The influence of C- 3-permutable subgroups on the structure of finite groups

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Abstract. Let 3 be a complete set of Sylow subgroups of a finite group G, that is, for each prime p dividing the order of G, 3 contains exactly one and only one Sylow p-subgroup of G, say Gp. Let C be a nonempty subset of G. A subgroup H of G is said to be C-3-permutable (conjugate-3-permutable) subgroup of G if there exists some x ∈ C such that HxGp = GpHx, for all Gp ∈ 3. We investigate the structure of the finite group G under the assumption that certain subgroups of prime power orders of G are C-3-permutable subgroups of G.

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1. Introduction

Throughout this article only finite groups are considered. We use conventional notions and notation, as in Doerk and Hawkes [2]. In addition, π(G) denotes the set of distinct primes dividing |G| and Gp is a Sylow p-subgroup of the group G for some prime p ∈ π(G). Two subgroups H and K of a group G are said to be permutable if HK = KH, that is, HK is a subgroup of G. Recall that a subgroup H of a group G is S-permutable (or S-quasinormal) in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel [7] in 1962.

Recently, in 2003, Asaad and Heliel [1] introduced the concept of 3-permutability which generalizes S-permutability as follows: Let 3 be a complete set of Sylow subgroups of a group G. A subgroup H of G is said to be 3-permutable in G if H permutes with every

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member in $\mathfrak{Z}$.

More recently, in 2013, Heliel and Al-Gafri [4] generalized the concept of $\mathfrak{Z}$-permutability by introducing a new subgroup embedding property, namely, the conjugate-$\mathfrak{Z}$-permutability.

Let $C$ be a nonempty subset of a group $G$ and $\mathfrak{Z}$ be a complete set of Sylow subgroups of $G$. A subgroup $H$ of $G$ is said to be $C$-$\mathfrak{Z}$-permutable subgroup of $G$ if there exists some $x \in C$ such that $H^xG_p = G_pH^x$, for all $G_p \in \mathfrak{Z}$. Remark 1.2 and Examples 1.3 and 1.4 in [4] show that $C$-$\mathfrak{Z}$-permutability is a nontrivial generalization of $\mathfrak{Z}$-permutability.

This article may be viewed as a continuation of Heliel and Al-Gafri [4]. In fact, we extend and improve the following theorem:

**Theorem 1.1.** [4], Theorem 3.11] Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$ and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$. Then the following two statements are equivalent:

(a) $G \in \mathfrak{F}$.

(b) There is a normal subgroup $H$ in $G$ and a solvable normal subgroup $C$ of $F^*(H)$ such that $G/H \in \mathfrak{F}$, and the maximal subgroups of $G_p \cap F^*(H)$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$, where $F^*(H)$ is the generalized Fitting subgroup of $H$.

More precisely, we prove the following theorem:

**Theorem 1.2.** Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$. Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group $G$ and let $C$ be a solvable normal subgroup of $G$. Then the following two statements are equivalent:

(a) $G \in \mathfrak{F}$.

(b) There is a normal subgroup $H$ in $G$ such that $G/H \in \mathfrak{F}$ and the maximal subgroups of $G_p \cap F^*(H)$ are $C$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$.

**Remark 1.3.** Let $S(F^*(H))$ denotes the solvable radical of $F^*(H)$, that is, $S(F^*(H))$ is the unique largest solvable normal subgroup of $F^*(H)$. In Theorem 1.1, $C$ is a solvable normal subgroup of $F^*(H)$. Therefore, $C$ is contained in $S(F^*(H))$. Since $S(F^*(H))$ is characteristic in $F^*(H)$ and $F^*(H)$ is normal in $G$, we have that $S(F^*(H))$ is normal in $G$. So, the maximal subgroups of $G_p \cap F^*(H)$ are $S(F^*(H))$-$\mathfrak{Z}$-permutable subgroups of $G$, for all $G_p \in \mathfrak{Z}$, where $S(F^*(H))$ is a solvable normal subgroup of $G$. Thus, Theorem 1.1 can be seen as an immediate consequence of Theorem 1.2.

2. Basic definitions and preliminaries

In this section, we list some definitions and known results from the literature that will be used in the sequel.
Let  be a saturated formation. Then the -residual, denoted by , is the unique smallest normal subgroup of  such that . Throughout, denotes the class of supersoluble groups which is a saturated formations, see [[5], Satz 8.6, p. 713].

A normal subgroup  of a group  is an -hypercentral subgroup of  provided  possesses a chain of subgroups  such that  is an -central chief factor of , see [[2], p. 387]. The product of all -hypercentral subgroups of  is again an -hypercentral subgroup, denoted by , and called the -hypercentre of , see [[2], IV, 6.8]. For the formation , the -hypercentre of a group , denoted by , is the product of all normal subgroups  of  such that each chief factor of  below  has prime order. For more details about saturated formations, see [[2], IV].

For any group , the generalized Fitting subgroup  is the unique maximal normal quasinilpotent subgroup of . In fact,  is an important characteristic subgroup of  and it is a natural generalization of  for each chief factor of  below  has prime order. For more details about saturated formations, see [[2], IV].

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Let be a complete set of Sylow subgroups of a group  and let  be a normal subgroup of . We denote the following families of subgroups of , , respectively:

\[
\begin{align*}
\mathcal{Z}N &= \{G_pN : G_p \in \mathcal{Z}\}, \\
\mathcal{Z}N/N &= \{G_pN/N : G_p \in \mathcal{Z}\}, \\
\mathcal{Z} \cap N &= \{G_p \cap N : G_p \in \mathcal{Z}\}.
\end{align*}
\]

The following lemmas will be used in the sequel.

**Lemma 2.1.** Let  be a complete set of Sylow subgroups of a group  and  be a nonempty subset of . Then:

(a)  and  are complete sets of Sylow subgroups of  and  respectively.

(b) If  is -permutable subgroup of , then  is -permutable subgroup of  for  and  for  and  respectively.

(c) If  is -permutable subgroup of , then  is -permutable subgroup of  for  and  respectively.

(d) Suppose that  is a subgroup of . Then  is -permutable subgroup of if and only if  is -permutable subgroup of  for  and  respectively.

(e) Let  be any member of . By hypothesis, there exists some  such that  is a subgroup of , that is,  for . Let  be a subgroup of  as  is normal in . Since  is a -subgroup of , it is clear that  is a -subgroup of  and
Obviously, we have that

Therefore,

\[ \text{Proof} \]

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Lemma 2

\[ \text{Lemma 2} \]

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\[ Z \]

\[ \text{Proof} \]


Let 3 be a complete set of Sylow subgroups of a group G and C be a nonempty subset of G. Assume that H is a normal subgroup of G such that the maximal subgroups of 3N/H are C-3-permutable subgroups of G. Then for any nontrivial normal subgroup N of G, the maximal subgroups of (3N/H) \( \cap (HN/N) \) are CN/N-3N/N-permutable subgroups of G/N.

**Proof.** See [4], Lemma 2.2.

**Lemma 2.3.** Let G be a group. Then:

(a) \( F^*(G) = F(G)E(G) \) and \( [F(G), E(G)] = 1 \), where \( E(G) \) is the layer subgroup of G.

(b) \( F^*(F^*(G)) = F^*(G) \geq F(G) \); if \( F^*(G) \) is solvable, then \( F^*(G) = F(G) \).

(c) \( C_G(F^*(G)) \leq F(G) \).

(d) Suppose that N is a normal subgroup of G contained in \( \Phi(G) \), then \( F^*(G/N) = F^*(G/N) \).

**Proof.** (a), (b) and (c) can be found in [6], X 13. For (d), see [10], Lemma 2.3 (8).

**Lemma 2.4.** Let 3 be a complete set of Sylow subgroups of a group G and C be a nonempty subset of G. Suppose that P is a normal p-subgroup of G, where p is a prime, and N is a minimal normal subgroup of G with \( N \leq P \). If N is complemented in P and the maximal subgroups of P are C-3-permutable subgroups of G, then the order of N is p.

**Proof.** By hypothesis, there exists a subgroup H of P such that \( P = NH \) and \( N \cap H = 1 \). Obviously, we have that \( N \leq G_p \in 3 \). Let \( N/M \) be a chief factor of \( G_p \). Then, the order of \( N/M \) is p. Clearly, \( MH \) is a subgroup of \( P \) as \( M \) is normal in \( P \). Since \( M \cap H = M \cap (N \cap H) = 1 \), then \( |P : MH| = |NH : MH| = |N : M| = p \) and hence \( MH \) is a maximal subgroup of \( P \). By hypothesis, \( MH \) is C-3-permutable subgroup of G. Therefore, \( M = M(H \cap N) = MH \cap N \) is C-3-permutable subgroup of G by Lemma
2.1(e). So, there exists some \( x \in C \) such that \( M^xG_q \) is a subgroup of \( G \), for all \( G_q \in \mathfrak{Z} \). This implies that \( MG_q^{-1} \) is a subgroup of \( G \), for all \( G_q \in \mathfrak{Z} \). Assume that \( q \neq p \). Since \( M = M(N \cap G_q^{-1}) = N \cap MG_q^{-1} \) and \( N \cap MG_q^{-1} \) is normal in \( MG_q^{\pm 1} \), it follows that \( G_q^{\pm 1} \leq N_G(M) \). If \( q = p \), then \( M \) is normal in \( G_p \) and so \( G_p \leq N_G(M) \). Therefore, \( N_G(M) = G \) and hence \( M \) is normal in \( G \). But \( N \) is a minimal normal subgroup of \( G \) and \( M \) is a maximal subgroup of \( N \), thus \( M = 1 \) and the order of \( N \) is \( p \).

**Lemma 2.5.** Let \( G \) be a group. Then:

(a) \( E(G) \), the layer subgroup of \( G \), is a perfect quasinilpotent characteristic subgroup of \( G \).

(b) If \( M \) is a perfect quasinilpotent subnormal subgroup of \( G \), then \( M \leq E(G) \).

(c) If \( M \) is a solvable subgroup of \( G \) and \( E(G) \leq N_G(M) \), then \([E(G), M] = 1 \).

**Proof.** For (a), see [6], Definition 13.14, p. 128. For (b) and (c), see [6], Theorem 13.15(a), p. 128 and Lemma 13.16(b), p. 128–129, respectively.

**Lemma 2.6.** Let \( \mathfrak{Z} \) be a complete set of Sylow subgroups of a group \( G \) and \( C \) be a solvable normal subgroup of \( G \). If \( p \) is the smallest prime dividing the order of \( G \) and the maximal subgroups of \( G_p \in \mathfrak{Z} \) are \( C \)-\( \mathfrak{Z} \)-permutable subgroups of \( G \), then \( G \) is \( p \)-nilpotent.

**Proof.** See [4], Theorem 3.1.

**Lemma 2.7.** Let \( \mathfrak{F} \) be a saturated formation containing the class of supersolvable groups \( \mathcal{U} \), \( \mathfrak{Z} \) be a complete set of Sylow subgroups of a group \( G \) and \( C \) be a solvable normal subgroup of \( G \). Then the following two statements are equivalent:

(a) \( G \in \mathfrak{Z} \).

(b) There is a normal subgroup \( H \) in \( G \) such that \( G/H \in \mathfrak{Z} \) and the maximal subgroups of \( G_p \cap H \) are \( C \)-\( \mathfrak{Z} \)-permutable subgroups of \( G \), for all \( G_p \in \mathfrak{Z} \).

**Proof.** See [4], Theorem 3.2.

**Lemma 2.8.** Suppose that \( G \) is a finite non-abelian simple group. Then there exists an odd prime \( r \in \pi(G) \) such that \( G \) has no Hall \( \{2, r\} \)-subgroup.

**Proof.** See [8], Lemma 2.6. [160-168]
3. Results

First, we prove the following lemma:

**Lemma 3.1.** Let $\mathcal{S}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a nonempty subset of $G$. Suppose that $P$ is a normal $p$-subgroup of $G$. If the maximal subgroups of $P$ are $C$-$3$-permutable subgroups of $G$, then $P \leq Z_{\mathfrak{U}}(G)$.

**Proof.** Assume that the result is false and let $G$ be a counterexample of minimal order. If $\Phi(P) \neq 1$, then the maximal subgroups of $P/\Phi(P)$ are $C\Phi(P)/\Phi(P)-3\Phi(P)/\Phi(P)$-permutable subgroups of $G/\Phi(P)$ by Lemma 2.1(d). Then, by the minimal choice of $G$, $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$. Hence, by [[11], Theorem 7.19, p. 39], $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Thus, we may assume that, $\Phi(P) = 1$ and so $P$ is elementary abelian $p$-group. Let $N$ be a minimal normal subgroup of $G$ contained in $P$. Since $N \cap \Phi(P) = 1$ as $\Phi(P) = 1$, it follows, by [[2], Theorem 9.2(f), p. 30], that $N$ is complemented in $P$. The hypothesis and Lemma 2.4 imply that the order of $N$ is $p$. If $N = P$, then $P \leq Z_{\mathfrak{U}}(G)$ by the definition of $Z_{\mathfrak{U}}(G)$, a contradiction. So, we may assume that $N \neq P$. It is easy to see that $\Phi(P/N) = 1$. Let $M/N$ be a maximal subgroup of $P/N$. Then $M$ is a maximal subgroup of $P$ as $|P : M| = |P/N : M/N| = p$. By hypothesis and Lemma 2.1(b), $M/N$ is $CN/N-3N/N$-permutable subgroup of $G/N$. So, the maximal subgroups of $P/N$ are $CN/N-3N/N$-permutable subgroups of $G/N$. Therefore, $P/N \leq Z_{\mathfrak{U}}(G/N)$ by the minimal choice of $G$. But $Z_{\mathfrak{U}}(G/N) = Z_{\mathfrak{U}}(G)/N$ by [[11], Lemma 7.1(ii), p. 30], then $P \leq Z_{\mathfrak{U}}(G)$, a contradiction completing the proof of the lemma.

Now we can prove:

**Theorem 3.2.** Let $\mathfrak{Z}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$, $\mathcal{S}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a solvable normal subgroup of $G$. Then the following two statements are equivalent:

(a) $G \in \mathfrak{Z}$.

(b) There is a normal subgroup $H$ in $G$ such that $G/H \in \mathfrak{Z}$, $F^*(H) = F(H)$ and the maximal subgroups of the Sylow subgroups of $F(H)$ are $C$-$3$-permutable subgroups of $G$, for all $G_p \in \mathcal{S}$.

**Proof.** We need only to prove $(b) \Rightarrow (a)$.

Let $P$ be any Sylow $p$-subgroup of $F(H)$. Clearly, $P$ is normal in $G$. The hypothesis and Lemma 3.1 imply that $P \leq Z_{\mathfrak{U}}(G)$. Since this is true for any Sylow $p$-subgroup of $F(G)$, we have that $F(H) \leq Z_{\mathfrak{U}}(G)$. Note that $[G_{\mathfrak{U}}, F(H)] \leq [G_{\mathfrak{U}}, Z_{\mathfrak{U}}(G)] = 1$ by [[2], Theorem 6.10, p. 390] and hence $G_{\mathfrak{U}} \leq C_G(F(H))$. Therefore, $G/C_G(F(H))$ is an epimorphic image of $G/G_{\mathfrak{U}} \in \mathfrak{U} \subseteq \mathfrak{Z}$ and so $G/C_G(F(H)) \in \mathfrak{U} \subseteq \mathfrak{Z}$. Consequently, $G/C_H(F(H)) = G/(C_G(F(H)) \cap H) \in \mathfrak{Z}$ as $G/C_G(F(H)) \in \mathfrak{Z}$ and $G/H \in \mathfrak{Z}$. But $C_H(F(H)) \leq F(H)$ holds by Lemma 2.3(c) and the fact that $F^*(H) = F(H)$, then
\(G/F(H)\) is an epimorphic image of \(G/C_H(F(H))\), thus \(G/F(H) \in \mathfrak{F}\). Applying Lemma 2.7 yields \(G \in \mathfrak{F}\). This completes the proof of the theorem.

The following lemma is a criterion for the solvability of finite groups:

**Lemma 3.3.** Let \(G\) be a group. Then the following two statements are equivalent:

(a) \(G\) is solvable.

(b) \(G\) has a complete set \(\mathfrak{F}\) of Sylow subgroups such that \(G_p \in \mathfrak{F}\) is \(\mathfrak{F}\)-permutable subgroup of \(G\), where \(p\) is the smallest prime dividing the order of \(G\).

**Proof.** (a) \(\Rightarrow\) (b). Since \(G\) is solvable, then \(G\) has a Sylow basis \(S\) by [9, Theorem 9.3.11, p. 229]. Let \(p\) be the smallest prime dividing the order of \(G\) and let \(G_p\) be the Sylow \(p\)-subgroup of \(G\) in \(S\). By the definition of the Sylow basis \(S\), we have that \(G_p G_q\) is a subgroup of \(G\), for all \(q \in S\), where \(q\) is a prime. Thus we can take \(\mathfrak{F} = S\) and we have \(G_p\) is \(\mathfrak{F}\)-permutable subgroup of \(G\).

(b) \(\Rightarrow\) (a). Assume that the result is false and let \(G\) be a counterexample of minimal order. By Feit-Thompson Theorem [3], we may assume that \(p = 2\). Since \(G_2 \in \mathfrak{F}\) is \(\mathfrak{F}\)-permutable subgroup of \(G\), it follows that \(G_2 G_q\) is a subgroup of \(G\), for every odd prime \(q\) dividing the order of \(G\), where \(q \in \mathfrak{F}\). Therefore, \(G\) is not simple by Lemma 2.8. Let \(N\) be a nontrivial proper normal subgroup of \(G\). Clearly, \(G_2 \cap N\) and \(G_2 N/N\) are Sylow 2-subgroups of \(N\) and \(G/N\), respectively. By Lemma 2.1(e), \(G_2 \cap N\) is \(\mathfrak{F}\)-permutable subgroup of \(G\). Therefore, \(G_2 \cap N\) is \(\mathfrak{F}\)-permutable subgroup of \(N\) by Lemma 2.1(c). Also, \(G_2 N/N\) is \(\mathfrak{F}\)-permutable subgroup of \(G/N\) by Lemma 2.1(b). If 2 divides the order of \(N\), then \(N\) is solvable by the minimal choice of \(G\) and if 2 does not divide the order of \(N\), then \(N\) is solvable by Feit-Thompson Theorem [3]. The same argument holds for \(G/N\), thus \(G/N\) is solvable. Since \(N\) and \(G/N\) are solvable, we have that \(G\) is solvable, a contradiction completing the proof of the lemma.

**Proof of Theorem 1.2.** We need only to prove (b) \(\Rightarrow\) (a) as (a) \(\Rightarrow\) (b) is true with \(H = 1\).

Let \(E(G)\) be the layer subgroup of \(G\). Since \(C\) is a solvable normal subgroup of \(G\), it follows, by Lemma 2.3(b), that \([E(G), C] = 1\) and so \(C \leq C_G(E(G))\). By Lemma 2.3(a), we have that \(F^*(H) = F(H) E(H)\). Moreover, \(E(H)\) is a perfect quasinilpotent characteristic subgroup of \(H\) by Lemma 2.5(a). Now \(E(H)\) char \(H\) and \(H\) is normal in \(G\), then \(E(H)\) is normal in \(G\). Note that \(E(H) \leq E(G)\) by Lemma 2.5(b), and hence \(C \leq C_G(E(H))\). Now we will show that \(E(H)\) is solvable. By Feit-Thompson Theorem [3], we may assume that 2 divides the order of \(E(H)\). Clearly, \(\mathfrak{F} \cap E(H)\) is a complete set of Sylow subgroups of \(E(H)\) by Lemma 2.1(a). Let \(U\) be a maximal subgroup of \(G_2 \cap F^*(H)\), where \(G_2 \in \mathfrak{F}\). The hypothesis and Lemma 2.1(e) imply that \(U \cap E(H)\) is \(\mathfrak{F}\)-3-permutable subgroup of \(G\). So, there exists some \(x \in C\) such that \((U \cap E(H))^x G_q\) is a subgroup of \(G\), for all \(G_q \in \mathfrak{F}\). But \((U \cap E(H))^x = U \cap E(H)\) as \(x \in C \leq C_G(E(H)) \leq C_G(U \cap E(H))\), then \((U \cap E(H)) G_q\)
is a subgroup of $G$, for all $G_q \in \mathcal{Z}$. Therefore, $U \cap E(H)$ is $\mathcal{Z}$-permutable subgroup of $G$. By Lemma 2.1(c), $U \cap E(H)$ is $\mathcal{Z} \cap E(H)$-permutable subgroup of $E(H)$. Suppose that $G_2 \cap E(H) = G_2 \cap F^*(H)$. By the hypothesis and the previous arguments, the maximal subgroups of $G_2 \cap E(H)$ are $\mathcal{Z} \cap E(H)$-permutable subgroups of $E(H)$. Consequently, $E(H)$ is 2-nilpotent by Lemma 2.6, where $C = 1$ in this case. So, $E(H) = (G_2 \cap E(H))K$, where $K$ is a normal Hall 2'-subgroup of $E(H)$. Because $K$ is solvable by Feit-Thompson Theorem [3] and $G/K \cong G_2 \cap E(H)$ is solvable, it follows that $E(H)$ is solvable. Thus, we may assume that $G_2 \cap E(H)$ is a proper subgroup of $G_2 \cap F^*(H)$. Then we can choose $U$ to be a maximal subgroup of $G_2 \cap F^*(H)$ such that $G_2 \cap E(H) \leq U$. Therefore, $G_2 \cap E(H) = U \cap E(H)$ as $G_2 \cap E(H)$ is a Sylow 2-subgroup of $E(H)$. Now we have that $G_2 \cap E(H) = U \cap E(H)$, as we proved in the beginning, is $\mathcal{Z} \cap E(H)$-permutable subgroup of $E(H)$. This implies that $E(H)$ is solvable by Lemma 3.3. So, in either case, $E(H)$ is solvable. But $E(H)$ is perfect, then $E(H) = 1$ and therefore $F^*(H) = F(H)$. Applying Theorem 3.2 yields $G \in \mathcal{F}$. This completes the proof of the theorem.

The next theorem is an improvement of Theorem 3.12 in [4]:

**Theorem 3.4.** Let $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups $\mathcal{U}$, $\mathcal{Z}$ be a complete set of Sylow subgroups of a group $G$ and $C$ be a solvable normal subgroup of $G$. Then the following two statements are equivalent:

(a) $G \in \mathcal{F}$.

(b) There is a normal subgroup $H$ in $G$ such that $G/H \in \mathcal{F}$ and the maximal subgroups of $G_p \cap F^*_n(H)$ are $C\mathcal{Z}$-permutable subgroups of $G$, for all $G_p \in \mathcal{Z}$, for some positive integer $n$.

**Proof.** We need only to prove (b) $\Rightarrow$ (a) as (a) $\Rightarrow$ (b) is true with $H = 1$.

If $n = 1$, then $G \in \mathcal{F}$ by Theorem 1.2. So, we may assume that $n > 1$. Let $K = F^*_n-1(H)$. It is clear that $(G/K)/(H/K) \cong G/H \in \mathcal{F}$. By Lemma 2.2, the maximal subgroups of $(\mathcal{Z}K/K) \cap (F^*_n(H)/K) = (\mathcal{Z}K/K) \cap F^*_n(H)/K$ are $C\mathcal{Z}$-permutable subgroups of $G/K$. Hence, $G/K \in \mathcal{F}$ by Theorem 1.2. Thus $G/F^*_n(H) \cong (G/K)/(F^*_n(H)/K) \in \mathcal{F}$ and the maximal subgroups of $G_p \cap F^*_n(H)$ are $C\mathcal{Z}$-permutable subgroups of $G$, for all $G_p \in \mathcal{Z}$. Applying Lemma 2.7 yields $G \in \mathcal{F}$. This completes the proof of the theorem.

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