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On McShane-Stieltjes integrals of interval-valued functions and fuzzy-number-valued functions on time scales

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Abstract. In this paper, we introduce the notion of the McShane-Stieltjes (MS) integrals of interval-valued functions and fuzzy-number-valued functions on time scales which are extensions of the McShane (M) integrals of interval-valued functions and fuzzy-number-valued functions on time scales [3] and investigate some of their properties.

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1. Introduction

The calculus on time scales was introduced for the first time in 1988 by Hilger [2] to unify the theory of difference equations and the theory of differential equations. In 2016, Hamid and Elmuiz [4] introduced the concept of the Henstock-Stieltjes (HS) integrals of interval-valued functions and fuzzy-number-valued functions and discussed a number of their properties. Very recently, Hamid et al. [5] introduced the thought of the AP-Henstock integrals of interval-valued functions and fuzzy-number-valued functions and obtained some of their properties.

In this paper, we introduce the notion of the (MS) delta integrals of interval-valued functions and fuzzy-number-valued functions on time scales and investigate some of their properties.

The paper is organized as follows, in Section 2 we provide the preliminary terminology used in this paper. Section 3 is dedicated to discuss the (MS) delta integral of interval-valued functions on time scales. In Section 4, we present the (MS) delta integral of fuzzy-number-valued functions on time scales. The last section provides Conclusions.

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2. Preliminaries

A time scale $\mathbf{T}$ is a nonempty closed subset of real number $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $t \in \mathbf{T}$ we define the forward jump operator $\sigma(t) = \inf\{s \in \mathbf{T} : s > t\}$ where $\inf \phi = \sup\{\mathbf{T}\}$, while the backward jump operator $\rho(t) = \sup\{s \in \mathbf{T} : s < t\}$ where $\sup \phi = \inf\{\mathbf{T}\}$. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. The forward graininess function $\mu(t)$ of $t \in \mathbf{T}$ is defined by $\mu(t) = \sigma(t) - t$, whlie the backward graininess function $\nu(t)$ of $t \in \mathbf{T}$ is defined by $\nu(t) = t - \rho(t)$. For $a, b \in \mathbf{T}$ we denote the closed interval $[a, b]_{\mathbf{T}} = \{t \in \mathbf{T} : a \leq t \leq b\}$.

Throughout this paper, all considered intervals will be intervals in $\mathbf{T}$. A division P of $[a, b]_{\mathbf{T}}$ is a finite collection of interval-point pairs $\{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$, where $\{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$ and $\xi_i \in [a, b]_{\mathbf{T}}$ for $i = 1, 2, \cdots, n$. By $\Delta t_i = t_i - t_{i-1}$ we denote the length of ith subinterval in the division P. $\delta(\xi) = (\delta_L(\xi), \delta_R(\xi))$ is a Δ- gauge for $[a, b]_{\mathbf{T}}$ provided $\delta_L(\xi) > 0$ on $(a, b]_{\mathbf{T}}$, $\delta_R(\xi) > 0$ on $[a, b)_{\mathbf{T}}$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$ and $\delta_R(b) \geq \mu(\xi)$ for all $\xi \in [a, b)_{\mathbf{T}}$. We say that $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a, b]_{\mathbf{T}}$ if $[t_{i-1}, t_i]_{\mathbf{T}} \subset \big(\xi_i - \delta_L(\xi_i), \xi_i + \delta_R(\xi_i)\big)_{\mathbf{T}}$ and $\xi_i \in [a, b]_{\mathbf{T}}$ for all $i = 1, 2, \cdots, n$.

Definition 1. *[10] Let $\alpha : [a, b] \to \mathbb{R}$ be an increasing function. A real-valued function $f : [a, b] \to \mathbb{R}$ is said to be McShane-Stieltjes (MS) integrable to B with respect to α on $[a, b]$ if for every $\varepsilon > 0$, there is a function $\delta(t) > 0$ such that for any δ-fine McShane division $P = \{[u_i, v_i]; \xi_i\}_{i=1}^n$ of $[a, b]$, we have*

$$\Big|\sum_{i=1}^{n} f(\xi_i)[\alpha(v_i) - \alpha(u_i)] - B\Big| < \varepsilon, \tag{2.1}$$

we write $(MS)\int_a^b f(t)d\alpha = B$, and $f \in MS_\alpha[a, b]$.

Definition 2. *Let $\alpha : [a, b]_{\mathbf{T}} \to \mathbb{R}$ be an increasing function. A function $f : [a, b]_{\mathbf{T}} \to \mathbb{R}$ is McShane-Stieltjes delta integrable (MS Δ-integrable) with respect to α on $[a, b]_{\mathbf{T}}$ if there exists a number $A \in \mathbb{R}$ such that for each $\varepsilon > 0$ there is a Δ-gauge, δ, on $[a, b]_{\mathbf{T}}$ such that*

$$\Big|\sum_{i=1}^{n} f(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})] - A\Big| < \varepsilon \tag{2.2}$$

for each δ-fine McShane division $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ of $[a, b]_{\mathbf{T}}$. A is called (MS Δ-integral) of f on $[a, b]_{\mathbf{T}}$, and we write $A = (MS_\Delta)\int_a^b f(t)d\alpha$.

Theorem 1. *Let $\alpha : [a, b]_{\mathbf{T}} \to \mathbb{R}$ be an increasing function. If $f(t)$ and $g(t)$ are (MS) Δ-integrable with respect to α on $[a, b]_{\mathbf{T}}$ and $f(t) \leq g(t)$ almost everywhere on $[a, b]_{\mathbf{T}}$, then*

$$(MS_\Delta)\int_a^b f(t)d\alpha \leq (MS_\Delta)\int_a^b g(t)d\alpha. \tag{2.3}$$

Proof. The proof is similar to Theorem 2.7 in [10].

3. The MS_Δ integral of interval-valued functions on time scales

This section introduces the notion of the MS_Δ integral of interval-valued functions on time scales and investigates some of their properties.

Definition 3. *[7] Let $I_\mathbb{R} = \{I = [I^-, I^+] : I$ is the closed bounded interval on the real line $\mathbb{R}\}$.*

For $A, B \in I_\mathbb{R}$, we define $A \leq B$ iff $A^- \leq B^-$ and $A^+ \leq B^+$, $A + B = C$ iff $C^- = A^- + B^-$ and $C^+ = A^+ + B^+$, and $A \cdot B = \{a \cdot b : a \in A, b \in B\}$, where

$$(A \cdot B)^- = \min\{A^- \cdot B^-, A^- \cdot B^+, A^+ \cdot B^-, A^+ \cdot B^+\} \tag{3.1}$$

and

$$(A \cdot B)^+ = \max\{A^- \cdot B^-, A^- \cdot B^+, A^+ \cdot B^-, A^+ \cdot B^+\}. \tag{3.2}$$

Define $d(A, B) = \max(|A^- - B^-|, |A^+ - B^+|)$ as the distance between intervals A and B.

Definition 4. *[3] An interval-valued function $F : [a, b]_\mathbf{T} \to I_\mathbb{R}$ is McShane delta (M_Δ) integrable to $I_0 \in I_\mathbb{R}$ on $[a, b]_\mathbf{T}$ if for every $\varepsilon > 0$ there exists a Δ-gauge, δ, on $[a, b]_\mathbf{T}$ such that*

$$d\Big(\sum_{i=1}^{n} F(\xi_i)(t_i - t_{i-1}), I_0\Big) < \varepsilon, \tag{3.3}$$

whenever $P = \{([t_{i-1}, t_i]_\mathbf{T}; \xi_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a, b]_\mathbf{T}$. We write $(IM_\Delta) \int_a^b F(t)\Delta t = I_0$ and $F \in IM_\Delta[a, b]_\mathbf{T}$.

Definition 5. *Let $\alpha : [a, b]_\mathbf{T} \to \mathbb{R}$ be an increasing function. An interval-valued function $F : [a, b]_\mathbf{T} \to I_\mathbb{R}$ is (MS_Δ) integrable to $I_0 \in I_\mathbb{R}$ with respect to α on $[a, b]_\mathbf{T}$ if for every $\varepsilon > 0$ there exists a Δ-gauge, δ, on $[a, b]_\mathbf{T}$ such that*

$$d\Big(\sum_{i=1}^{n} F(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})], I_0\Big) < \varepsilon, \tag{3.4}$$

whenever $P = \{([t_{i-1}, t_i]_\mathbf{T}; \xi_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a, b]_\mathbf{T}$. We write $(IMS_\Delta) \int_a^b F(t)d\alpha = I_0$ and $F \in IMS^\alpha_\Delta[a, b]_\mathbf{T}$.

Remark 1. *Let $\alpha : [a, b]_\mathbf{T} \to \mathbb{R}$ be an increasing function. If $F(t) \in IMS^\alpha_\Delta[a, b]_\mathbf{T}$, then the integral value is unique.*

Theorem 2. *Let $\alpha : [a, b]_\mathbf{T} \to \mathbb{R}$ be an increasing function. An interval-valued function $F : [a, b]_\mathbf{T} \to I_\mathbb{R}$ is (MS_Δ) integrable with respect to α on $[a, b]_\mathbf{T}$ if and only if $F^-, F^+ \in MS^\alpha_\Delta[a, b]_\mathbf{T}$ and*

$$(IMS_\Delta)\int_a^b F(t)d\alpha = \Big[(MS_\Delta)\int_a^b F^-(t)d\alpha, (MS_\Delta)\int_a^b F^+(t)d\alpha\Big]. \tag{3.5}$$

Proof. Let $F \in IMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$, then there exists an interval $I_0 = [I_0^-, I_0^+]$ with the property that for any $\varepsilon > 0$ there exists a Δ-gauge, δ with respect to α on $[a,b]_{\mathbf{T}}$ such that

$$d\big(\sum_{i=1}^{n} F(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})], I_0\big) < \varepsilon, \tag{3.6}$$

whenever $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a,b]_{\mathbf{T}}$.

Since $\alpha(t_i) - \alpha(t_{i-1}) \geq 0$ for $1 \leq i \leq n$, we have

$$d\big(\sum_{i=1}^{n} F(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})], I_0\big)$$

$$= \max\left(\left|\Big[\sum_{i=1}^{n} F(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})]\Big]^- - I_0^-\right|, \left|\Big[\sum_{i=1}^{n} F(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})]\Big]^+ - I_0^+\right|\right) < \varepsilon.$$

$$= \max\left(\left|\sum_{i=1}^{n} F^-(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})] - I_0^-\right|, \left|\sum_{i=1}^{n} F^+(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})] - I_0^+\right|\right) < \varepsilon. \tag{3.7}$$

Hence $\left|\sum_{i=1}^{n} F^-(\xi_i)[\alpha(t_i)-\alpha(t_{i-1})]-I_0^-\right| < \varepsilon$, $\left|\sum_{i=1}^{n} F^+(\xi_i)[\alpha(t_i)-\alpha(t_{i-1})]-I_0^+\right| < \varepsilon$ whenever $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a,b]_{\mathbf{T}}$. Thus $F^-, F^+ \in MS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$ and

$$(IMS_\Delta)\int_a^b F(t)\mathrm{d}\alpha = \Big[(MS_\Delta)\int_a^b F^-(t)\mathrm{d}\alpha, (MS_\Delta)\int_a^b F^+(t)\mathrm{d}\alpha\Big]. \tag{3.8}$$

Conversely, let $F^-, F^+ \in MS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$. Then there exists $M_1, M_2 \in \mathbb{R}$ with the property that given $\varepsilon > 0$ there exists a Δ-gauge, δ with respect to α on $[a,b]_{\mathbf{T}}$ such that

$$\left|\sum_{i=1}^{n} F^-(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})] - M_1\right| < \varepsilon, \qquad \left|\sum_{i=1}^{n} F^+(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})] - M_2\right| < \varepsilon$$

whenever $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a,b]_{\mathbf{T}}$. We define $I_0 = [M_1, M_2]$, then if $P = \{([t_{i-1}, t_i]_{\mathbf{T}}; \xi_i)\}_{i=1}^n$ is a δ-fine McShane division of $[a,b]_{\mathbf{T}}$, we have

$$d\big(\sum_{i=1}^{n} F(\xi_i)[\alpha(t_i) - \alpha(t_{i-1})], I_0\big) < \varepsilon. \tag{3.9}$$

Hence $F : [a,b]_{\mathbf{T}} \to I_{\mathbb{R}}$ is (MS_Δ) integrable with respect to α on $[a,b]_{\mathbf{T}}$.

Theorem 3. *Let $\alpha : [a,b]_{\mathbf{T}} \to \mathbb{R}$ be an increasing function. If $F(t), G(t) \in IMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$ and $\beta, \gamma \in \mathbb{R}$. Then $\big[\beta F(t) + \gamma G(t)\big] \in IMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$ and*

$$(IMS_\Delta)\int_a^b (\beta F(t) + \gamma G(t))\, d\alpha = \beta(IMS_\Delta)\int_a^b F(t)\, d\alpha + \gamma(IMS_\Delta)\int_a^b G(t)\, d\alpha. \tag{3.10}$$

Proof. If $F(t), G(t) \in IMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$, then $F^{-}(t), F^{+}(t), G^{-}(t), G^{+}(t) \in MS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$ by Theorem 2. Hence $\beta F^{-}(t)+\gamma G^{-}(t), \beta F^{-}(t)+\gamma G^{+}(t), \beta F^{+}(t)+\gamma G^{-}(t), \beta F^{+}(t)+\gamma G^{+}(t) \in MS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$.

(1) If $\beta > 0$ and $\gamma > 0$, then

$$\begin{aligned}
(MS_{\Delta})\int_a^b (\beta F(t)+\gamma G(t))^{-}\mathrm{d}\alpha &= (MS_{\Delta})\int_a^b (\beta F^{-}(t)+\gamma G^{-}(t))\mathrm{d}\alpha \\
&= \beta (MS_{\Delta})\int_a^b F^{-}(t)\mathrm{d}\alpha + \gamma (MS_{\Delta})\int_a^b G^{-}(t)\mathrm{d}\alpha \\
&= \beta\Big((IMS_{\Delta})\int_a^b F(t)\mathrm{d}\alpha\Big)^{-} + \gamma\Big((IMS_{\Delta})\int_a^b G(t)\mathrm{d}\alpha\Big)^{-} \\
&= \Big(\beta(IMS_{\Delta})\int_a^b F(t)\mathrm{d}\alpha + \gamma(IMS_{\Delta})\int_a^b G(t)\mathrm{d}\alpha\Big)^{-}.
\end{aligned}$$

(2) If $\beta < 0$ and $\gamma < 0$, then

$$\begin{aligned}
(MS_{\Delta})\int_a^b (\beta F(t)+\gamma G(t))^{-}\mathrm{d}\alpha &= (MS_{\Delta})\int_a^b (\beta F^{+}(t)+\gamma G^{+}(t))\mathrm{d}\alpha \\
&= \beta (MS_{\Delta})\int_a^b F^{+}(t)\mathrm{d}\alpha + \gamma (MS_{\Delta})\int_a^b G^{+}(t)\mathrm{d}\alpha \\
&= \beta\Big((IMS_{\Delta})\int_a^b F(t)\mathrm{d}\alpha\Big)^{+} + \gamma\Big((IMS_{\Delta})\int_a^b G(t)\mathrm{d}\alpha\Big)^{+} \\
&= \Big(\beta(IMS_{\Delta})\int_a^b F(t)\mathrm{d}\alpha + \gamma(IMS_{\Delta})\int_a^b G(t)\mathrm{d}\alpha\Big)^{-}.
\end{aligned}$$

(3) If $\beta > 0$ and $\gamma < 0$, (or $\beta < 0$ and $\gamma > 0$), then

$$\begin{aligned}
(MS_{\Delta})\int_a^b (\beta F(t)+\gamma G(t))^{-}\mathrm{d}\alpha &= (MS_{\Delta})\int_a^b (\beta F^{-}(t)+\gamma G^{+}(t))\mathrm{d}\alpha \\
&= \beta (MS_{\Delta})\int_a^b F^{-}(t)\mathrm{d}\alpha + \gamma (MS_{\Delta})\int_a^b G^{+}(t)\mathrm{d}\alpha \\
&= \beta\Big((IMS_{\Delta})\int_a^b F(t)\mathrm{d}\alpha\Big)^{-} + \gamma\Big((IMS_{\Delta})\int_a^b G(t)\mathrm{d}\alpha\Big)^{+}
\end{aligned}$$

$$= \left(\beta(IMS_\Delta)\int_a^b F(t)\mathrm{d}\alpha + \gamma(IMS_\Delta)\int_a^b G(t)\mathrm{d}\alpha\right)^-.$$

Similarly, for four cases above we have

$$(MS_\Delta)\int_a^b (\beta F(t)+\gamma G(t))^+\mathrm{d}\alpha = \left(\beta(IMS_\Delta)\int_a^b F(t)\mathrm{d}\alpha + \gamma(IMS_\Delta)\int_a^b G(t)\mathrm{d}\alpha\right)^+. \quad (3.11)$$

Hence by Theorem 2 $\beta F(t)+\gamma G(t) \in IMS^\alpha_\Delta[a,b]_{\mathbf{T}}$ and

$$(IMS_\Delta)\int_a^b (\beta F(t)+\gamma G(t))\mathrm{d}\alpha = \beta(IMS_\Delta)\int_a^b F(t)\mathrm{d}\alpha + \gamma(IMS_\Delta)\int_a^b G(t)\mathrm{d}\alpha. \quad (3.12)$$

Theorem 4. *Let $\alpha : [a,b]_{\mathbf{T}} \to \mathbb{R}$ be an increasing function. If $F(t) \in IMS^\alpha_\Delta[a,c]_{\mathbf{T}}$ and $F(t) \in IMS^\alpha_\Delta[c,b]_{\mathbf{T}}$, then $F(t) \in IMS^\alpha_\Delta[a,b]_{\mathbf{T}}$ and*

$$(IMS_\Delta)\int_a^b F(t)d\alpha = (IMS_\Delta)\int_a^c F(t)d\alpha + (IMS_\Delta)\int_c^b F(t)d\alpha. \quad (3.13)$$

Proof. If $F(t) \in IMS^\alpha_\Delta[a,c]_{\mathbf{T}}$ and $F(t) \in IMS^\alpha_\Delta[c,b]_{\mathbf{T}}$, then by Theorem 2 $F^-(t), F^+(t) \in MS^\alpha_\Delta[a,c]_{\mathbf{T}}$ and $F^-(t), F^+(t) \in MS^\alpha_\Delta[c,b]_{\mathbf{T}}$. Hence $F^-(t), F^+(t) \in MS^\alpha_\Delta[a,b]_{\mathbf{T}}$ and

$$\begin{aligned}(MS_\Delta)\int_a^b F^-(t)\mathrm{d}\alpha &= (MS_\Delta)\int_a^c F^-(t)\mathrm{d}\alpha + (MS_\Delta)\int_c^b F^-(t)\mathrm{d}\alpha \\ &= \left((IMS_\Delta)\int_a^c F(t)\mathrm{d}\alpha + (IMS_\Delta)\int_c^b F(t)\mathrm{d}\alpha\right)^-.\end{aligned}$$

Similarly, $(MS_\Delta)\int_a^b F^+(t)\mathrm{d}\alpha = \left((IMS_\Delta)\int_a^c F(t)\mathrm{d}\alpha + (IMS_\Delta)\int_c^b F(t)\mathrm{d}\alpha\right)^+$. Hence by Theorem 2 $F(t) \in IMS^\alpha_\Delta[a,b]_{\mathbf{T}}$ and

$$(IMS_\Delta)\int_a^b F(t)\mathrm{d}\alpha = (IMS_\Delta)\int_a^c F(t)\mathrm{d}\alpha + (IMS_\Delta)\int_c^b F(t)\mathrm{d}\alpha. \quad (3.14)$$

Theorem 5. *Let $\alpha : [a,b]_{\mathbf{T}} \to \mathbb{R}$ be an increasing function. If $F(t) \leq G(t)$ almost everywhere with respect to α on $[a,b]_{\mathbf{T}}$ and $F(t), G(t) \in IMS^\alpha_\Delta[a,b]_{\mathbf{T}}$, then*

$$(IMS_\Delta)\int_a^b F(t)d\alpha \leq (IMS_\Delta)\int_a^b G(t)d\alpha. \quad (3.15)$$

Proof. Let $F(t) \leq G(t)$ almost everywhere with respect to α on $[a,b]_{\mathbf{T}}$ and $F(t), G(t) \in IMS_{\Delta}^{\alpha}[a,b]_{\mathbf{T}}$. Then $F^{-}(t), F^{+}(t), G^{-}(t), G^{+}(t) \in MS_{\Delta}^{\alpha}[a,b]_{\mathbf{T}}$ and $F^{-}(t) \leq G^{-}(t)$, $F^{+}(t) \leq G^{+}(t)$ nearly everywhere with respect to α on $[a,b]_{\mathbf{T}}$. By Theorem 1 $(MS_{\Delta}) \int_a^b F^{-}(t)\mathrm{d}\alpha \leq (MS_{\Delta}) \int_a^b G^{-}(t)\mathrm{d}\alpha$ and $(MS_{\Delta}) \int_a^b F^{+}(t)\mathrm{d}\alpha \leq (MS_{\Delta}) \int_a^b G^{+}(t)\mathrm{d}\alpha$. Hence

$$(IMS_{\Delta}) \int_a^b F(t)\mathrm{d}\alpha \leq (IMS_{\Delta}) \int_a^b G(t)\mathrm{d}\alpha, \tag{3.16}$$

by Theorem 2.

Theorem 6. *Let $\alpha : [a,b]_{\mathbf{T}} \to \mathbb{R}$ be an increasing function. Let $F(t), G(t) \in IMS_{\Delta}^{\alpha}[a,b]_{\mathbf{T}}$ and $d(F(t), G(t))$ is (MS_{Δ}) integrable with respect to α on $[a,b]_{\mathbf{T}}$. Then*

$$d\big((IMS_{\Delta}) \int_a^b F(t)\, d\alpha, (IMS_{\Delta}) \int_a^b G(t)\, d\alpha\big) \leq (MS_{\Delta}) \int_a^b d\big(F(t), G(t)\big)\, d\alpha. \tag{3.17}$$

Proof. By definition of distance,

$$\begin{aligned}
&d\big((IMS_{\Delta}) \int_a^b F(t)\mathrm{d}\alpha, (IMS_{\Delta}) \int_a^b G(t)\mathrm{d}\alpha\big) \\
&= \max\left(\left|\left((IMS_{\Delta}) \int_a^b F(t)\mathrm{d}\alpha\right)^{-} - \left((IMS_{\Delta}) \int_a^b G(t)\mathrm{d}\alpha\right)^{-}\right|, \left|\left((IMS_{\Delta}) \int_a^b F(t)\mathrm{d}\alpha\right)^{+} - \left((IMS_{\Delta}) \int_a^b G(t)\mathrm{d}\alpha\right)^{+}\right|\right) \\
&= \max\left(\left|(MS_{\Delta}) \int_a^b \big(F^{-}(t) - G^{-}(t)\big)\mathrm{d}\alpha\right|, \left|(MS_{\Delta}) \int_a^b \big(F^{+}(t) - G^{+}(t)\big)\mathrm{d}\alpha\right|\right) \\
&\leq \max\left((MS_{\Delta}) \int_a^b \left|F^{-}(t) - G^{-}(t)\right|\mathrm{d}\alpha, (MS_{\Delta}) \int_a^b \left|F^{+}(t) - G^{+}(t)\right|\mathrm{d}\alpha\right) \\
&\leq (MS_{\Delta}) \int_a^b \max\left(\left|F^{-}(t) - G^{-}(t)\right|\mathrm{d}\alpha, \left|F^{+}(t) - G^{+}(t)\right|\mathrm{d}\alpha\right) \\
&= (MS_{\Delta}) \int_a^b d\big(F(t), G(t)\big)\mathrm{d}\alpha.
\end{aligned} \tag{3.18}$$

4. The MS_Δ integral of fuzzy-number-valued functions on time scales

In this section, we introduce the notion of the (MS_Δ) integral of fuzzy-number-valued functions on time scales and discusses some of their properties.

Definition 6. *[6, 8, 9] Let* $\tilde{A} \in F(\mathbb{R})$ *be a fuzzy subset on* $\mathbb{R}$*. If for any* $\lambda \in [0,1]$, $A_\lambda = [A_\lambda^-, A_\lambda^+]$ *and* $A_1 \neq \phi$*, where* $A_\lambda = \{t : \tilde{A}(t) \geq \lambda\}$*, then* $\tilde{A}$ *is called a fuzzy number. If* $\tilde{A}$ *is (1) convex, (2) normal, (3) upper semi-continuous, (4) has the compact support, we say that* $\tilde{A}$ *is a compact fuzzy number.*

Let $\tilde{\mathbb{R}}$ denote the set of all compact.

Definition 7. *[6] Let* $\tilde{A}, \tilde{B} \in \tilde{\mathbb{R}}$*, we define (1)* $\tilde{A} \leq \tilde{B}$ *iff* $A_\lambda \leq B_\lambda$ *for all* $\lambda \in (0,1]$*, (2)* $\tilde{A} + \tilde{B} = \tilde{C}$ *iff* $A_\lambda + B_\lambda = C_\lambda$ *for any* $\lambda \in (0,1]$*, (3)* $\tilde{A} \cdot \tilde{B} = \tilde{D}$ *iff* $A_\lambda \cdot B_\lambda = D_\lambda$ *for any* $\lambda \in (0,1]$*.*

For $\tilde{A}, \tilde{B} \in \tilde{\mathbb{R}}^C$*, then*

$$D(\tilde{A}, \tilde{B}) = \sup_{\lambda \in [0,1]} d(A_\lambda, B_\lambda), \tag{4.1}$$

is called the distance between $\tilde{A}$ *and* $\tilde{B}$*.*

Lemma 1. *[1] If a mapping* $H : [0,1] \to I_{\mathbb{R}}$, $\lambda \to H(\lambda) = [m_\lambda, n_\lambda]$*, satisfies* $[m_{\lambda_1}, n_{\lambda_1}] \supset [m_{\lambda_2}, n_{\lambda_2}]$ *when* $\lambda_1 < \lambda_2$*, then*

$$\tilde{A} := \bigcup_{\lambda \in (0,1]} \lambda H(\lambda) \in \tilde{\mathbb{R}} \tag{4.2}$$

and

$$A_\lambda = \bigcap_{n=1}^{\infty} H(\lambda_n), \tag{4.3}$$

where $\lambda_n = [1 - \frac{1}{(n+1)}]\lambda$*.*

Definition 8. *Let* $\alpha : [a,b]_{\mathbf{T}} \to \mathbb{R}$ *be an increasing function and let* $\tilde{F} : [a,b]_{\mathbf{T}} \to \tilde{\mathbb{R}}$*. If the interval-valued function* $F_\lambda(t) = [F_\lambda^-(t), F_\lambda^+(t)]$ *is* (MS_Δ) *integrable with respect to* α *on* $[a,b]_{\mathbf{T}}$ *for any* $\lambda \in (0,1]$*, then* $\tilde{F}(t)$ *is called* (MS_Δ) *integrable with respect to* α *on* $[a,b]_{\mathbf{T}}$ *and the integral is defined by* (MS_Δ) *integral as follow:*

$$\begin{aligned} (FMS_\Delta)\int_a^b \tilde{F}(t)d\alpha \quad &:= \bigcup_{\lambda \in (0,1]} \lambda (IMS_\Delta)\int_a^b F_\lambda(t)d\alpha \\ &= \bigcup_{\lambda \in (0,1]} \lambda \Big[(MS_\Delta)\int_a^b F_\lambda^-(t)d\alpha, (MS_\Delta)\int_a^b F_\lambda^+(t)d\alpha\Big]. \end{aligned}$$

We write $\tilde{F}(t) \in FMS_\Delta^\alpha[a,b]_{\mathbf{T}}$*.*

Theorem 7. *If* $\tilde{F}(t) \in FMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$, *then* $(FMS_{\Delta})\int_a^b \tilde{F}(t)d\alpha \in \tilde{\mathbb{R}}$ *and*

$$\left[(FMS_{\Delta})\int_a^b \tilde{F}(t)d\alpha\right]_{\lambda} = \bigcap_{n=1}^{\infty}(IMS_{\Delta})\int_a^b F_{\lambda_n}(t)d\alpha, \tag{4.4}$$

where $\lambda_n = [1 - \frac{1}{(n+1)}]\lambda$.

Proof. Let $H : (0,1] \to I_{\mathbb{R}}$, be defined by $H(\lambda) = \left[(MS_{\Delta})\int_a^b F^-_{\lambda}(t)\mathrm{d}\alpha, (MS_{\Delta})\int_a^b F^+_{\lambda}(t)\mathrm{d}\alpha\right]$.

Since $F^-_{\lambda}(t)$ and $F^+_{\lambda}(t)$ are increasing and decreasing on λ respectively, therefore, when $0 < \lambda_1 \leq \lambda_2 \leq 1$, we have $F^-_{\lambda_1}(t) \leq F^-_{\lambda_2}(t)$, $F^+_{\lambda_1}(t) \geq F^+_{\lambda_2}(t)$, on $[a,b]_{\mathbf{T}}$. From Theorem 5 we have

$$\left[(MS_{\Delta})\int_a^b F^-_{\lambda_1}(t)\mathrm{d}\alpha, (MS_{\Delta})\int_a^b F^+_{\lambda_1}(t)\mathrm{d}\alpha\right] \supset \left[(MS_{\Delta})\int_a^b F^-_{\lambda_2}(t)\mathrm{d}\alpha, (MS_{\Delta})\int_a^b F^+_{\lambda_2}(t)\mathrm{d}\alpha\right]. \tag{4.5}$$

Using Theorem 2 and Lemma 1 we obtain

$$(FMS_{\Delta})\int_a^b \tilde{F}(t)\mathrm{d}\alpha := \bigcup_{\lambda\in(0,1]} \lambda\left[(MS_{\Delta})\int_a^b F^-_{\lambda}(t)\mathrm{d}\alpha, (MS_{\Delta})\int_a^b F^+_{\lambda}(t)\mathrm{d}\alpha\right] \in \tilde{\mathbb{R}} \tag{4.6}$$

and for all $\lambda \in (0,1]$,

$$\left[(FMS_{\Delta})\int_a^b \tilde{F}(t)\mathrm{d}\alpha\right]_{\lambda} = \bigcap_{n=1}^{\infty}(IMS_{\Delta})\int_a^b F_{\lambda_n}(t)\mathrm{d}\alpha, \tag{4.7}$$

where $\lambda_n = [1 - \frac{1}{(n+1)}]\lambda$.

Theorem 8. *Let* $\alpha : [a,b]_{\mathbf{T}} \to \mathbb{R}$ *be an increasing function. If* $\tilde{F}(t), \tilde{G}(t) \in FMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$ *and* $\beta, \gamma \in \mathbb{R}$. *Then* $\beta\tilde{F}(t) + \gamma\tilde{G}(t) \in FMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$ *and*

$$(FMS_{\Delta})\int_a^b \left(\beta\tilde{F}(t) + \gamma\tilde{G}(t)\right)d\alpha = \beta(FMS_{\Delta})\int_a^b \tilde{F}(t)d\alpha + \gamma(FMS_{\Delta})\int_a^b \tilde{G}(t)d\alpha. \tag{4.8}$$

Proof. If $\tilde{F}(t), \tilde{G}(t) \in FMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$, then the interval-valued function $F_{\lambda}(t) = [F^-_{\lambda}(t), F^+_{\lambda}(t)]$ and $G_{\lambda}(t) = [G^-_{\lambda}(t), G^+_{\lambda}(t)]$ are (MS_{Δ}) integrable with respect to α on $[a,b]_{\mathbf{T}}$ for any $\lambda \in (0,1]$ and $(FMS_{\Delta})\int_a^b \tilde{F}(t)\mathrm{d}\alpha = \bigcup_{\lambda\in(0,1]} \lambda(IMS_{\Delta})\int_a^b F_{\lambda}(t)\mathrm{d}\alpha$ and $(FMS_{\Delta})\int_a^b \tilde{G}(t)\mathrm{d}\alpha =$

$\bigcup\limits_{\lambda\in(0,1]} \lambda (IMS_\Delta) \int\limits_a^b G_\lambda(t)\mathrm{d}\alpha$. From Theorem 3 we have $\beta F_\lambda(t)+\gamma G_\lambda(t) \in IMS^\alpha_\Delta[a,b]_\mathbf{T}$ and $(IMS_\Delta)\int\limits_a^b \big(\beta F_\lambda(t)+\gamma G_\lambda(t)\big)\mathrm{d}\alpha = \beta(IMS_\Delta)\int\limits_a^b F_\lambda(t)\mathrm{d}\alpha+\gamma(IMS_\Delta)\int\limits_a^b G_\lambda(t)\mathrm{d}\alpha$ for any $\lambda\in(0,1]$. Hence $\beta\tilde{F}(t)+\gamma\tilde{G}(t)\in FMS^\alpha_\Delta[a,b]_\mathbf{T}$ and

$$
\begin{aligned}
(FMS_\Delta)\int\limits_a^b \big(\beta\tilde{F}(t)+\gamma\tilde{G}(t)\big)\mathrm{d}\alpha &= \bigcup_{\lambda\in(0,1]} \lambda(IMS_\Delta)\int\limits_a^b \big(\beta F_\lambda(t)+\gamma G_\lambda(t)\big)\mathrm{d}\alpha \\
&= \bigcup_{\lambda\in(0,1]} \lambda\Big(\beta(IMS_\Delta)\int\limits_a^b F_\lambda(t)\mathrm{d}\alpha+\gamma(IMS_\Delta)\int\limits_a^b G_\lambda(t)\mathrm{d}\alpha\Big) \\
&= \beta\bigcup_{\lambda\in(0,1]} \lambda(IMS_\Delta)\int\limits_a^b F_\lambda(t)\mathrm{d}\alpha+\gamma\bigcup_{\lambda\in(0,1]} \lambda(IMS_\Delta)\int\limits_a^b G_\lambda(t)\mathrm{d}\alpha \\
&= \beta(FMS_\Delta)\int\limits_a^b \tilde{F}(t)\mathrm{d}\alpha+\gamma(FMS_\Delta)\int\limits_a^b \tilde{G}(t)\mathrm{d}\alpha.
\end{aligned}
$$

Theorem 9. *Let $\alpha:[a,b]_\mathbf{T}\to\mathbb{R}$ be an increasing function. If $\tilde{F}(t)\in FMS^\alpha_\Delta[a,c]_\mathbf{T}$ and $\tilde{F}(t)\in FMS^\alpha_\Delta[c,b]_\mathbf{T}$, then $\tilde{F}(t)\in FMS^\alpha_\Delta[a,b]_\mathbf{T}$ and*

$$(FMS_\Delta)\int\limits_a^b \tilde{F}(t)\,d\alpha = (FMS_\Delta)\int\limits_a^c \tilde{F}(t)\,d\alpha+(FMS_\Delta)\int\limits_c^b \tilde{F}(t)\,d\alpha. \tag{4.9}$$

Proof. If $\tilde{F}(t)\in FMS^\alpha_\Delta[a,c]_\mathbf{T}$ and $\tilde{F}(t)\in FMS^\alpha_\Delta[c,b]_\mathbf{T}$, then the interval-valued function $F_\lambda(t)=[F^-_\lambda(t),F^+_\lambda(t)]$ is (MS_Δ) integrable with respect to α on $[a,c]_\mathbf{T}$ and $[c,b]_\mathbf{T}$ for any $\lambda\in(0,1]$ and $(FMS_\Delta)\int\limits_a^c \tilde{F}(t)\mathrm{d}\alpha = \bigcup\limits_{\lambda\in(0,1]}\lambda(IMS_\Delta)\int\limits_a^c F_\lambda(t)\mathrm{d}\alpha$ and $(FMS_\Delta)\int\limits_c^b \tilde{F}(t)\mathrm{d}\alpha = \bigcup\limits_{\lambda\in(0,1]}\lambda(IMS_\Delta)\int\limits_c^b F_\lambda(t)\mathrm{d}\alpha$. From Theorem 4 we have $F_\lambda(t)\in IMS^\alpha_\Delta[a,b]_\mathbf{T}$ and $(IMS_\Delta)\int\limits_a^b F_\lambda(t)\mathrm{d}\alpha = (IMS_\Delta)\int\limits_a^c F_\lambda(t)\mathrm{d}\alpha+(IMS_\Delta)\int\limits_c^b F_\lambda(t)\mathrm{d}\alpha$ for any $\lambda\in(0,1]$. Hence $\tilde{F}(t)\in FMS^\alpha_\Delta[a,b]_\mathbf{T}$ and

$$
\begin{aligned}
(FMS_\Delta)\int\limits_a^b \tilde{F}(t)\mathrm{d}\alpha &= \bigcup_{\lambda\in(0,1]}\lambda(IMS_\Delta)\int\limits_a^b F_\lambda(t)\mathrm{d}\alpha \\
&= \bigcup_{\lambda\in(0,1]}\lambda\Big((IMS_\Delta)\int\limits_a^c F_\lambda(t)\mathrm{d}\alpha+(IMS_\Delta)\int\limits_c^b F_\lambda(t)\mathrm{d}\alpha\Big)
\end{aligned}
$$

$$= \bigcup_{\lambda\in(0,1]} \lambda(IMS_\Delta)\int_a^c F_\lambda(t)\mathrm{d}\alpha + \bigcup_{\lambda\in(0,1]} \lambda(IMS_\Delta)\int_c^b F_\lambda(t)\mathrm{d}\alpha$$

$$= (FMS_\Delta)\int_a^c \tilde{F}(t)\mathrm{d}\alpha + (FMS_\Delta)\int_c^b \tilde{F}(t)\mathrm{d}\alpha.$$

Theorem 10. *Let $\alpha : [a,b]_{\mathbf{T}} \to \mathbb{R}$ be an increasing function. If $\tilde{F}(t) \le \tilde{G}(t)$ almost everywhere with respect to α on $[a,b]_{\mathbf{T}}$ and $\tilde{F}(t), \widetilde{G}(t) \in FMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$, then*

$$(FMS_\Delta)\int_a^b \tilde{F}(t)\,d\alpha \le (FMS_\Delta)\int_a^b \tilde{G}(t)\,d\alpha. \tag{4.10}$$

Proof. If $\tilde{F}(t) \le \tilde{G}(t)$ almost everywhere with respect to α on $[a,b]_{\mathbf{T}}$ and $\tilde{F}(t), \widetilde{G}(t) \in FMS^{\alpha}_{\Delta}[a,b]_{\mathbf{T}}$, then $F_\lambda(t) \le G_\lambda(t)$ nearly everywhere with respect to α on $[a,b]_{\mathbf{T}}$ for any $\lambda \in (0,1]$ and $F_\lambda(t)$ and $G_\lambda(t)$ are (MS_Δ) integrable with respect to α on $[a,b]_{\mathbf{T}}$ for any $\lambda \in (0,1]$ and $(FMS_\Delta)\int_a^b \tilde{F}(t)\mathrm{d}\alpha = \bigcup_{\lambda\in(0,1]} \lambda(IMS_\Delta)\int_a^b F_\lambda(t)\mathrm{d}\alpha$ and $(FMS_\Delta)\int_a^b \tilde{G}(t)\mathrm{d}\alpha = \bigcup_{\lambda\in(0,1]} \lambda(IMS_\Delta)\int_a^b G_\lambda(t)\mathrm{d}\alpha$. From Theorem 5 we have $(IMS_\Delta)\int_a^b F_\lambda(t)\mathrm{d}\alpha \le (IMS_\Delta)\int_a^b G_\lambda(t)\mathrm{d}\alpha$ for any $\lambda \in (0,1]$. Hence

$$\begin{aligned}(FMS_\Delta)\int_a^b \tilde{F}(t)\mathrm{d}\alpha &= \bigcup_{\lambda\in(0,1]} \lambda(IMS_\Delta)\int_a^b F_\lambda(t)\mathrm{d}\alpha \\ &\le \bigcup_{\lambda\in(0,1]} \lambda(IMS_\Delta)\int_a^b G_\lambda(t)\mathrm{d}\alpha \\ &= (FMS_\Delta)\int_a^b \tilde{G}(t)\mathrm{d}\alpha.\end{aligned}$$

5. Conclusions

In this paper, we introduced the concept of the (MS_Δ) integrals of interval-valued functions and fuzzy number- valued functions on time scales and investigated some properties of those integrals.

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