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Exponential Stability of Almost Periodic Solution for Shunting Inhibitory Cellular Neural Networks with Time-Varying and Distributed Delays

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Abstract. In this paper, shunting inhibitory cellular neural networks (SICNNs) with time-varying and distributed delays are considered. Without assuming the global Lipschitz conditions of activation functions, some new sufficient conditions for the existence and exponential stability of the almost periodic solutions are established. Finally, a numerical example is given to demonstrate the effectiveness of the obtained result.

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1. Introduction

Recently, the dynamical behaviors of almost periodic solutions for shunting inhibitory cellular neural networks (SICNNs) have been extensively studied (see [1 −

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11]), due to SICNNs have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Many important results have been established and successfully applied to signal processing, pattern recognition, associative memories, and so on. However, in the existing literatures (see $[1-3,5-9]$), almost all results on the stability of almost periodic solutions for SICNNs are obtained under global Lipschitz neuron activations. When neuron activation functions do not satisfy global Lipschitz conditions, people want to know whether the SICNNs is stable. In practical engineering applications, people also need to present new neural networks. Therefore, developing a new class of SICNNs without global Lipschitz neuron activation functions and giving the conditions of the stability of new SICNNs are very interesting and valuable.

Consider the following SICNNs with time-varying and distributed delays:

$$\begin{aligned} x_{ij}'(t) = &- a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) f(x_{kl}(t-\tau(t)))x_{ij}(t) \\ &- \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_0^{\infty} K_{ij}(u) g(x_{kl}(t-u))\mathrm{d}u x_{ij}(t) + L_{ij}(t), \end{aligned} \tag{1.1}$$

where $i=1,\cdots,m$, $j=1,\cdots,n$, C_{ij} is the cell at the (i,j) position of the lattice, the r-neighborhood $N_r(i,j)$ of C_{ij} is

$$N_r(i,j) = \left\{C_{kl} : \max(|k-i|, \left|l-j\right|) \le r, 1\le k\le m, 1\le l\le n\right\},$$

$N_q(i,j)$ is similarly specified. x_{ij} is the activity of the cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, $a_{ij}(t) > 0$ is the passive decay rate of the cell activity, $C_{ij}^{kl}(t) \ge 0$ and $B_{ij}^{kl}(t) \ge 0$ are the connections or coupling strengths of postsynaptic activity of the cells in $N_r(i,j)$ and $N_q(i,j)$ transmitted to the cell C_{ij}, respectively. The activity functions $f(x_{kl})$ and $g(x_{kl})$ are continuous functions representing the output or firing rate of cell C_{kl}, and $\tau(t) \ge 0$ is the transmission delay.

Throughout this paper, we will assume that $\tau(t) : R \to R$ is an almost periodic function, and $0 \le \tau(t) \le \overline{\tau}$, where $\overline{\tau} \ge 0$ is a constant.

Set $\left\{x_{ij}(t)\right\} = (x_{11}(t),\cdots,x_{1n}(t),\cdots,x_{m1}(t),\cdots,x_{mn}(t))$, for $\forall x = \left\{x_{ij}(t)\right\} \in R^{m\times n}$, we define the norm $\|x\| = \max_{(i,j)}\left\{\left|x_{ij}(t)\right|\right\}$.

Set $B = \left\{\varphi \mid \varphi = \left\{\varphi_{ij}(t)\right\} = (\varphi_{11}(t),\cdots,\varphi_{1n}(t),\cdots,\varphi_{m1}(t),\cdots,\varphi_{mn}(t))\right\}$, where φ is is an almost periodic function on R. For $\forall\varphi \in B$, we define the norm $\left\|\varphi\right\|_B = \sup_{t\in R}\left\|\varphi(t)\right\|$, then B is a Banach space.

The initial conditions associated with system (1.1) are of the form

$$x_{ij}(s) = \varphi_{ij}(s), s \in (-\infty, 0], i = 1,\cdots,m, j = 1,\cdots,n, \tag{1.1}$$

where $\varphi = \left\{\varphi_{ij}(t)\right\} \in C((-\infty,0], R^{m\times n})$.

Definition 1.1. *Let $k \in Z^+$. A continuous function $u : R \to R^k$ is called almost periodic if for each $\varepsilon > 0$ there exists a constant $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number δ with the property that*

$$\|u(t+\delta) - u(t)\| < \varepsilon \quad \text{for all} \quad t \in R.$$

Definition 1.2. *Let $x \in R^n$ and $Q(t)$ be a $n \times n$ continuous matrix defined on R. The linear system*

$$x'(t) = Q(t)x(t) \tag{1.3}$$

is said to admit an exponential dichotomy on R if there exist positive constants k, α, projection P and the fundamental solution matrix $X(t)$ of (1.3) satisfying

$$\left\|X(t)PX^{-1}(s)\right\| \le ke^{-\alpha(t-s)} \quad \text{for} \quad t \ge s,$$

$$\left\|X(t)(I-P)X^{-1}(s)\right\| \le ke^{-\alpha(s-t)} \quad \text{for} \quad t \le s.$$

Lemma 1.1. [12]. *If the linear system (1.3) admits an exponential dichotomy, then almost periodic system*

$$x'(t) = Q(t)x(t) + g(t) \tag{1.4}$$

has a unique almost periodic solution $x(t)$, and

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)g(s)\mathrm{d}s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)g(s)\mathrm{d}s.$$

Lemma 1.2. [12]. *Let $c_i(t)$ be an almost periodic function on R and*

$$M[c_i] = \lim_{T\to+\infty} \frac{1}{T}\int_{t}^{t+T} c_i(s)\mathrm{d}s > 0, \quad i = 1,\cdots,n.$$

Then the linear system $x'(t) = diag(-c_1(t),\cdots,-c_n(t))x(t)$ admits an exponential dichotomy on R.

2. Existence of Almost Periodic Solutions

Theorem 2.1. *Assume that*

(H_1) *For $i = 1,\cdots,m$, $j = 1,\cdots,n$, the delay kernels $K_{ij} : [0,\infty) \to R$ are continuous and integrable, $a_{ij}, C_{ij}^{kl}, B_{ij}^{kl}, L_{ij} \in B$;*

(H_2) *there exists a continuous function $L : R^+ \to R^+$ such that for each $r > 0$,*

$$\left|f(u) - f(v)\right| \le L(r)\left|u - v\right|, \quad |u|,|v| \le r;$$

$$\left|g(u) - g(v)\right| \le L(r)\left|u - v\right|, \quad |u|,|v| \le r.$$

(H_3) *there exists a constant $r_0 > 0$ such that*

$$D[F(0)r_0 + L(r_0)r_0^2] + L \le r_0, \quad DF(0) + 2DL(r_0)r_0 < 1,$$

where $D = \max_{(i,j)} \{ \frac{\sum_{C_{kl} \in N_r(i,j)} \overline{C}_{ij}^{kl} + \sum_{C_{kl} \in N_q(i,j)} \overline{B}_{ij}^{kl} \int_0^\infty |K_{ij}(u)| \, du}{\underline{a}_{ij}} \} > 0,$

$F(0) = \max \{ |f(0)|, |g(0)| \}$, $L = \max_{(i,j)} \frac{\overline{L}_{ij}}{\underline{a}_{ij}}$, $\overline{L}_{ij} = \sup_{t \in R} |L_{ij}(t)|$, $\overline{C}_{ij}^{kl} = \sup_{t \in R} C_{ij}^{kl}(t)$, $\overline{B}_{ij}^{kl} = \sup_{t \in R} B_{ij}^{kl}(t)$,
$\underline{a}_{ij} = \inf_{t \in R} a_{ij}(t) > 0$.

Then SICNNs (1.1) has a unique almost periodic solution in the region
$E := \{ \varphi \in B : \|\varphi\|_B \le r_0 \}$.

Proof. For any given $\varphi \in B$, we consider the following almost periodic differential equation:

$$\begin{aligned} x'_{ij}(t) = & -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f(\varphi_{kl}(t - \tau(t)))\varphi_{ij}(t) \\ & - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(t) \int_0^\infty K_{ij}(u) g(\varphi_{kl}(t-u)) du \varphi_{ij}(t) + L_{ij}(t). \end{aligned} \tag{2.1}$$

Then, notice that $M[a_{ij}] > 0$, from Lemma 1.2, the linear system

$$x'_{ij}(t) = -a_{ij}(t)x_{ij}(t), \quad i = 1, \cdots, m, j = 1, \cdots, n, \tag{2.2}$$

admits an exponential dichotomy on R. Thus, by Lemma 1.1, we obtain that the system (2.1) has exactly one almost periodic solution:

$$\begin{aligned} x_\varphi(t) = & \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} \Big(- \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) f(\varphi_{kl}(s - \tau(s)))\varphi_{ij}(s) \\ & - \sum_{C_{kl} \in N_q(i,j)} B_{ij}^{kl}(s) \int_0^\infty K_{ij}(u) g(\varphi_{kl}(s-u)) du \varphi_{ij}(s) + L_{ij}(s) \Big) ds. \end{aligned}$$

Now, we define a nonlinear operator on B by $T(\varphi)(t) = x_\varphi(t), \forall \varphi \in B$. Next, we will prove $T(E) \subset E$. For any given $\varphi \in E$, it suffices to prove that $\|T(\varphi)\|_B \le r_0$. By (H_2) and (H_3), we have

$$\|T(\varphi)\|_B = \sup_{t \in R} \max_{(i,j)} \{ | \int_{-\infty}^t e^{-\int_s^t a_{ij}(u) du} \Big(- \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(s) f(\varphi_{kl}(s - \tau(s)))\varphi_{ij}(s)$$

$$- \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_0^{\infty} K_{ij}(u)g(\varphi_{kl}(s-u))\mathrm{d}u\varphi_{ij}(s)+L_{ij}(s)\Big)\mathrm{d}s\,|\}$$

$$\le \sup_{t\in R}\max_{(i,j)}\{|\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl} f(\varphi_{kl}(s-\tau(s)))\varphi_{ij}(s)\mathrm{d}s$$

$$+\sum_{C_{kl}\in N_q(i,j)}\overline{B}_{ij}^{kl}\int_0^{\infty}K_{ij}(u)g(\varphi_{kl}(s-u))\mathrm{d}u\varphi_{ij}(s)\mathrm{d}s\Big)\,|\}+\max_{(i,j)}\frac{\overline{L}_{ij}}{\underline{a}_{ij}}$$

$$\le \sup_{t\in R}\max_{(i,j)}\{|\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}(\left|f(0)\right|+L(r_0)\,|\,\varphi_{kl}(s$$

$$-\tau(s))\,|)\left|\varphi_{ij}(s)\right|\mathrm{d}s+\sum_{C_{kl}\in N_q(i,j)}\overline{B}_{ij}^{kl}(\left|g(0)\right|+L(r_0)\left|\varphi_{kl}(s-\tau(s))\right|)\cdot$$

$$\int_0^{\infty}\left|K_{ij}(u)\right|\mathrm{d}u\left|\varphi_{ij}(s)\right|\mathrm{d}s\Big)\,|\}+L$$

$$\le \sup_{t\in R}\max_{(i,j)}\{|\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}(\left|f(0)\right|+L(r_0)r_0)r_0\mathrm{d}s$$

$$+\sum_{C_{kl}\in N_q(i,j)}\overline{B}_{ij}^{kl}(\left|g(0)\right|+L(r_0)r_0)r_0\int_0^{\infty}\left|K_{ij}(u)\right|\mathrm{d}u\mathrm{d}s\Big)\,|\}+L$$

$$\le D[F(0)r_0+L(r_0)r_0^2]+L\le r_0.$$

Therefore, $T(E)\subset E$.

Taking $\varphi,\psi\in E$, combining (H_2) and (H_3), we deduce that

$$\left\|T(\varphi)-T(\psi)\right\|_B=\sup_{t\in R}\left\|T(\varphi)(t)-T(\psi)(t)\right\|$$

$$=\sup_{t\in R}\max_{(i,j)}\{|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)\mathrm{d}u}\sum_{C_{kl}\in N_r(i,j)}-C_{ij}^{kl}(s)\Big(f(\varphi_{kl}(s-\tau(s)))\varphi_{ij}(s)$$

$$-f(\psi_{kl}(s-\tau(s)))\psi_{ij}(s)\Big)\mathrm{d}s\,|+|\int_{-\infty}^{t} e^{-\int_s^t a_{ij}(u)\mathrm{d}u}\sum_{C_{kl}\in N_q(i,j)}-B_{ij}^{kl}(s)\cdot$$

$$\int_0^{\infty}K_{ij}(u)\Big(g(\varphi_{kl}(s-u))\mathrm{d}u\varphi_{ij}(s)-g(\psi_{kl}(s-u))\mathrm{d}u\psi_{ij}(s)\Big)\mathrm{d}s\,|\}$$

$$\le \sup_{t\in R}\max_{(i,j)}\{\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}\,|\,f(\varphi_{kl}(s-\tau(s)))\,||\,\varphi_{ij}(s)-\psi_{ij}(s)\,|\,\mathrm{d}s\}$$

$$
\begin{aligned}
&+\sup_{t\in R}\max_{(i,j)}\{\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}\mid f(\varphi_{kl}(s-\tau(s)))-f(\psi_{kl}(s\\
&-\tau(s)))\mid\cdot\mid\psi_{ij}(s)\mid \mathrm{d}s\}+\sup_{t\in R}\max_{(i,j)}\{\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\sum_{C_{kl}\in N_q(i,j)}\overline{B}_{ij}^{kl}\cdot\\
&\int_0^{\infty}\mid K_{ij}(u)g(\varphi_{kl}(s-u))\mathrm{d}u\mid\cdot\mid\varphi_{ij}(s)-\psi_{ij}(s)\mid \mathrm{d}s\}\\
&+\sup_{t\in R}\max_{(i,j)}\{\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\sum_{C_{kl}\in N_q(i,j)}\overline{B}_{ij}^{kl}\int_0^{\infty}\mid K_{ij}(u)\Big(g(\varphi_{kl}(s-u))\\
&-g(\psi_{kl}(s-u))\Big)\mathrm{d}u\mid\cdot\mid\psi_{ij}(s)\mid \mathrm{d}s\}\\
\le&\sup_{t\in R}\max_{(i,j)}\{\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}(\left|f(0)\right|+L(r_0)r_0)\mathrm{d}s\}\cdot\left\|\varphi-\psi\right\|_B\\
&+\sup_{t\in R}\max_{(i,j)}\{\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}L(r_0)r_0\mathrm{d}s\}\cdot\left\|\varphi-\psi\right\|_B\\
&+\sup_{t\in R}\max_{(i,j)}\{\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\sum_{C_{kl}\in N_q(i,j)}\overline{B}_{ij}^{kl}(\left|g(0)\right|+L(r_0)r_0)\int_0^{\infty}\left|K_{ij}(u)\right|\mathrm{d}u\mathrm{d}s\}\\
&\quad\cdot\left\|\varphi-\psi\right\|_B\\
&+\sup_{t\in R}\max_{(i,j)}\{\int_{-\infty}^{t} e^{-\underline{a}_{ij}(t-s)}\sum_{C_{kl}\in N_q(i,j)}\overline{B}_{ij}^{kl}L(r_0)r_0\int_0^{\infty}\left|K_{ij}(u)\right|\mathrm{d}u\mathrm{d}s\}\cdot\left\|\varphi-\psi\right\|_B\\
\le& D(F(0)+L(r_0)r_0)\cdot\left\|\varphi-\psi\right\|_B+DL(r_0)r_0\cdot\left\|\varphi-\psi\right\|_B\\
\le&[DF(0)+2DL(r_0)r_0]\cdot\left\|\varphi-\psi\right\|_B<\left\|\varphi-\psi\right\|_B
\end{aligned}
$$

So T is a contraction from E to E. Since E is a closed subset of B, T has a unique fixed point in E, which means system (1.1) has a unique almost periodic solution in E.

3. Exponential Stability of the Almost Periodic Solution

Theorem 3.1. *Suppose* $(H_1)-(H_3)$ *hold, let* $x^*(t)=\left\{x_{ij}^*(t)\right\}$ *be the unique almost periodic solution of SICNNs (1.1) in the region* $\left\|\varphi\right\|_B\le r_0$. *Further we assume that*

(H_4) there exists a constant $r_1 \geq r_0$ such that

$$F(0) + L(r_0)r_0 + L(r_1)r_1 < \frac{1}{D},$$

where $F(0) = \max\left\{\left|f(0)\right|, \left|g(0)\right|\right\}$;

(H_5) For $i = 1, \cdots, m,\ j = 1, \cdots, n$, there exists a constant $\lambda_0 > 0$ such that

$$\int_0^{\infty} \left|K_{ij}(s)\right| e^{\lambda_0 s} \mathrm{d}s < +\infty.$$

Then there exists a constant $\lambda > 0$ such that for any solution $x(t) = \left\{x_{ij}(t)\right\}$ of SICNNs (1.1) with initial value $\sup_{t\in(-\infty,0]} \left\|\varphi(t)\right\| \leq r_1$,

$$\|x(t) - x^*(t)\| \leq Me^{-\lambda t}, \quad \forall t > 0,$$

where $M = \sup_{t\in(-\infty,0]} \left\|\varphi(t) - x^*(t)\right\|$.

Proof. Set

$$\Gamma_{ij}(\alpha) = \alpha - \underline{a}_{ij} + \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}(\left|f(0)\right| + L(r_0)r_0 + L(r_1)r_1 e^{\alpha\overline{\tau}}) + \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl} \cdot \\ \left[(\left|g(0)\right| + L(r_0)r_0) \int_0^{\infty} \left|K_{ij}(s)\right| \mathrm{d}s + L(r_1)r_1 \int_0^{\infty} \left|K_{ij}(s)\right| e^{\alpha s} \mathrm{d}s\right]$$

where $i = 1, \cdots, m,\ j = 1, \cdots, n$. It is easy to prove that Γ_{ij} are continuous functions on $[0, \lambda_0]$. Moreover, by (H_4) and (H_5), we have

$$\begin{aligned}
\Gamma_{ij}(0) = & -\underline{a}_{ij} + \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} \left(\left|f(0)\right| + L(r_0)r_0 + L(r_1)r_1\right) \\
& + \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl} \left[\left|g(0)\right| + L(r_0)r_0 + L(r_1)r_1\right] \int_0^{\infty} \left|K_{ij}(s)\right| \mathrm{d}s \\
\leq & -\underline{a}_{ij} + \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} \left(F(0) + L(r_0)r_0 + L(r_1)r_1\right) \\
& + \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl} \left[F(0) + L(r_0)r_0 + L(r_1)r_1\right] \int_0^{\infty} \left|K_{ij}(s)\right| \mathrm{d}s < 0.
\end{aligned}$$

Thus, there exists a sufficiently small constant $\lambda > 0$ such that

$$\Gamma_{ij}(\lambda) < 0, \quad i = 1, \cdots, m, j = 1, \cdots, n. \tag{3.1}$$

Take $\varepsilon > 0$. Set $Z_{ij}(t) = \left| x_{ij}(t) - x^*_{ij}(t) \right| e^{\lambda t}$, $i = 1, \cdots, m$, $j = 1, \cdots, n$. It follows that: $Z_{ij}(t) \leq M < M + \varepsilon, \forall\, t \in (-\infty, 0]$, $i = 1, \cdots, m$, $j = 1, \cdots, n$. In the following, we will prove that

$$Z_{ij}(t) \leq M + \varepsilon, \forall\, t > 0, i = 1, \cdots, m, j = 1, \cdots, n. \tag{3.2}$$

If this is not true, then there exist $i_0 \in \{1, \cdots, m\}$ and $j_0 \in \{1, \cdots, n\}$ such that

$$\left\{ t > 0 \mid Z_{i_0 j_0}(t) > M + \varepsilon \right\} \neq \emptyset. \tag{3.3}$$

Let

$$t_{ij} = \begin{cases} \inf\left\{ t > 0 \mid Z_{ij}(t) > M + \varepsilon \right\}, & \left\{ t > 0 \mid Z_{ij}(t) > M + \varepsilon \right\} \neq \emptyset, \\ +\infty, & \left\{ t > 0 \mid Z_{ij}(t) > M + \varepsilon \right\} = \emptyset. \end{cases}$$

Then $t_{ij} > 0$ and

$$Z_{ij}(t) \leq M + \varepsilon, \forall\, t \in (-\infty, t_{ij}], i = 1, \cdots, m, j = 1, \cdots, n. \tag{3.4}$$

We denote $t_{ph} = \min\limits_{(i,j)} t_{ij}$, where $p \in \{1, \cdots, m\}$ and $h \in \{1, \cdots, n\}$. From (3.3), we have $0 < t_{ph} < +\infty$. It follows from (3.4), we have

$$Z_{ij}(t) \leq M + \varepsilon, \forall\, t \in (-\infty, t_{ph}], i = 1, \cdots, m, j = 1, \cdots, n. \tag{3.5}$$

In addition, noticing that $t_{ph} = \inf\left\{ t > 0 \mid Z_{ph}(t) > M + \varepsilon \right\}$, we obtain

$$Z_{ph}(t_{ph}) = M + \varepsilon, \quad \text{and} \quad D^+ Z_{ph}(t_{ph}) \geq 0. \tag{3.6}$$

Since $x(t)$ and $x^*(t)$ are solutions of Eq.(1.1), combining with (3.5)-(3.6), (H_2) and (H_3), we have

$$0 \leq D^+ Z_{ph}(t_{ph}) = D^+ [\left| x_{ph}(t) - x^*_{ph}(t) \right| e^{\lambda t}] \mid_{t = t_{ph}}$$

$$
\begin{aligned}
&\leq \left|x_{ph}(t_{ph})-x^*_{ph}(t_{ph})\right| \lambda e^{\lambda t_{ph}} - \underline{a}_{ph}\left|x_{ph}(t_{ph})-x^*_{ph}(t_{ph})\right| e^{\lambda t_{ph}} \\
&\quad + \sum_{C_{kl}\in N_r(p,h)} \overline{C}^{kl}_{ph} \mid f(x_{kl}(t_{ph}-\tau(t_{ph})))x_{ph}(t_{ph}) - f(x^*_{kl}(t_{ph}-\tau(t_{ph})))x^*_{ph}(t_{ph}) \mid \cdot \\
&\quad\ e^{\lambda t_{ph}} + \sum_{C_{kl}\in N_q(p,h)} \overline{B}^{kl}_{ph} \mid \int_0^\infty K_{ij}(u) g(x_{kl}(t_{ph}-u))\mathrm{d}u x_{ph}(t_{ph}) \\
&\quad - \int_0^\infty K_{ij}(u) g(x^*_{kl}(t_{ph}-u))\mathrm{d}u x^*_{ph}(t_{ph}) \mid e^{\lambda t_{ph}} \\
&\leq (\lambda - \underline{a}_{ph}) Z_{ph}(t_{ph}) + \sum_{C_{kl}\in N_r(p,h)} \overline{C}^{kl}_{ph} \mid f(x^*_{kl}(t_{ph}-\tau(t_{ph}))) \mid \cdot \mid x_{ph}(t_{ph}) \\
&\quad\ - x^*_{ph}(t_{ph}) \mid e^{\lambda t_{ph}} + \sum_{C_{kl}\in N_r(p,h)} \overline{C}^{kl}_{ph} \mid f(x_{kl}(t_{ph}-\tau(t_{ph}))) \\
&\quad\ - f(x^*_{kl}(t_{ph}-\tau(t_{ph}))) \mid \cdot \mid x_{ph}(t_{ph}) \mid e^{\lambda t_{ph}} + \sum_{C_{kl}\in N_q(p,h)} \overline{B}^{kl}_{ph} \int_0^\infty \left|K_{ij}(u)\right| \cdot \\
&\quad\ \left|g(x^*_{kl}(t_{ph}-u))\right| \mathrm{d}u \cdot \mid x_{ph}(t_{ph}) - x^*_{ph}(t_{ph}) \mid e^{\lambda t_{ph}} + \sum_{C_{kl}\in N_q(p,h)} \overline{B}^{kl}_{ph} \cdot \\
&\quad\ \int_0^\infty \left|K_{ij}(u)\right| \cdot \mid g(x_{kl}(t_{ph}-u)) - g(x^*_{kl}(t_{ph}-u)) \mid \mathrm{d}u \cdot \mid x_{ph}(t_{ph}) \mid e^{\lambda t_{ph}} \\
&\leq (\lambda - \underline{a}_{ph})(M+\varepsilon) + \sum_{C_{kl}\in N_r(p,h)} \overline{C}^{kl}_{ph} (\left|f(0)\right| + L(r_0) r_0) \cdot Z_{ph}(t_{ph}) \\
&\quad\ + \sum_{C_{kl}\in N_r(p,h)} \overline{C}^{kl}_{ph} L(r_1) \left|x_{kl}(t_{ph}-\tau(t_{ph})) - x^*_{kl}(t_{ph}-\tau(t_{ph}))\right| \cdot \\
&\quad\ e^{\lambda(t_{ph}-\tau(t_{ph}))} e^{\lambda \tau(t_{ph})} \cdot r_1 + \sum_{C_{kl}\in N_q(p,h)} \overline{B}^{kl}_{ph} (\left|g(0)\right| + L(r_0) r_0) \cdot \\
&\quad\ \int_0^\infty \left|K_{ij}(u)\right| \mathrm{d}u \cdot Z_{ph}(t_{ph}) + \sum_{C_{kl}\in N_q(p,h)} \overline{B}^{kl}_{ph} L(r_1) \int_0^\infty \left|K_{ij}(u)\right| \cdot \\
&\quad\ \left|x_{kl}(t_{ph}-u) - x^*_{kl}(t_{ph}-u)\right| e^{\lambda(t_{ph}-u)} e^{\lambda u} \mathrm{d}u \cdot r_1 \\
&\leq (\lambda - \underline{a}_{ph})(M+\varepsilon) + \sum_{C_{kl}\in N_r(p,h)} \overline{C}^{kl}_{ph} (\left|f(0)\right| + L(r_0) r_0) \cdot (M+\varepsilon) \\
&\quad\ + \sum_{C_{kl}\in N_r(p,h)} \overline{C}^{kl}_{ph} L(r_1) r_1 e^{\lambda \overline{\tau}} \cdot (M+\varepsilon) + \sum_{C_{kl}\in N_q(p,h)} \overline{B}^{kl}_{ph} (\left|g(0)\right|
\end{aligned}
$$

$$+ L(r_0)r_0) \cdot \int_0^{\infty} \left|K_{ij}(u)\right| \mathrm{d}u \cdot (M+\varepsilon)$$
$$+ \sum_{C_{kl}\in N_q(p,h)} \overline{B}_{ph}^{kl} L(r_1)r_1 \int_0^{\infty} \left|K_{ij}(u)\right| e^{\lambda u}\mathrm{d}u \cdot (M+\varepsilon)$$
$$\leq (\lambda - \underline{a}_{ph})(M+\varepsilon) + \sum_{C_{kl}\in N_r(p,h)} \overline{C}_{ph}^{kl}(F(0) + L(r_0)r_0) \cdot (M+\varepsilon)$$
$$+ \sum_{C_{kl}\in N_r(p,h)} \overline{C}_{ph}^{kl} L(r_1)r_1 e^{\lambda\overline{\tau}} \cdot (M+\varepsilon) + \sum_{C_{kl}\in N_q(p,h)} \overline{B}_{ph}^{kl}\big(F(0)$$

$$+ L(r_0)r_0) \cdot \int_0^{\infty} \left|K_{ij}(u)\right| \mathrm{d}u \cdot (M+\varepsilon)$$
$$+ \sum_{C_{kl}\in N_q(p,h)} \overline{B}_{ph}^{kl} L(r_1)r_1 \int_0^{\infty} \left|K_{ij}(u)\right| e^{\lambda u}\mathrm{d}u \cdot (M+\varepsilon)$$

It follows that:

$$\lambda - \underline{a}_{ph} + \sum_{C_{kl}\in N_r(p,h)} \overline{C}_{ph}^{kl}(F(0) + L(r_0)r_0 + L(r_1)r_1 e^{\lambda\overline{\tau}}) + \sum_{C_{kl}\in N_q(p,h)} \overline{B}_{ph}^{kl} \cdot$$
$$\big[(F(0) + L(r_0)r_0) \int_0^{\infty} \left|K_{ij}(u)\right| \mathrm{d}u + L(r_1)r_1 \int_0^{\infty} \left|K_{ij}(u)\right| e^{\lambda u}\mathrm{d}u\big] \geq 0,$$

that is $\Gamma_{ph}(\lambda) \geq 0$. This contradicts with (3.1). Hence, (3.2) holds, i.e.,

$$\left|x_{ij}(t) - x_{ij}^*(t)\right| e^{\lambda t} = Z_{ij}(t) \leq M + \varepsilon, \forall\, t > 0, i = 1, \cdots, m, j = 1, \cdots, n.$$

Therefore,

$$\|x(t) - x^*(t)\| = \max_{(i,j)} \left|x_{ij}(t) - x_{ij}^*(t)\right| \leq (M+\varepsilon)e^{-\lambda t}, \forall\, t > 0.$$

Let $\varepsilon \to 0$, we get

$$\|x(t) - x^*(t)\| \leq M e^{-\lambda t}, \forall\, t > 0.$$

4. Illustrative Example

Consider SICNNs (1.1) described by $i,j = 1,2,3$, $\tau(t) = \cos^2 t$, $f(x) = g(x) = \frac{x^4+1}{6}$, $K_{ij}(u) = e^{-u}\sin u$, $a_{ij}(t) = \begin{pmatrix} 5+|\sin t| & 5+\left|\sin\sqrt{2}t\right| & 9+|\sin t| \\ 6+|\sin t| & 6+|\sin t| & 7+|\sin t| \\ 8+|\sin t| & 8+|\sin t| & 5+\left|\sin\sqrt{3}t\right| \end{pmatrix}$,

$$C_{ij}(t) = B_{ij}(t) = \left|\sin\sqrt{3}t\right| \begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{10} & \frac{1}{5} & \frac{1}{10} \end{pmatrix},$$

$$L_{ij}(t) = \begin{pmatrix} \sin t & \sin t & \cos t \\ \frac{\sin t+\sin\sqrt{2}t}{2} & \cos t & \cos t \\ \cos t & \frac{\cos t+\cos\sqrt{3}t}{2} & \sin t \end{pmatrix}.$$

Obviously , let $L(r) = \frac{2}{3}r^3$ and $r_0 = 1$, then we get $D \leq 0.6, L = 0.2$, so $D[F(0)r_0 + L(r_0)r_0^2] + L \leq 0.7 < 1 = r_0, DF(0) + 2DL(r_0)r_0 \leq 0.81 < 1$. From Theorem 2.1, the system in example has a unique almost periodic solution in the region $\left\|\varphi\right\|_B \leq 1$.

Take $r_1 = \sqrt[4]{\frac{51}{50}}$, then $D[F(0) + L(r_0)r_0 + L(r_1)r_1] < 1$. From Theorem 3.1, all the solutions with initial value $\sup\limits_{t\in[-1,0]} \left\|\varphi(t)\right\| \leq r_1$ converge exponentially to the unique almost periodic solution in the region $\left\|\varphi\right\|_B \leq 1$ as $t \to +\infty$.

5. Conclusion

In this paper, some new sufficient conditions are established to ensure the existence and exponential stability of almost periodic solutions for SICNNs with time-varying and distributed delays. Since we do not need the neuron activations to satisfy global Lipschitz conditions, the result in this paper is new, and it is also valuable in the design of neural networks which is used to solve efficiently problems arising in practical engineering applications.

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