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A Novel Power Series Method for Solving Second Order Partial Differential Equations

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Abstract. In this article, a new approach is proposed to solve partial differential equation. This method is based on generalized Taylor's formula. The solution of partial differential equations can be expanded using MAPLE. Doing some simple mathematical operations on these equations, we can get a closed form series solution or approximate solution quickly. PDE problems with constant and variable coefficients are solved by the present method. With this method, we can reach same results simpler way than other analytical or approximate methods.

Key words: Second order partial differential equations; Power Series.

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1. Introduction

Study about partial differential algebraic equation were done by Marszalek. Marszalek studied analysis of the partial differential algebraic equations [1]. Lucht ve Strehmel [2, 3] studied numerical solution and indexes of the linear partial differential equations with constant coefficients. A study about characteristics analysis and differential index of the partial differential algebraic equations were given by Martinson and Barton [5, 6]. Debrabant and Strehmel investigated convergence of the Runge-Kutta Method for linear partial differential algebnraic equations [4].

Chen and Ho [7] proposed methods to Solving partial differential equations by two-dimensional differential transform method. The method was applied to the partial differential equation [8–10].Second-order linear Partial differential equations problems were solved by Yang, Liu and Bai [11]. Chebyshev polynomial solutions of second-order linear partial differential equations [12].

In this paper, we propose, by making full use of the properties of power series, a generral scheme for solving the second-order partial differential equation

$$\begin{aligned} a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} \\ + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u + g(x,y) = 0 \end{aligned} \tag{1.1}$$

with the initial conditions

$$u(x,0) = p(x), \frac{\partial}{\partial y}u(x,0) = q(x\pi) \tag{1.2}$$

Finally, three PDE problems are solved by the present method, and the calculated results are compared very well with those obtained by other analytical or approximate methods.

2. Preliminary Knowledge

The basic theory of power series is stated below. In what follows, we assume that for each function involved in our study, all its derivatives are existent and continuous in the region of interest.

The Taylor series may also be generalized to functions of more than one variable with

$$T(x_1,...,x_d) = \sum_{n_1=0}^{\infty} ... \sum_{n_d=0}^{\infty} \frac{\partial^n}{\partial x_d^{n_1}} ... \frac{\partial^{nd}}{\partial x_d^{nd}} \frac{f(a_1,...,a_d)}{n_{1,...,}n_d}(x_1 - a_1)^{n_1}...(x_d - a_d)^{n_d} \qquad (2.1)$$

For example, for a function that depends on two variables, x and y, the Taylor series to second order about the point (a, b) is:

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$= \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right] + \dots \qquad (2.2)$$

where the subscripts denote the respective partial derivatives.

Definition 1. The coefficient of variables of a function $w(x,y)$ is defined as

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{\substack{x=0\\y=0}}, \qquad (2.3)$$

where it is be noted that upper case symbol $W(k,h)$ is used to denote the coefficients of variables in (2.2) which represented by a corresponding lower case symbol $w(x,y)$.

Definition 2. The inverse of $W(k,h)$ is defined as

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k y^h. \qquad (2.4)$$

From Equations (2.3) and (2.4), we obtain

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{x^k y^h}{k!h!}\left[\frac{\partial^{k+h}w(x,y)}{\partial x^k\partial y^h}\right]_{\substack{x=0\\y=0}} \tag{2.5}$$

Theorem 1. If $w(x,y)=u(x,y)\pm v(x,y)$, then $W(k,h)=U(k,h)\pm V(k,h)$.

Proof. By definition 1 we have

$$U(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial y^h}u(x,y)\right]_{\substack{x=0\\y=0}} \tag{2.6}$$

$$V(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial y^h}v(x,y)\right]_{\substack{x=0\\y=0}} \tag{2.7}$$

$$W(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial y^h}u(x,y)\pm v(x,y)\right]_{\substack{x=0\\y=0}} \tag{2.8}$$

Using equations (2.6)-(2.8), we have

$$W(k,h)=U(k,h)\pm V(k,h) \tag{2.9}$$

Theorem 2. If $w(x,y)=\lambda u(x,y)$, then $W(k,h)=\lambda U(k,h)$.

Proof. By definition 1 we have

$$U(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial y^h}u(x,y)\right]_{\substack{x=0\\y=0}} \tag{2.10}$$

$$W(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial y^h}\lambda w(x,y)\right]_{\substack{x=0\\y=0}} \tag{2.11}$$

Using equations (2.10)-(2.11), we have

$$W(k,h)=\lambda U(k,h). \tag{2.12}$$

3. Numerical Solution of Second-Order Partial Differential Equations

This section aim at describing a numerical solution of partial differential equations by power series. We write power series in the form

$$w(x,y) = W(0,0) + W(1,0)x + W(0,1)y + W(1,1)xy + \ldots + ax^m y^n \tag{3.1}$$

where $W(0,0), W(1,0), W(0,1), W(1,1)\ldots$ are known constants but is unknown constant. Substituting (3.1) into (1.1), we can get the following:

$$W(m,n) = (\mu a + \lambda)x^m y^{n-i} = 0 \tag{3.2}$$

where μ and λ are constant and is order of partial differential equation. From (3.2), we have a constant. Substituting (3.2) into (3.1), we get solution arbitrary order of partial differential equation. Repeating this procedure we can get the arbitrary order power series of the solution for PDEs in (1.1).

4. Applications

Example 1.

The test problem consider the fallowing partial differential equation [7],

$$\frac{\partial^2 u(x.t)}{\partial t^2} - c^2 \frac{\partial^2 u(x.t)}{\partial x^2} = 0 \tag{4.1}$$

with initial condition

$$u(x,0) = x^3 \tag{4.2}$$

$$\frac{\partial u(x,t)}{\partial t} = x. \tag{4.3}$$

Using equation (2.4), and initial condition equation (4.2),we obtain

$$U(i,0)=0, \qquad i=0,1,2,4,...,m,$$
$$U(3,0)=1. \tag{4.4}$$

Using equation (2.4), and initial condition equation (4.3),we obtain

$$U(i,1)=0, \qquad i=0,2,...,n,$$
$$U(1,1)=1. \tag{4.5}$$

Substituting equations (4.4)-(4.5) into equations (2.4), we have

$$u_1(x,t)=xt+x^3+axt^2 \tag{4.6}$$

Substituting equations (4.6) into equations (4.1), and by recursive method,the results corresponding to $m\to\infty, \quad n\to\infty$ are listed as follows

$$U(1,2)=3c^2 \tag{4.7}$$

and the others are zero.

We obtain the closed form series solution as follows.

$$u(x,t)=xt+x^3+3c^2xt^2 \tag{4.8}$$

By analitik method [7] ,a closed form solution is obtained as follows.

$$u\hat{}(x,t)=xt+x^3+3c^2xt^2 \tag{4.9}$$

From last two equations, we have $u(x,t)=u\hat{}(x,t)$.

Example 2. We now consider the problem [12]

$$u_{tt} = u_{xx} + 6 \tag{4.10}$$

With initial condition

$$\begin{aligned} u(x,0) &= x^2, \\ u_t(x,0) &= 4x. \end{aligned} \tag{4.11}$$

Using equation (2.4), and initial condition equation (4.11),we obtain

$$\begin{aligned} &U(i,0) = 0, \qquad i = 0,1,3,4,...,m, \\ &U(2,0) = 1 \end{aligned} \tag{4.12}$$

Using equation (2.4), and initial condition equation (4.12),we obtain

$$\begin{aligned} &U(i,1) = 0, \qquad i = 0,2,...,n, \\ &U(1,1) = 4 \end{aligned} \tag{4.13}$$

Substituting equations (4.12)-(4.13) into equations (2.4), we have

$$u_1(x,t) = 4xt + x^2 + at^2 \tag{4.14}$$

Substituting equations(4.14) into equations (4.10), we get

the results corresponding to $m \to \infty, \quad n \to \infty$ are listed as follows

$$U(0,2) = 4 \tag{4.15}$$

and the others are zero.

Substituting all $U(k,h)$ into equation (2.4),we obtain the closed form series solution as follows.

$$u(x,t) = 4xt + x^2 + 4t^2 \tag{4.16}$$

Again, we have obtained the exact solution of the initial value problem stated previously.

Example 3. [7]

$$\frac{\partial^2 u(x.t)}{\partial t^2} - \frac{\partial^2 u(x.t)}{\partial x^2} - x^2 u(x,t) = x \tag{4.17}$$

with initial condition

$$\begin{aligned} u(x,0) &= 0, \\ \frac{\partial u(x,0)}{\partial t} &= 0. \end{aligned} \tag{4.18}$$

Using equation (2.4) and initial conditions(4.18) , we have

$$U(i,0) = 0, \qquad\qquad i = 0,1,2,...,m. \tag{4.19}$$

$$U(i,1) = 0, \qquad\qquad i = 0,1,2,...,n. \tag{4.20}$$

the results corresponding to $m \to \infty, \quad n \to \infty$ are listed as follows

$$\begin{aligned} U(1,2) &= \frac{1}{2}, \\ U(3,4) &= \frac{1}{24}, \\ U(1,6) &= \frac{1}{120} \end{aligned}$$

and the others are zero.

Substituting all $U(k,h)$ into equation(2.4) ,we obtain

$$u(x,t) = \frac{xt^2}{2} + \frac{x^3t^4}{24} + \frac{xt^6}{120}. \tag{4.21}$$

In this case, we have obtained the exact solution of the targeted equation with the specified initial conditions

5. Conclusion

Analytic solutions of the second-order linear partial differential equations with variable coefficients are usually difficult. In many cases, it is required toapproximate solutions. For this purpose, power series method can be proposed.In this study, the usefulness power series method presented for the approximate solution of the second-order linear partial differential equations is discussed. Also, the method can be applied to the non-homogeneous (Example 2) and homogeneous (Examples 1) cases. This method is very simple an effective for most of linear partial differential equations.

References

[1] Marszalek, W. (1997), Analysis of partial differential algebraic equations, Ph.D. Thesis, North Carolina State University, Raleigh.

[2] Lucht, W. , Strehmel, K. , Eichler-Liebenow, C. (1997a), “Linear partial differential algebraic equations, Part I: Indexes, consistent boundary/initial conditions”, Report 17, Fachbereich Mathematik und Informatik, Martin-Luther-Universitat Halle.

[3] Lucht, W. , Strehmel, K. , Eichler-Liebenow, C. (1997b), “Linear partial differential algebraic equations, Part II: Numerical solution”, Report 18, Fachbereich Mathematik und Informatik, Martin-Luther-Universitat Halle.

[4] Debrabant, K. ve Strehmel K. (2005), “Convergence of Runge-Kutta methods applied to linear partial differential-algebraic equations”, Applied Numerical Mathematics, 53:213-229.

[5] Martinson, W. , Barton, P. (2000), “A differentiation index for partial differential-algebraic equations”, SIAM J. Sci. Comput., Vol. 21, 6:2295-2315.

[6] Martinson, W. , Barton, P. (2002), “Index and characteristic analysis of linear PDAE systems”, SIAM J. Sci. Comput., Vol. 24, 3:905-923.

[7] Chen, C.K. , Ho, S.H. (1999), “Solving partial differential equations by two-dimensional differential transform method”, Applied Mathematics and Computation,106:171-179.

[8] Hassan, I.H. (2002), “On solving some eigenvalue problems by using a differential transformation”, Applied Mathematics and Computation, 127:1-22.

[9] Jang, M.J. , Chen, C.L. , Liy, L.C. (2000), “On solving the initial-value problems using the differential transformation method”, Applied Mathematics and Computation, 115:145-160.

[10] Liu, H. and Song, Y. (2007), “Differential transform method applied to high index differential-algebraic equations”, Applied Mathematics and Computation, 184:748-753.

[11] X Yang, Y. Liu, S. Bai, (2006). “ A numerical solution of second-order linear partial differential equations by differential transform”, Applied Mathematics and Computation 173 ;792-802.

[12] C.Kesan, (2003) ,“ Chebyshev polynomial solutions of second-order linear partial differential equations”, Applied Mathematics and Computation 134 ;109–124.

[13] Bayram, M., Çelik, E., (2003), “Arbitrary Order Numerical Method for Solving Differential- Algebraic Equations by Padé Series”, Applied Mathematics and Computation. 137;57-65.