element $k$ hence it a CRDSA*

**Theorem 2.3.** *If $A$ is CRDSA with core element $k$, then every element $x$ of $A$ can be written as $x = x^{**} \wedge (x^{++} \vee k)$ and $x = x^{++} \vee (x^{**} \wedge k)$*

*Proof.* Let $y = x^{**} \wedge (x^{++} \vee k)$. Then $y^{**} = (x^{**} \wedge (x^{++} \vee k))^{**} = x^{**}$ and $y^{++} = (x^{++} \wedge (x^{++} \vee k))^{++} = x^{++}$. Thus by regularity $x = y$. Other one follows from duality.

**Definition 5.** *Suppose that $A$ and $B$ are two CRDSAs with core elements $k_1, k_2$ respectively. A mapping $f : A \longrightarrow B$ is called a homomorphism from $A$ to $B$ if*

(i) *$f$ is lattice homomorphism from $A$ to $B$*

(ii) *for $a \in A, f(a^*) = f(a)^*$ and $f(a^+) = f(a)^+$*

(iii) $f(k_1) = k_2$

*A necessary and sufficient condition for two CRDSAs is isomorphic is discussed in the following theorem.*

**Theorem 2.4.** *Two CRDSAs are isomorphic if and only if their centers are isomorphic*

*Proof.* Let $A_1, A_2$ be CRDSAs with Core elements $k_1, k_2$ respectively. First suppose that $f : C(A_1) \longrightarrow C(A_2)$ is an isomorphism. Define the map $\phi$ on $A_1$ to $A_2$ by

$$\phi(x) = f(x^{**}) \wedge (f(x^{++}) \vee k_2).$$

By using distributive property and the fact that $f$ is a homomorphism it can be easily verify that

$$\phi(x) = f(x^{++}) \vee (f(x^{**}) \wedge k_2).$$

And also observe that, for $x \in C(A_1)$,

$\phi(x) = f(x^{**}) \wedge (f(x^{++}) \vee k_2) = f(x) \vee (f(x) \wedge k_2) = f(x)$, i.e. $\phi$ coincides with $f$ on $C(A_1)$

To show that $\phi$ is one-one suppose that $\phi(x) = \phi(y)$ for $x, y$ in $A_1$. Then $(\phi(x))^* = (\phi(y))^*$ and $(\phi(x))^+ = (\phi(y))^+$, by using the definition of $\phi$ and the fact that $f$ is one-to-one , it gives $x^* = y^*$ and $x^+ = y^+$ and by regularity $x = y$. Hence $\phi$ is one-one.

To show that $\phi$ is onto, $y \in A_2$ and consider the following cases

*Case (i) :* $y \in C(A_2)$.
*Since* $f$ *is onto from* $C(A_1)$ *to* $C(A_2)$*, there exists an element* $x \in C(A_1)$ *such that* $f(x) = y$ *and*

$$\begin{aligned}\phi(x) &= f(x^{**}) \wedge \big(f(x^{++}) \vee k_2\big)\\ &= f(x) \wedge (f(x) \vee k_2)\\ &= f(x)\\ &= y\end{aligned}$$

*Case (ii) :* $y = k_2$ *Then*

$$\begin{aligned}\phi(k_1) &= f(k_1^{**}) \wedge \big(f(k_1^{++}) \vee k_2\big)\\ &= 1 \wedge k_2\\ &= k_2\end{aligned}$$

*Case (iii) :* $k_2 \neq y$ *and* $y \notin C(A_2)$
*Then* $y^{**}, y^{++} \in C(A_2)$ *and from the fact that* $f$ *is onto there exists* $x_1, x_2 \in C(A_1)$ *such that* $\phi(x_1) = f(x_1) = y^{**}$ *and* $\phi(x_2) = f(x_2) = y^{++}$*, now*

$$\begin{aligned}&\phi(x_1 \wedge (x_2 \vee k_1))\\ &\qquad = f(x_1) \wedge (f(x_1 \wedge f(x_2)) \vee k_2)\\ &\qquad = f(x_1) \wedge (f(x_2) \vee k_2)\\ &\qquad = y^{**} \wedge \big(y^{++} \vee k_2\big)\\ &\qquad = y\end{aligned}$$

*Hence* $\phi$ *is onto. The remaining conditions which verifies that* $\phi$ *is homomorphism is straightforward.*

*Conversely suppose that* $\phi : A_1 \longrightarrow A_2$ *is an isomorphism. Let* $x \in C(A_1)$ *be any element, then* $(\phi(x))^{**} = \phi(x^{**}) = \phi(x)$*. Therefore* $\phi(x) \in C(A_2)$*, and hence* $\phi(C(A_1)) \subseteq C(A_2)$*. On the other hand, if* $y \in C(A_2)$ *then there exist* $x \in A_1$ *and* $\phi(x) = y$*. Now*

$$\begin{aligned}\phi(x^{**}) &= (\phi(x))^{**}\\ &= y^{**}\\ &= y\\ &= \phi(x)\end{aligned}$$

*As* $\phi$ *is one-one, we get* $x^{**} = x$ *and hence* $x \in C(A_1)$*. Therefore* $\phi(C(A_1)) = C(A_2)$ *and hence they are isomorphic.*

*Hence Boolean isomorphism between centre of a CRDSA can be extended to whole algebra so that core elements are mapped each other.*

*LetA be a regular double Stone algebra. For* $a \in A$ *the* $^*-$ *centralizer of* $a$ *is denoted by* $A_a^*$ *and defined as* $A_a^* = \{x^{**} \mid x \leq a\} = \{x^{**} \wedge a^{**} \mid x \in A\}$.

**Definition 7.** *LetA be a regular double Stone algebra. For $a \in A$ the $^{+}-$ centralizer of a is denoted by $A^{+}_{a}$ and defined as $A^{+}_{a} = \{x^{++} \mid x \geq a\} == \{x^{++} \vee a^{++} \mid x \in A\}$.*

**Theorem 2.5.** *Let A be a Core Regular double Stone algebra. The relativized algebra $A^{*}_{a} =< A^{*}_{a}, \wedge, \vee,' 0, a^{**} >$ is a Boolean algebra*

*Proof.* Let $x, y \in A^{*}_{a}$. Then $x = p^{**}, y = q^{**}$ for some $p, q \in A$ and $p, q \leq a$. Which gives $p \vee q \leq a, p \wedge q \leq a$. Hence $(p \vee q)^{**} = p^{**} \vee q^{**} \leq a^{**}$ and $(p \wedge q)^{**} = p^{**} \wedge q^{**} \leq a^{**}$ . Therefore $x \vee y, x \wedge y \in A^{*}_{a}$. Therefore $A^{*}_{a}$ is closed with respect to $\vee$ and $\wedge$. It is a routine verification that $< A^{*}_{a}, \wedge, \vee >$ is distributive lattice.
Clearly $0^{**} = 0 \leq a$, so $0 \in A^{*}_{a}$. Since $a \leq a$ we get $a^{**} \in A^{*}_{a}$. Let $x \in A^{*}_{a}$ be any element then $x = p^{**}$ for some $p \in A$ and $p \leq a$ which gives $p^{**} = x \leq a^{**}$. Therefore $a^{**}$ is the greatest element of $A^{*}_{a}$. Hence $< A^{*}_{a}, \wedge, \vee, 0, a^{**} >$ is a bounded distributive lattice.
Finally for $x = p^{**} \in A^{*}_{a}$ we have $p \leq a$ and $x^{*} \wedge a = p^{*} \wedge a \leq a$ which gives $(x^{*} \wedge a)^{**} = x^{*} \wedge a^{**} \in A^{*}_{a}$ and $x \wedge (x^{*} \wedge a^{**}) = 0$ and $x \vee (x^{*} \wedge a^{**}) = a^{**}$. Therefore $x^{*} \wedge a^{**}$ is the compliment of $x$ in $A^{*}_{a}$ i.e. $x' = x^{*} \wedge a^{**}$. Hence $A^{*}_{a} =< A^{*}_{a}, \wedge, \vee,' 0, a^{**} >$ is a Boolean algebra

**Theorem 2.6.** *Let A be a Core Regular double Stone algebra and k is the core element of A. Then $A^{*}_{k} = A^{+}_{k}$*

*Proof.* Let $x^{**} \in A^{*}_{k}$ and $y = k \vee x^{**}$. Then $y \geq k$ and $y^{++} = (k \vee x^{**})^{++} = x^{**}$.Therefore $x^{**} \in A^{+}$ and hence $A^{*}_{k} \subseteq A^{+}_{k}$. Now take $x^{++} \in A^{+}_{k}$ put $y = k \wedge x^{++}$ then $y \leq k$ and hence $y^{**} = (k \wedge x^{++})^{**} = x^{++} \in A^{*}_{k}$. So $A^{+}_{k} \subseteq A^{*}_{k}$ and hence $A^{*}_{k} = A^{+}_{k}$.

*In fact we have the stronger result in the following theorem.*

**Theorem 2.7.** *Let A be a Core Regular double Stone algebra and k is the core element of A then $A^{*}_{a} = A^{+}_{a}$ if and only if $a = k$*

*Proof.* First suppose that for some $a \in A, A^{*}_{a} = A^{+}_{a}$. Since $0 \in A^{*}_{a} = A^{+}_{a}$ there exists $b \in A$ such that $a \leq b$ and $0 = b^{++}$. So $b^{+} = 1$ and $b^{+} \leq a^{+}$. Hence $a^{+} = 1$. As $1 \in A^{+}_{a} = A^{*}_{a}$ there exists $c \in A$ such that $c \leq a$ and $1 = c^{**}$. So $c^{*} = 0$ and $a^{*} \leq C^{*}$. Hence $a^{*} = 0$. Therefore $a \in K(A) = k$. Other part is clear from theorem 2.6.

*Throughout this paper we denote $A^{*}_{k} = A^{+}_{k}$ by $k(A)$. The following theorem discuss the relation between centralizer of core and centre of CRDSA .*

**Theorem 2.8.** *Let A be a Core Regular double Stone algebra and k is the core element of A then $k(A) = C(A)$ .*

*Proof.* Clearly $k(A) \subseteq C(A)$. Let $a$ be any element of $C(A)$, then $(a \wedge k)^{**} = a^{**} = a \in k(A)$. Hence $k(A) = C(A)$.

*Theorem 2.8 gives another characterization for centre of a CRDSA based on the core element.*

## 3. Boolean centre

*In [6], Swamy and Murthy introduced the concept of 'Balanced congruence' on any algebraA and showed the set $B(A)$ of all balanced (direct) factor congruences which admit a balanced complement as a permutable Boolean sublattice of the lattice $C(A)$ of all congruences on $A$. They referred to $B(A)$ as the 'Boolean centre' of $A$. The main goal of this Section is to characterize the Boolean centre of a CRDSA $A$ in terms of central elements.*

*Let $A$ be a Core Regular double Stone algebra. Let $\theta_x$ denote the equivalence relation associated to the function $x \to x \wedge p$ from $A$ to itself: $\theta_x = \{(p,q) \in A \times A \,|x \wedge p = x \wedge q\}$. We will write $p\theta_x q$ to indicate $(p,q) \in \theta_x$.*

**Theorem 3.1.** *Let $A$ be a Core Regular double Stone algebra and $x, y \in A$ Then*

(i) $\theta_y \subseteq \theta_x$ *if and only if* $x = x \wedge y$.

(ii) $\theta_y = \theta_x$ *if and only if* $x = y$.

(iii) $\theta_x$ *is compatible with* $\wedge, \vee, *$

(iv) $\theta_x$ *is compatible with* $+$ *if and only if* $x \in k(A)$

(v) $\theta_x$ *congruence on* $A$ *if and only if* $x \in k(A)$.

(vi) $\theta_0 = A \times A$

(vii) $\theta_1 = \Delta_A$

(viii) $\theta_x \cap \theta_y = \theta_{x \vee y}$

(ix) $\theta_x \circ \theta_y = \theta_y \circ \theta_x$

(x) $\theta_x \circ \theta_y = \theta_{x \wedge y}$

(xi) $\theta_x \circ \theta_{x^*} = \theta_{x^*} \circ \theta_x = A \times A$

(xii) $\theta_{x \vee x^*} = \Delta_A$ *if and only if* $\theta_x$ *is a congruence relation*

(xiii) *for* $x \in k(A), \theta_x$ *is the smallest congruence containing* $(1, x)$

*Proof.*

*(i) Let $x, y \in A$ and suppose that $\theta_y \subseteq \theta_x$. Since $y \wedge (x \vee y) = y = y \wedge y$ we have $(y, x \vee y) \in \theta_y$, by our supposition $(y, x \vee y) \in \theta_x$ ; that is $x \wedge y = x \wedge (x \vee y)$ or $x \wedge y = x$. Conversely suppose that $x \wedge y = x$.*
*Let $(p,q) \in \theta_y$. Then $y \wedge p = y \wedge q$. Now,*

$$\begin{aligned} x \wedge p &= (x \wedge y) \wedge p \\ &= x \wedge (y \wedge p) \end{aligned}$$

$$\begin{aligned} &= x \wedge (y \wedge q) \\ &= (x \wedge y) \wedge q \\ &= x \wedge q \end{aligned}$$

*Therefore,* $(p, q) \in \theta_x$ *and hence* $\theta_y \subseteq \theta_x$.

*(ii) Clear from (i).*

*(iii) If* $p, q, r, s \in A$ *satisfy* $(p, q) \in \theta_x$ *and* $(r, s) \in \theta_x$. *From associativity and distributivity in* $A$ *it follows that* $((p \wedge r), (q \wedge s)) \in \theta_x$ *and* $((p \vee r), (q \vee s)) \in \theta_x$. *Also if* $p, q \in A$ *and* $(p, q) \in \theta_x$, *it follows that* $(x^* \vee p^*) = (x^* \vee q^*)$, *so that* $x \wedge (x^* \vee p^*) = x \wedge (x^* \vee q^*)$ *using distributivity we conclude that* $(p^*, q^*) \in \theta_x$.

*(iv) Suppose that* $\theta_x$ *is compatible with* $+$. *Put* $y = x \vee k$ *then* $y^{++} = x^{++}$. *As* $(1, x)$ *and* $(k, k) \in \theta_x$ *which gives* $(1, x \vee k) = (1, y) \in \theta_x$. *Since* $\theta_x$ *is compatible with* $+$, *we get*$(1, y^{++}) \in \theta_x$. *Hence* $x = x \wedge y^{++} = y^{++}$. *Therefor* $x \in k(A)$.
*Conversely suppose that* $x \in k(A)$ *then* $x = y^{++}$ *for some* $y \geq k$, *if* $p, q \in A$ *satisfy* $(p, q) \in \theta_x = \theta_{y^{++}}$ *then* $y^{++} \wedge p = y^{++} \wedge q$ *and hence* $y^+ \vee p^+ = y^+ \vee q^+$ *, it follows that* $y^{++} \wedge (y^+ \vee p^+) = y^{++} \wedge (y^+ \vee q^+)$ *which gives* $(y^{++} \wedge p^+) = (y^{++} \wedge q^+)$ *hence* $(p^+, q^+) \in \theta_{y^{++}} = \theta_x$.

*(v) is clear from (iii) and (iv).*

*(vi) and (vii) are clear from the definition of* $\theta_x$.

*(viii) Let* $(p, q) \in \theta_x \cap \theta_y$. *Then* $x \wedge p = x \wedge q$ *and* $y \wedge p = y \wedge q$.
*Now,*

$$\begin{aligned} (x \vee y) \wedge p &= (x \wedge p) \vee (y \wedge p) \\ &= (x \wedge q) \vee (y \wedge q) \\ &= (x \vee y) \wedge q \end{aligned}$$

*Therefore* $(p, q) \in \theta_{x \vee y}$. *Hence* $\theta_x \cap \theta_y \subseteq \theta_{x \vee y}$.

*Conversely suppose that* $(p, q) \in \theta_{x \vee y}$ *then* $(x \vee y) \wedge p = (x \vee y) \wedge q$.
*Now,*

$$\begin{aligned} x \wedge ((x \vee y) \wedge p) &= x \wedge ((x \vee y) \wedge q) \\ (x \wedge (x \vee y)) \wedge p &= (x \wedge (x \vee y)) \wedge q) \\ x \wedge p &= x \wedge q \end{aligned}$$

*Therefore* $(p, q) \in \theta_x$*, similarly it can be shown that* $(p, q) \in \theta_y$*. So* $(p, q) \in \theta_x \cap \theta_y$ *and hence* $\theta_{x \vee y} \subseteq \theta_x \cap \theta_y$*. Therefore* $\theta_x \cap \theta_y = \theta_{x \vee y}$*.*

*(ix) Let* $(p, r) \in \theta_x \circ \theta_y$*. Then there exists* $q \in A$ *such that* $(p, q) \in \theta_x$ *and* $(q, r) \in \theta_y$*. So* $x \wedge p = x \wedge q$ *and* $y \wedge q = y \wedge r$*. Put* $t = (x \wedge r) \vee (y \wedge p)$*.*
*Then,*

$$\begin{aligned} x \wedge t &= x \wedge ((x \wedge r) \vee (y \wedge p)) \\ &= (x \wedge r) \vee (x \wedge (y \wedge p)) \\ &= (x \wedge r) \vee x \wedge (y \wedge p) \\ &= (x \wedge r) \vee (x \wedge p \wedge p) \\ &= (x \wedge r) \vee (x \wedge q \wedge y), \text{since } x \wedge p = x \wedge q \\ &= (x \wedge r) \vee (x \wedge y \wedge r), \text{since } q \wedge y = y \wedge q = y \wedge r \\ &= x \wedge r \end{aligned}$$

*Hence* $(t, r) \in \theta_x$*. Similarly it can be shown that* $y \wedge t = y \wedge p$ *which gives* $(p, t) \in \theta_y$*. Therefore* $(p, r) \in \theta_y \circ \theta_x$ *i.e.* $\theta_x \circ \theta_y \subseteq \theta_y \circ \theta_x$*.*

*Conversely suppose that* $(p, r) \in \theta_y \circ \theta_x$*. Now by setting* $t = (y \wedge r) \vee (x \wedge p)$ *and proceeding as above it can be shown that* $\theta_y \circ \theta_x \subseteq \theta_x \circ \theta_y$*. Finally it gives* $\theta_x \circ \theta_y = \theta_y \circ \theta_x$*.*

*(x) Let* $(p, r) \in \theta_x \circ \theta_y$*. Then there exists* $q \in A$ *such that* $(p, q) \in \theta_x$ *and* $(q, r) \in \theta_y$*. So* $x \wedge p = x \wedge q$ *and* $y \wedge q = y \wedge r$
*Now,*

$$\begin{aligned} (x \wedge y) \wedge p &= (x \wedge p) \wedge y \\ &= (x \wedge q) \wedge y, \text{since } x \wedge p = x \wedge q \\ &= x \wedge (y \wedge q) \\ &= x \wedge (y \wedge r), \text{since } y \wedge q = y \wedge r \\ &= (x \wedge y) \wedge r. \end{aligned}$$

*Therefore* $(p, r) \in \theta_{x \wedge y}$ *and hence* $\theta_x \circ \theta_y \subseteq \theta_{x \wedge y}$*.*

*Conversely suppose that* $(p, r) \in \theta_{x \wedge y}$ *then* $(x \wedge y) \wedge p = (x \wedge y) \wedge r$*. Put* $q = (x \wedge p) \vee (y \wedge r)$*.*
*Then,*

$$\begin{aligned} x \wedge q &= x \wedge (x \wedge p) \vee (y \wedge r) \\ &= (x \wedge p) \vee (x \wedge y \wedge r) \\ &= (x \wedge p) \vee (x \wedge y \wedge p), \text{since } (x \wedge y) \wedge p = (x \wedge y) \wedge r \\ &= (x \wedge p) \end{aligned}$$

*hence* $(p,q) \in \theta_x$. *By considering* $y \wedge q$ *and proceeding as above it can shown that* $y \wedge q = y \wedge r$, *so* $(q,r) \in \theta_y$ *and hence* $(p,r) \in \theta_x \circ \theta_y$. *Therefore* $\theta_{x\wedge y} \subseteq \theta_x \circ \theta_y$, *which completes the proof.*

*(xi) is clear from (x), (vi) and the fact that* $x \wedge x^* = 0$.

*(xii) follows from (v), (ii) and (vii).*

*(xiii) Let* $x \in k(A)$. *Then* $\theta_x$ *is congruence and* $(1,x) \in \theta_x$. *Suppose that* $\theta$ *be any congruence containing* $(1,x)$ *and* $(a,b) \in \theta_x$ *i.e.* $a \wedge x = b \wedge x$. *Since* $\theta$ *is reflexive and* $(1,x) \in \theta$ *we get* $(a,a) \wedge (1,x) = (a, a \wedge x) \in \theta$ *and* $(b,b) \wedge (1,x) = (b, b \wedge x) \in \theta$ *which in turn gives* $(a, b \wedge x) \in \theta$ *and* $(b, b \wedge x) \in \theta$. *Hence* $(a,b) \in \theta$ *and* $\theta_x \subseteq \theta$. *Therefore* $\theta_x$ *is the smallest congruence containing* $(1,x)$ *for* $x \in k(A)$.

*Recall that a congruence* $\theta$ *on an algebra* $A$ *is said to be factor congruence if there is a congruence* $\psi$ *on* $A$ *such that*

$$\begin{aligned} \theta \wedge \psi &= \Delta \\ \theta \vee \psi &= A \times A \end{aligned}$$

*and* $\theta$ *permutes with* $\psi$

*In the following theorem the factor congruences of core regular double stone algebras were characterized.*

**Theorem 3.2.** *Let* $A$ *be a Core Regular double Stone algebra and* $\theta$ *be congruence on* $A$. *Then* $\theta$ *is factor congruence on* $A$ *if and only if* $\theta = \theta_x$ *for some* $x \in k(A)$.

*Proof.* *Suppose that* $\theta = \theta_x$ *for some* $x \in k(A)$, *then from (vi) ,(vii), (viii) and (x) of theorem 3.1 and theorem 2.8, we have* $\theta_x \wedge \theta_{x^*} = \Delta$ *and* $\theta_x \vee \theta_{x^*} = A \times A$ *and hence* $\theta = \theta_x$ *is factor congruence on* $A$

*Conversely suppose that* $\theta$ *is factor congruence on* $A$. *Then there exists a congruence* $\psi$ *on* $A$ *such that* $\theta \wedge \psi = \Delta$ *and* $\theta \vee \psi = A \times A$. *Since* $(1,0) \in A \times A = \theta \vee \psi$, *there exists* $x \in A$ *such that* $(1,x) \in \theta$ *and* $(x,0) \in \psi$. *Now put* $y = x \wedge k$ *then* $y^{**} \in k(A)$ *and from the fact that* $(1,x),(k,k) \in \theta$ *it follows that* $(1,y) \in \theta$ *and hence* $(1,y^{**}) \in \theta$. *Also observe that* $(x,0),(k,k) \in \psi$ *gives* $(y,0) \in \psi$ *and hence* $(y^{**},0) \in \psi$.

*Now we show that* $\theta = \theta_{y^{**}}$. *Since* $(1,y^{**}) \in \theta$, *by (xiii) of theorem 3.1, we have* $\theta_{y^{**}} \subseteq \theta$. *Next suppose that* $(p,q) \in \theta$ *then* $(y^{**} \wedge p, y^{**} \wedge q) \in \theta$. *Since* $(y^{**},0),(p,p)$ *and* $(q,q) \in \psi$, *we have* $(y^{**} \wedge p, 0 \wedge p)$ *and* $(y^{**} \wedge q, 0 \wedge q) \in \psi$; *that is* $(y^{**} \wedge p, 0)$ *and* $(0, y^{**} \wedge q) \in \psi$ *which imply that* $(y^{**} \wedge p, y^{**} \wedge q) \in \psi$. *Therefore,*$(y^{**} \wedge p, y^{**} \wedge q) \in \theta \cap \psi = \Delta$ *and hence* $y^{**} \wedge p = y^{**} \wedge q$. *Therefore* $(p,q) \in \theta_{y^{**}}$, *hence* $\theta \subseteq \theta_{y^{**}}$. *Thus* $\theta = \theta_{y^{**}}$.

*Recall that a congruence* $\theta$ *on any Universal algebra* $A$ *of any type, is called balanced if* $(\theta \vee \psi) \cap (\theta \vee \psi') = \theta$ *for all factor congruence* $\psi$ *and its complements* $\psi'$ *and the set* $B(A)$

*of all balanced factor congruences which admit a balanced complement is called the Boolean centre of A. Now we conclude this section by proving that, if A is a Core regular double stone algebra, then the Boolean centre B(A) is precisely the set $D = \{\theta_x \mid x \in k(A)\}$ and that the map $x \longmapsto \theta_x$ is an isomorphism of k(A) onto B(A). First we prove the following.*

**Lemma 1.** *Let A be a CRDSA and $x \in k(A)$. Then $\theta_x$ is balanced.*

*Proof.* *Let $\psi$ be a factor congruence on A and $\psi'$ be its complement. Then there exist $y, z \in k(A)$ such that* $\psi = \theta_y$ and $\psi' = \theta_z$.
*Now,*

$$\begin{aligned}(\theta_x \vee \psi) \cap (\theta_x \vee \psi') &= (\theta_x \vee \theta_y) \cap (\theta_x \vee \theta_z)\\ &= \theta_{x\wedge y} \cap \theta_{x\wedge z}\\ &= \theta_{(x\wedge y)\vee(x\wedge z)}\\ &= \theta_{x\wedge(y\vee z)}\\ &= \theta_x \vee \theta_{y\vee z}\\ &= \theta_x \vee (\theta_y \cap \theta_z)\\ &= \theta_x \vee (\psi \cap \psi')\\ &= \theta_x \vee \Delta_A\\ &= \theta_x\end{aligned}$$

*Therefore, $\theta_x$ is balanced.*

*Thus we have proved the following.*

**Theorem 3.3.** *Let A be a CRDSA. Then the Boolean centre B(A) of A is precisely the set $\{\theta_x \mid x \in k(A)\}$.*

*The following theorem is a consequence of lemma 1 and above theorem 3.3*

**Theorem 3.4.** *Let A be a CRDSA. Then the Boolean centre $B(A) = \{\theta_x \mid x \in k(A)\}$ of A, is a Boolean algebra and the map $x \longmapsto \theta_x$ is an isomorphism of k(A) onto B(A).*

## 4. Birkhoff Centre

*An element a of a bounded poset P is called a 'central element' of P if there exist bounded posets $P_1$ and $P_2$ and an order isomorphism of P onto $P_1 \times P_2$ such that a is mapped onto (1, 0). The set of all central elements of P are called the 'Birkhoff centre' of P and is denoted by BC(P). It is known that BC(P) is a Boolean algebra in which the operations are g.l.b and l.u.b with respect to the partial order in P. In this section we extend the concept of Birkhoff centre to core regualr double Stone algebra.*

**Definition 8.** *An element a of an RDSA A is called a Birkhoff central element if there exist RDSAs $A_1$ and $A_2$ and an isomorphism A onto $A_1 \times A_2$ such that a is mapped onto $(1, 0)$. The set BC(A) of all central elements of P is called the Birkhoff centre.*

*Recall that the ideal generated by an element x of A in a RDSA is called a relativized algebra and is denoted by $(x]_A$. In [9] it is proved that if A in a CRDSA with core element k then for $x \in At(C(A))$,the relativized algebra $(x]_A$ is a three element chain i.e. a discrete CRDSA. In fact we have the following theorem.*

**Theorem 4.1.** *Let A be a Core Regular double Stone algebra. The relativized algebra $(a]_A$ is a CRDSA if and only if $a \in k(A)$.*

*Proof.* Assume that $a \in k(A)$. Then $a^* \vee a = 1$ and $a^+ \wedge a = 0$. It is a routine verification that $(a]_A = ((a], \wedge, \vee, *_a, +_a, 0, a)$ is a double Stone algebra where a is the greatest element and for $x \in (a], x^{*_a} = x^* \wedge a$ and $x^{+_a} = x^+ \wedge a$.

To prove that $(a]_A$ is regular consider $x, y \in (a]_A$ such that $x^{*_a} = y^{*_a}$ and $x^{+_a} = y^{+_a}$, that is, $x^* \wedge a = y^* \wedge a$ and $x^+ \wedge a = y^+ \wedge a$. Then

$$\begin{aligned} (x^* \wedge a) \vee a^* &= (y^* \wedge a) \vee a^* \text{ and } (x^+ \wedge a) \vee a^+ = y^+ \wedge a \\ \Rightarrow (x^* \vee a^*) \wedge (a \vee a^*) &= (y^* \vee a^*) \wedge (a \vee a^*) \end{aligned}$$

and $(x^+ \vee a^+) \wedge (a \vee a^+) = (y^+ \vee a^+) \wedge (a \vee a^+) - (*)$
Since $x, y \in (a]$ we have $x, y \le a \Rightarrow a^* \le x^*, y^*$ and $a^+ \le x^+, y^+$. Also since $a^* \vee a = 1$ Therefore (*) gives $x^* = y^*$ and $x^+ = y^+$ and by regularity in $A, x = y$. Hence $(a]_A$ is a regular double Stone algebra. Moreover $a \wedge k \in (a]$ and $(a \wedge k)^{*_a} = a \wedge k^* = 0, (a \wedge k)^{+_a} = a \wedge k^+ = a$. Theretofore $a \wedge k$ is the core element of $(a]_A$. So $(a]_A$ is a CRDSA.

Conversely suppose that for $a \in A$, $(a]_A = ((a], \wedge, \vee, *_a, +_a, 0, a)$ is a CRDSA with the above defined operations. Since a is the greatest element of $(a]_A$, we have $a^{*_a} = a^{+_a}$ and therefore $a^+ \wedge a = 0$. Hence a is complimented element. So $a \in C(A) = k(A)$

By the principle of duality and theorem 2.6 we have the following theorem.

**Theorem 4.2.** *Let A be a Core Regular double Stone algebra. Then relativized algebra $[a)_A = ([a), \wedge, \vee, *_a, +_a, a, 1)$ is a CRDSA ifand only if $a \in k(A)$.*

**Theorem 4.3.** *Let A be a Core Regular double Stone algebra. $a \in BC(A)$ if and only if $a \in k(A)$ .*

*Proof.* Let $a \in BC(A)$. Then there exist CRDSAs $A_1$ and $A_2$ and an isomorphism f from A onto $A_1 \times A_2$ such that a is mapped onto $(1, 0)$. By theorem 2.4, $C(A)$ is isomorphic to $C(A_1) \times C(A_2)$ and $(1, 0) \in C(A_1) \times C(A_2)$ which in turn gives $a \in C(A)$ and hence by theorem 2.8 $a \in k(A)$.

*Conversely suppose that* $a \in k(A)$. *By theorems 4.1 and 4.2* $(a]_A$ *and* $[a)_A$ *are CRDSAs. Now define a map* $f : A \to (a]_A \times [a)_A$ *by* $f(x) = (a \wedge x, a \vee x)$. *Then* $f$ *is a isomorphism from* $A$ *onto* $(a] \times [a)_A$, *such that* $f(a) = (a, a) = (1, 0)$. *Hence* $a \in BC(A)$.

*Thus we have proved the following.*

**Theorem 4.4.** *Let* $A$ *be a CRDSA. Then the Birkhoff centre* $BC(A)$ *of* $A$ *is precisely the set* $\{a \mid a \in k(A)\} = \{a \mid a \in C(A)\}$.

*The following theorem is a consequence of theorem 3.4 and above theorem 4.4.*

**Theorem 4.5.** *Let* $A$ *be a CRDSA. Then the Boolean centre* $B(A)$ *of* $A$, *is isomorphic to Birkhoff centre* $BC(A)$ *of* $A$ .

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