Almost prime ideal in gamma near ring

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Abstract. In this manuscript we introduce the notion of almost prime ideals in Γ-near-rings along with few of their characterizations. We also present the interesting relations among almost prime, prime and primary ideal in Γ-nearrings.

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1. Introduction and Preliminaries

Recently, the generalization of prime ideal i.e., almost prime ideal in commutative rings has been introduced and discussed by Srikant M. Bhatwadekar and Pramod K. Sharma (See [3]). Following [3], an ideal $I$ of a ring $R$ is said to be an almost prime if for all $a, b \in R$ implies $ab \in I - I^2$ either $a \in I$ or $b \in I$. All prime and idempotent ideals are almost prime [3]. It has been proved that every almost prime ideal in a noetherian domain $R$ is primary [3]. Further to this, almost primary ideals in rings have been introduced by A. K. Jabbar and C. A. Ahmed in [12], a proper ideal $A$ of a ring $R$ is an almost primary ideal if for $a, b \in R$ such that $ab \in A - A^2$, then $a \in A$ or $b \in A$, for some positive integer $n$ [12]. In [12], authors have also discussed several characterizations of almost primary ideals. It is evident that primary ideals, almost prime ideals and idempotent ideals of a ring $R$ are almost primary ideals, but the converse is not true in each case. Notion of weakly prime element (author called it a prime) was introduced by Steven Galovich while studying the property of unique factorization of rings with zero divisors [10]. Following [10], let $r \neq 0$ be in $R$ than $r$ is prime if, whenever $r$ divides $ab$ where $ab \neq 0$, then $r$ divides $a$ or $r$ divides $b$. Author established the fundamental results: (i) In [10], author also

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showed that every irreducible is a prime, (ii) every irreducible in $R$ is a zero divisor [10], (iii) every irreducible element of $R$ is nilpotent, and (iv) every nonunit in $R$ is nilpotent. Consequently the author declared the unique maximal ideal consists of nonunit elements [10]. In [1], authors declare that (which was named prime by Galovich in [10]) a nonzero nonunit $p \in R$ is weakly prime if $p|ab \neq 0$ implies $p|a$ or $p|b$. Consequently, an ideal $I$ of a commutative ring $R$ is called a weakly prime if $0 \neq ab \in I$ implies $a \in I$ or $b \in I$, and also $p$ is weakly prime iff $(p)$ is weakly prime [1]. Following [2], $P$ is weakly prime ideal if and only if $0 \neq AB \subseteq P$, $A$ and $B$ ideals of $R$, implies $A \subseteq P$ or $B \subseteq P$. Further to this, every weakly prime ideal is an almost prime ideal.

We call an algebraic system $N$ with two binary operation ” + ” and ”.” (right) near-ring if it is a group (not necessarily abelian) under addition, and $N$ is associative group under multiplication and distribution of multiplication over addition on the right holds i.e., for any $x, y, z \in N$, it satisfies that $x + (y + z) = (x + y) + z$[15]. Likewise, a left near-ring can be defined by replacing the right distributive law by the equivalent left distributive law. Suppose $N$ is a left near-ring with binary operation ” + ” and ”.” then a subset $I$ is said to be an ideal if (i) $(I, +)$ is a normal subgroup of a $(N, +)$, (ii) For each $n \in N$, $i \in I$, $n_i \in I$ i.e., $N_1 \subseteq I$, and (iii) $(n_1 + i)n_2 \in I$ for each $n_1, n_2 \in N$ and $i \in I$. But A. Frohlich [9] showed that for d.g. near-rings the third condition is equivalent to $in \in I$ i.e., $IN \subseteq I$. Hence a subset $I$ is a right (left) ideal if $N$ satisfies the first and third (second) conditions. A proper ideal $P$ of a near ring $N$ is prime if for ideals $A$ and $B$ of $N$, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ . An ideal $P$ of a near-ring $N$ is a completely prime (prime ideal of type-2) if for all $x, y \in N$, $xy \in P$ implies $x \in P$ or $y \in P$. Almost prime ideals in near rings have been endorsed by B. Elavarasan (see [8]). A proper ideal $P$ of a near ring $N$ is said to be almost prime if for any ideals $A$ and $B$ of $N$ such that $AB \subseteq P$ and $AB \notin P$, we have $A \subseteq P$ or $B \subseteq P$[8]. The author established few relationships between almost prime and prime ideals [8]. Weakly prime ideals in near rings have been introduced by P. Dheena and B. Elavarasan [6], a proper ideal $P$ of a near ring $N$ is said to be weakly prime if $0 \neq AB \subseteq P$, $A$ and $B$ are ideals of $N$, implies $A \subseteq P$ or $B \subseteq P$. Clearly, every prime ideal is weakly prime and (0) is always weakly prime ideal of a near ring $N$. Also every prime ideal is a weakly prime, and a weakly prime ideal is an almost prime ideal. An ideal $I$ of a near ring $N$ is said to be a completely prime ideal if $x, y \in N$, $xy \in I$ implies $x \in I$ or $y \in I$ [11]. Similarly, an ideal of a near ring $N$ is said to be primary ideal of $N$ if $x, y \in N$, $xy \in I$ implies $x \in I$ or $y^m \in I$ for some $m \in Z$. An ideal $I$ of a near ring $N$ is called a completely semiprime ideal of a near ring $N$ if $y^2 \in I$ implies $y \in I$ for all $y \in N$[11]. Further to this, almost prime ideals in near rings have been endorsed by B. Elavarasan (see [8]). A proper ideal $P$ of a near ring $N$ is said to be almost prime if for any ideals $A$ and $B$ of $N$ such that $AB \subseteq P$ and $AB \notin P^2$, we have $A \subseteq P$ or $B \subseteq P$[8]. The author established few relationships between almost prime and prime ideals [8]. Number of ideals in near ring have been introduced and discussed such as completely prime, primary, completely primary and so on. Following [11], an ideal $I$ of a near ring $N$ is said to be a completely prime ideal if $x, y \in N$, $xy \in I$ implies $x \in I$ or $y \in I$ [11]. Similarly, an ideal of a near ring $N$ is said to be primary ideal of $N$ if $x, y \in N$, $xy \in I$ implies $x \in I$ or $y^m \in I$ for some $m \in Z$. An ideal $I$ of a near ring $N$ is called a
completely semiprime ideal of a near ring $N$ if $y^2 \in I$ implies $y \in I$ for all $y \in N$ [11].

The ideal theory is the most important part of algebra, different types of ideals in rings have been discussed in the literature. A right (left) ideal of a Γ-ring $M$ is an additive subgroup $I$ of $M$ such that $I \Gamma M \subseteq I$ ($M I \subseteq I$). If $I$ is both a right and a left ideal, then we say that $I$ is an ideal or a two-sided ideal of $M$. In rings, an ideal $P$ is prime ideal if and only if $A$ and $B$ are ideals in $M$ such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$ [13]. The prime ideals of the $\Gamma_{n,m}$-ring $M_{n,m}$ are the sets $P_{m,n}$ corresponding to the prime ideals $P$ of the Γ-ring $M$ [13]. If $P$ is an ideal in a Γ-ring $M$ then, (i) Ideal $P$ is a prime ideal of $M$, (ii) If $a, b \in M$ and $a \Gamma M \Gamma b \subseteq P$ then either $a \in P$ or $b \in P$, (iii) If ideal generated by $<a>$ and $<b>$ are called principal ideals in $M$ and $<a> \Gamma <b> \subseteq P$, then $a \in P$ or $b \in P$, (iv) If $U$ and $V$ are right ideals in $M$ with $U \Gamma V \subseteq P$, then $U \subseteq P$ or $V \subseteq P$, (v) If $U$ and $V$ are left ideals in $M$ with $U \Gamma V \subseteq P$, either $U \subseteq P$ or $V \subseteq P$ [16].

Γ-near rings were introduced by Satyanarayana Bhavanari (see [14], [15]). A subset $A$ of a Γ-near-ring $M$ is called a left (resp. right) ideal of $M$ if $(A,+)$ is a normal divisor of $(M,+)$, $u \alpha (x + v) - u \alpha v \in A$ (resp. $x \alpha u \in A$) for all $x \in A$, $\alpha \in \Gamma$ and $u, v \in M$. An ideal $P$ of Γ-near ring $(M,+,(\cdot)_\Gamma)$ is called prime, if for every two ideals $I, J$ of $M$, $I \Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. An ideal $P$ of a Γ-near-ring $N$ is called a completely primary ideal if for $a, b \in N$ and $\gamma \in \Gamma$ such that $a \gamma b \in P$ implies that $a \in P$ or $b \in P$, for some positive integer $n$ [17]. If an ideal $I$ of Γ-near-ring $M$ is maximal, then it is prime or $M \Gamma M = I$ [7]. If $(M,+,(\cdot)_\Gamma)$ is a Γ-near-ring such that for any $\gamma \in \Gamma$ there is an element which is $\Gamma$-unit, then every maximal ideal $I$ of $M$ is prime [7]. For every ideal $I$ of Γ-near-ring $M$ exists prime minimal ideal of $I$ [7]. In this note first we introduce the notion of almost prime ideals in Γ-near-rings along with few of their characterizations. Finally, we present the interesting relations of an almost prime with the prime and primary ideal in Γ-near-rings.

2. Almost prime ideal in Γ-near-ring

In this section we introduce almost prime ideal in Γ-near-rings. Furthermore, we also present its implications with the same ideals, we start with the following definition.

**Definition 1.** Let $M$ be Γ-near-ring and $P$ be a prime ideal of $M$ then $P$ is almost prime ideal if $a, b \in R$, $ab \in P - PT\overline{P}$, either $a \in P$ or $b \in P$.

**Example 1.** Suppose $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $\Gamma = \{0, 2, 4\}$. Let $P = 2Z_8 = \{0, 2, 4\}$ be a prime ideal in $Z_8$ and consider $PT\overline{P} = \{0, 6\}$, $P - PT\overline{P} = \{2, 4\}$. Here $2, 3 \in Z_8$ and $2.2.3 = 4 \in P - PT\overline{P}$ where $2 \in P$ and $3 \notin P$. Similarly we can check for other elements as well. Hence $P$ is an almost prime ideal in Γ-near ring.

**Example 2.** Suppose $R$ is a Γ-near ring of algebraic integers such that the integral closure of $Z$ in $C$. Suppose that $I$ be a radical ideal of $R$ say $\overline{I}I = I$, if $a \in I$ then $\beta \in R$ exist such that $\beta \Gamma \beta = \alpha$. Since $\beta \Gamma \beta = \alpha \in I$, $\beta \in I$ implies $I = \overline{I}I$.

**Example 3.** Consider the near ring $N = \{0, 1, 2, 3\}$ and $\Gamma = \{0, 2\}$ such that addition and multiplication defined as follow.
Suppose \( P = \{0, 2\} = 2N \) be a prime ideal of \( N \) because for all \( a, b \in N \) and \( a\gamma b \in P \) implies \( a \in P \) or \( b \in P \). As \( PTP = \{0\} \) then \( P - PTP = \{2\} \), then for all \( a, b \in N \) such that \( a\gamma b \in P - PTP \) either \( a \in P \) or \( b \in P \) which is almost prime ideal.

**Preposition 1.** Every prime ideal in a \( \Gamma \)-near ring is almost prime ideal. Proof. Suppose \( P \) be a prime ideal of \( \Gamma \)-near ring but not an almost prime. Assume \( a\gamma b \in P - PTP \), implies \( a\gamma b \in P \). If \( a\gamma b \notin PTP \) implies \( a \in P \) or \( b \in P \) then contradiction arise to our supposition. Hence \( P \) must be a prime.

**Remark 1.** If \( I \) is a maximal ideal of \( \Gamma \)-near-ring \( M \) then it is prime or \( M\Gamma M = I \).

Supporting the above remark 1, we present the below example.

**Example 4.** Let \( M = \{0, 1, 2, 3\} \) is a \( \Gamma \)-near-ring where \( \Gamma = \{0, 2\} \) and ideal \( I = 2M = \{0, 2\} \) that is maximal in \( M \). Obviously \( I \) is prime ideal in \( M \) also \( M\Gamma M = I \).

**Lemma 1.** Suppose \( N \) is a \( \Gamma \)-near-ring and for any \( \gamma \in \Gamma \) there is an element which is \( \Gamma \)-unit then every maximal ideal \( I \) of \( M \) is prime.

Proof. If for one \( \gamma \in \Gamma \) the element \( c \) is \( \gamma \)-one of \( M \) then \( M\gamma M = \{m_1\gamma m_2 : m_1; m_2 \in M\} = M \) since for any \( m \in M \), \( m = m\gamma e \). Because \( M \neq I \) the equation is not true \( M\Gamma M = I \). When \( M = I \) or \( M = 0 \) then equation is true so \( M \) is simple and \( M\Gamma M \neq 0 \), as a result \( M \) is prime.

**Preposition 2.** Suppose \( I \) be a \( P \)-primary ideal of a \( \Gamma \)-near ring such that \( PTP = I\Gamma I \) implies \( I \) is an almost prime ideal.

Proof. Suppose \( a, b \in R \), \( a\gamma b \in I - I\Gamma I \), \( a \notin I \) and \( b \notin I \). As \( a \notin I \) and \( I \) is a \( P \)-primary ideal it implies that \( b \in P \). Also \( a \in P \) thus \( a\gamma b \in PTP = I\Gamma I \), which is a contradiction.

**Lemma 2.** Suppose that \( R \) be a near integral domain and \( c \) be a nonzero nonunit element of \( R \). If element \( c \) is other than prime element then there exist \( a \notin R\Gamma c \), \( b \notin R\Gamma c \) such that \( a\gamma b \in R\Gamma c \) but \( a\gamma b \notin R\Gamma c^2 \).

Proof. Suppose an ideal \( Rc \) is not prime then there exist \( a \notin R\Gamma c \), \( b \notin R\Gamma c \) such that \( a\gamma b \in R\Gamma c \). If the case \( a\gamma b \in R\Gamma c^2 \) then for \( d = (b + c)\gamma \notin R\Gamma c \) and \( a\gamma d \in R\Gamma c \). If \( a\gamma d \in R\Gamma c^2 \), implies \( a\gamma c \in R\Gamma c^2 \) as \( a\gamma b \in R\Gamma c^2 \) implies \( a \in R\Gamma c \), a contradiction to our supposition. Hence the result follows.

**Example 5.** Let \( Z \) be a \( \Gamma \)-near ring and \( \Gamma = \{0, 1, 2, 3\} \) consider \( c = 6 \) be an non prime element of \( Z \) then \( Z\Gamma 6 \) is non prime ideal because \( 3 \notin Z\Gamma 6 \) and \( 4 \notin Z\Gamma 6 \) but \( 12 \in Z\Gamma 6 \) and \( 12 \notin Z\Gamma 6^2 \).

In the below proposition, we reverse the situation occurring in lemma 2.

**Preposition 3.** Suppose that \( R \) be \( \Gamma \)-near integral domain and \( c \) be a nonzero nonunit element of \( R \). If \( c \) is not a prime element then there exists \( a \in R\Gamma c \) and \( b \in R\Gamma c \) such that \( a\gamma b \in R\Gamma c \) and \( a\gamma b \in R\Gamma c^2 \).

Proof. Suppose an ideal \( R\Gamma c \) is not prime and consider \( a \in R\Gamma c \), \( b \in R\Gamma c \) such that \( a\gamma b \in R\Gamma c \). If the case, \( a\gamma b \notin R\Gamma c^2 \) then for \( d = (b + c) \in R\Gamma c \) and \( a\gamma d \in R\Gamma c \). Consider \( a\gamma d \notin R\Gamma c^2 \) implies \( ac \notin R\Gamma c^2 \) and because \( a\gamma b \notin R\Gamma c^2 \) implies \( a \notin R\Gamma c \), a contradiction.
to our hypothesis. Hence the result is valid. Supporting the above lemma we present the below example.

**Example 6.** Let \( Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\} \) and \( \Gamma = \{0, 2, 4\} \) consider a non-prime element of \( Z_8 \) i.e., \( c = 6 \) implies \( 6Z_8 = \{0, 2, 4\} \). Consider \( 6, 4 \in 6Z_8 \) such that \( 6.2.4 = 0 \in 6Z_8 \) and \( c^2 = 6^2 \) and \( 6^2Z_8 = \{0, 4\} \), hence \( 6.2.4 = 0 \in 6^2Z_8 \). Further we consider \( 6.4.4 = 4 \in 6^2Z \) and take \( 4, 2 \in 6Z_8 \) then \( 4.2.2 = 0 \in 6Z_8 \), and again we get \( 4.2.2 = 0 \in 6^2Z_8 \), similarly \( 4.4.2 = 0 \in 6Z_8 \) and \( 4.4.2 = 0 \in 6^2Z_8 \).

**Theorem 1.** Suppose \( N \) be a \( \Gamma \)-near-ring with identity and \( P \) be an almost prime ideal of \( N \). If \( P \) is not prime then \( 
P \Gamma P \neq P \).

**Proof.** Let us assume that \( P \subseteq P \Gamma \Gamma P \). We have to prove that \( P \) is prime. Let us suppose that two ideals \( A \) and \( B \) contained in \( N \) such that \( A \Gamma B \subseteq P \). If \( A \Gamma B \not\subseteq P \Gamma P \) then \( A \not\subseteq P \) or \( B \not\subseteq P \). We assume that \( A \Gamma B \not\subseteq P \Gamma P \). Since \( P \not\subseteq P \Gamma P \) as a result \( p \in P \) such that \( \langle p \rangle \not\subseteq P \Gamma P \) hence \( \langle A + \langle p \rangle \rangle \Gamma(B + N) \not\subseteq P \Gamma P \). Consider \( \langle A + \langle p \rangle \rangle \Gamma(B + N) \not\subseteq P \), there exist an element \( a \in A, b \in B, p_0 \in \langle p \rangle \) and \( q_0 \in N \) such that \( (a + p_0)\gamma(b + q_0) \not\subseteq P \). We assume that \( \gamma(b + q_0) \not\subseteq P \), but \( \gamma(b + q_0) = \gamma(b + q_0) - a\gamma(b) + c \subseteq P \) as \( A \Gamma B \subseteq P \), a contradiction. Hence \( \langle A + \langle p \rangle \rangle \Gamma(B + N) \subseteq P \) implies \( A \subseteq P \).

**Corollary 1.** Consider \( N \) a \( \Gamma \)-near-ring having identity and containing an ideal \( P \). If \( P \Gamma P \neq P \) then \( P \) is prime if and only if \( P \) is almost prime.

**Proposition 4.** If \( P \neq 0 \) be a proper ideal of a \( \Gamma \)-near-ring \( N \) such that \( P \) is almost prime and \( (P \Gamma P : P) \subseteq P \) then \( P \) is prime.

**Proof.** We suppose that \( P \) is not a prime ideal of \( N \). Then there exist \( x \) and \( y \in P \) such that \( \langle x \rangle \Gamma \langle y \rangle \not\subseteq P \). If \( \langle x \rangle \Gamma \langle y \rangle \neq \langle y \rangle \Gamma P \Delta P \), then the result holds. Hence \( \langle x \rangle \Gamma \langle y \rangle \subseteq P \Gamma P \). Suppose \( \langle x \rangle \Gamma \langle y \rangle + P \subseteq P \). If \( \langle x \rangle \Gamma \langle y \rangle + P \subseteq P \) then we have \( x \in P \) or \( y \in P \), a contradiction to our assumption, or else \( \langle x \rangle \Gamma \langle y \rangle + P \subseteq P \Gamma P \). Thus \( \langle x \rangle \Gamma \Gamma P \subseteq P \Gamma P \) implies \( x \in (P \Gamma P : \Gamma) \subseteq P \).

**Theorem 2.** Suppose \( N \) be a \( \Gamma \)-near-ring and let \( P \) be an ideal of \( N \). Then the following statements are equivalent:

i) If elements \( a,b,c \in N \) with \( a\gamma(b) + c \in P \) and \( a\gamma(b) + c \not\subseteq P \Gamma P \) then \( a \in P \) or \( b \in P \) or \( c \in P \).

ii) If \( x \in N - P \), then \( (P : \Gamma : x) + (y) = P \cup (P \Gamma P : \Gamma : x) + (y) \) for some \( y \in N \).

iii) If \( x \in NP \), then \( (P : \Gamma : x) + (y) = P \) or \( (P : \Gamma : x) + (y) = (P \Gamma P : \Gamma : x) + (y) \).

iv) \( P \) is an almost prime.

**Proof.** (i) implies (ii). Consider \( t \in (P : \Gamma : x) + (y) \) for some \( x \in N - P, y \in N \). After that \( t\Gamma(x) + (y) \subseteq P \). If \( t\Gamma(x) + (y) \not\subseteq P \Gamma P \) subsequently \( t^2\Gamma(P \Gamma P : \Gamma : x) + (y) \). If \( t\Gamma(x) + (y) \not\subseteq P \Gamma P \), then \( t \in P \) by assumption.

(ii) implies (iii) holds from the truth that if union of two ideal is an ideal then it is equal to one of them.(iii) implies (iv). Imagine \( A \) and \( B \) be ideals of \( N \) such that \( A \Gamma B \subseteq P \). Assume \( A \not\subseteq P \) and \( B \not\subseteq P \). If \( a \in A \) and \( b \in B \) exist with \( a,b \not\subseteq P \). Now we say that \( A \Gamma B \not\subseteq P \Gamma P \) and consider \( b_1 \in B \). In that case \( A \Gamma (x) + (b_1) \not\subseteq P \Gamma P \). Then by supposition \( A \subseteq (x) + (b_1) \Gamma P \Gamma P \) implies \( A \Gamma b_1 \subseteq P \Gamma P \). Consequently \( AB \subseteq P \Gamma P \) and therefore \( P \) is an almost prime ideal of \( N \).
(iv) implies (i) is obvious.

**Theorem 3.** Suppose $N_1, N_2$ be any two $\Gamma$-near-rings with identity and let $P$ be a proper ideal of $N_1$. Then $P$ is almost prime if and only if $(P \times N_2)$ is an almost prime ideal of $N_1 \times N_2$.

**Proof.** Suppose $P$ be an almost prime ideal of $N_1$ and consider $(A_1 \times B_1)$ and $(A_2 \times B_2)$ be ideals of $N_1 \times N_2$ such that $(A_1 \times B_1)\Gamma(A_2 \times B_2) \subseteq (P \times N_2)$ and $(A_1 \times B_1)\Gamma(A_2 \times B_2) \nsubseteq (P \times N_2)\Gamma(P \times N_2)$. In this case $(A_1\Gamma A_2 \times B_1\Gamma B_2) \subseteq (P \times N_2)$ and $(A_1\Gamma A_2 \times B_1\Gamma B_2) \nsubseteq (P\Gamma P \times N\Gamma N)$. Therefore, $A_1\Gamma A_2 \times P$ and $A_1\Gamma A_2 \nsubseteq P\Gamma P$ implies $A_1 \nsubseteq P$ or $A_2 \nsubseteq P$.

Conversely, assume that $(P \times N_2)$ be an almost prime ideal of $N_1 \times N_2$ and consider $I$ and $J$ be ideals of $N_1$ such that $IJ \subseteq P$ and $IJ \nsubseteq P\Gamma P$. Then $(I \times N_2)\Gamma(J \times N_2) \subseteq (P \times N_2)$ and $(I \times N_2)\Gamma(J \times N_2) \nsubseteq (P \times N_2)\Gamma(P \times N_2)$. By hypothesis, we have $(I \times N_2) \subseteq (P \times N_2)$ or $(J \times N_2) \subseteq (P \times N_2)$. Thus $I \subseteq P$ or $J \subseteq P$.

**Lemma 3.** If $c \neq 0$ is a nonunit element in $\Gamma$-near integral domain $R$ then ideal $R\Gamma c$ is prime if and only if $R\Gamma c$ is an almost prime.

**Proof.** Let $c \neq 0$ be a nonunit element in an $\Gamma$-near integral domain $R$. Assume that ideal $R\Gamma c$ is an almost prime we need to prove that $R\Gamma c$ is prime. As we known that ideal $R\Gamma c$ is an almost prime for some $a, b \in R$ and $a\gamma b \in R\Gamma c - R\Gamma c\Gamma R\Gamma c$ implies either $a \in R\Gamma c$ or $b \in R\Gamma c$ where $a\gamma b \not\in R\Gamma c\Gamma R\Gamma c$ implies $a\gamma b \in R\Gamma c$. Hence $R\Gamma c$ is a prime ideal.

Conversely, suppose that ideal $R\Gamma c$ is prime and we use a result that every prime ideal is almost prime then $R\Gamma c$ is almost prime ideal which is immediate from Lemma 2.

**Lemma 4.** Suppose $I$ be an almost prime ideal in a $\Gamma$-near integral domain $R$. Then the below statements hold:

(i) If element $b$ is a zero divisor in $R/I$, in that case $b\Gamma I \subseteq \Gamma I$.

(ii) If for any ideal $J$ of $R$ such that $I \subseteq J$ where $J$ consists of zero divisors on $R/I$ then $JI = \Gamma I$.

(iii) If $I$ is an invertible ideal then $I$ is prime.

**Proof.** (i) Let us suppose that there is an element $c \in I$ such that $b\gamma c \in I$. If $b \in I$ then obviously $b\Gamma I \subseteq \Gamma I$, so let $b \in I$. Since we have $b \notin I, c \notin I$ and $b\gamma c \in I$. Furthermore $I$ is an almost prime and $b\gamma c \in \Gamma I$. Also, for any $x \in I, x + c \notin I$ and $b\gamma (x + c) \in I$. Thus, as $I$ is almost prime, $b\gamma (x + c) \in \Gamma I$. As a result $b\gamma c \in \Gamma I, b\gamma x \in \Gamma I$. Therefore $b\Gamma I \subseteq \Gamma I$. (ii) This is obvious from (i). (iii) Let $x\gamma y \in I$ and $x \in I$. Then from (i) $y\Gamma I \subseteq \Gamma I$. Since $I$ is invertible it is immediate that $y \in I$. Thus $I$ is a prime ideal.

**Lemma 5.** Let $S^{-1}R$ be an almost prime in the ring $S^{-1}R$, where $R$ be a $\Gamma$-near integral domain. Then $I$ be an almost prime ideal in $R$ and $S$ be a multiplicatively closed subset of $R$ disjoint from $I$.

**Proof.** Suppose for $x, y \in R$ and $s, t \in S$, $x\gamma y/s\gamma t \in S^{-1}(I - \Gamma I)$. Then there exists $u, w \in S$ such that $w\gamma x\gamma y \in I$ and $w\gamma x\gamma y \not\in \Gamma I$. Therefore, $w\gamma x\gamma y \in I - \Gamma I$. Since $I$ is almost prime so $w\gamma x \in I$ or $y \in I$. Therefore, either $x/s \in S^{-1}I$ or $y/t \in S^{-1}I$ implies $S^{-1}I$ is an almost prime ideal.


References


