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Transmuted Modified Weibull distribution: Properties and Application

Muhammad Shuaib Khan[1,*], Robert King[1], Irene L. Hudson[2]

[1] *School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, Australia*
[2] *Department of Statistics, Data Science and Epidemiology, Swinburne University of Technology, Hawthorn, VIC 3122, Australia*

Abstract. This research investigates the potential usefulness of the transmuted modified Weibull distribution for modelling lifetime data. We attain the diagnostic shapes of the density and hazard functions. We formulate the expressions for the moments, incomplete moments, Rnyi entropy and q- entropy. We estimate the mean, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis based on moment approach. The method of maximum likelihood is used for estimating the model parameters. We illustrate the use of transmuted modified Weibull distribution with an application to survival data.

2010 Mathematics Subject Classifications: 62N05, 90B25

Key Words and Phrases: Reliability functions, moments, entropies, maximum likelihood estimation

1. Introduction

A considerable literature discussing the transmuting approach for developing a new family of lifetime distributions by using the baseline model. Shaw and Buckley [14] introduced the the quadratic rank transmutation map (QRTM) approach for adding a new parameter to an existing distribution. According to this approach a random variable X is said to have a transmuted distribution if its cumulative distribution function (cdf) satisfies the following relationship.

$$F(x) = (1+\lambda)\ G(x) - \lambda\ G(x)^2, \qquad |\lambda| \le 1 \tag{1}$$

and

$$f(x) = \ g(x)\left\{(1+\lambda) - 2\lambda\ G(x)\right\}, \tag{2}$$

*Corresponding author.

Email addresses: Shuaib.stat@gmail.com (M. Shuaib)
robert.king@newcastle.edu.au (R. King), lhudson@swin.edu.au (Irene L. Hudson)

where $G(x)$ is the cdf of the base distribution, $g(x)$ and $f(x)$ are the corresponding probability density functions (pdf) associated with $G(x)$ and $F(x)$, respectively. Recently Khan and King [5] proposed and studied the transmuted modified Weibull distribution as an important competitive model with eleven lifetime distributions as special sub-models and discussed some of its properties. This research investigates the potential usefulness of the transmuted modified Weibull distribution for modelling lifetime data and formulate some of its theoretical properties. The modified Weibull (MW) distribution was pioneered by Sarhan and Zaindiu [13] and the cdf of the MW distribution is given by

$$G(x) \quad = \quad 1 - \exp\left\{-\alpha x - \eta x^{\beta}\right\}, \qquad x > 0 \tag{3}$$

where $\beta > 0$ is the shape parameters and $\alpha, \eta > 0$ are the scale parameters. The probability density function corresponding to (3) is given by

$$g(x) \quad = \quad \left(\alpha + \eta\beta x^{\beta-1}\right) \exp\left\{-\alpha x - \eta x^{\beta}\right\}, \tag{4}$$

A significant amount of work has been attributed towards developing the new transmuted family of lifetime distribution and provides more flexibility comparing with baseline model. Aryal and Tsokos [1] studied the transmuted Weibull distribution to analyse reliability data. More recently Khan et al. [8] proposed the transmuted Chen distribution and investigated various structural properties with application. Recently Khan et al. [5], [6], [7] proposed the transmuted modified Weibull distribution and the transmuted inverse Weibull distribution with applications. Merovci [9] proposed and studied the transmuted Rayleigh distribution among several other distributions using QRTM technique.

The rest of the article is organized as follows, In Section 2, we present the analytical shapes of the probability density and hazard functions of the TMW model. Some structural properties are considered in Section 2, such as moments and incomplete moments. Rnyi entropy and q-entropy are formulated in Section 3. Maximum likelihood estimates (MLE) of the unknown parameters are discussed in Section 4. Application to the real data set is illustrated in Section 5. In Section 6, concluding remarks are addressed.

2. Transmuted Modified Weibull Distribution

The transmuted modified Weibull distribution (TMWD) was recently proposed by Khan and King [5] by using QRTM technique for modeling reliability data and discussed some theoretical properties of this distribution. The TMWD with four parameters $\alpha, \eta, \beta > 0$ and $|\lambda| \leq 1$, is given by

$$f(x) = \left(\alpha + \eta\beta x^{\beta-1}\right) \exp\left\{-\alpha x - \eta x^{\beta}\right\} \left\{1 - \lambda + 2\lambda \exp\left\{-\alpha x - \eta x^{\beta}\right\}\right\}, \tag{5}$$

The CDF corresponding to (5) is given by

$$F(x) = \left[1 - \exp\left\{-\alpha x - \eta x^{\beta}\right\}\right] \left\{1 + \lambda \exp\left\{-\alpha x - \eta x^{\beta}\right\}\right\}, \tag{6}$$

respectively. where α and η are the scale parameters, β is the shape parameter and λ is the transmuted parameter. We obtain the baseline model when the transmuting parameter $\lambda = 0$. This distribution has nice relationship with some other well-known distributions such as transmuted extreme value family, transmuted additive weibull, transmuted linear failure rate, transmuted Weibull, transmuted Rayleigh and transmuted exponential distributions. An important feature of the TMWD is that its hazard function has increasing, decreasing and constant hazard function. The motivation of this study is to investigate the potential usefulness of the TMW distribution which has a bathtub shaped hazard function. If X is a random variable with density function (5), we write this model as X $\sim$ TMW$(x; \alpha, \beta, \eta, \lambda)$.

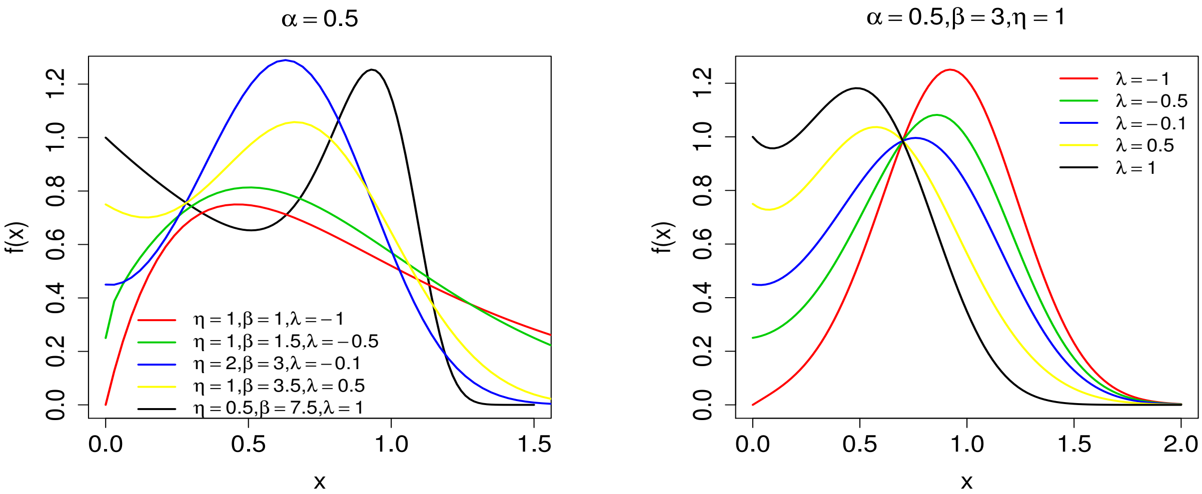

Figure 1: Plots of the TMW pdf for some parameter values.

Figure 1 shows the plots of the transmuted modified Weibull distribution for some selected values of parameters. The reliability and hazard functions of the transmuted modified Weibull distribution are given by

$$R(x) = 1 - \left[1 - \exp\left\{-\alpha x - \eta x^{\beta}\right\}\right]\left\{1 + \lambda \exp\left\{-\alpha x - \eta x^{\beta}\right\}\right\}. \tag{7}$$

and

$$h(x) = \frac{\left(\alpha + \eta\beta x^{\beta-1}\right)\exp\left\{-\alpha x - \eta x^{\beta}\right\}\left\{1 - \lambda + 2\lambda\exp\left\{-\alpha x - \eta x^{\beta}\right\}\right\}}{1 - \left[1 - \exp\left\{-\alpha x - \eta x^{\beta}\right\}\right]\left\{1 + \lambda\exp\left\{-\alpha x - \eta x^{\beta}\right\}\right\}}, \tag{8}$$

Plots of the TMW hazard function for some selected values of parameters are displayed in Figure 2. This figure also illustrate the effect of the shape parameter β and the transmuted parameter λ for hazard rate function.

This section presents the k^{th} moments and incomplete moments and discuss the mean, variance, coefficient of variation, skewness and kurtosis measures.

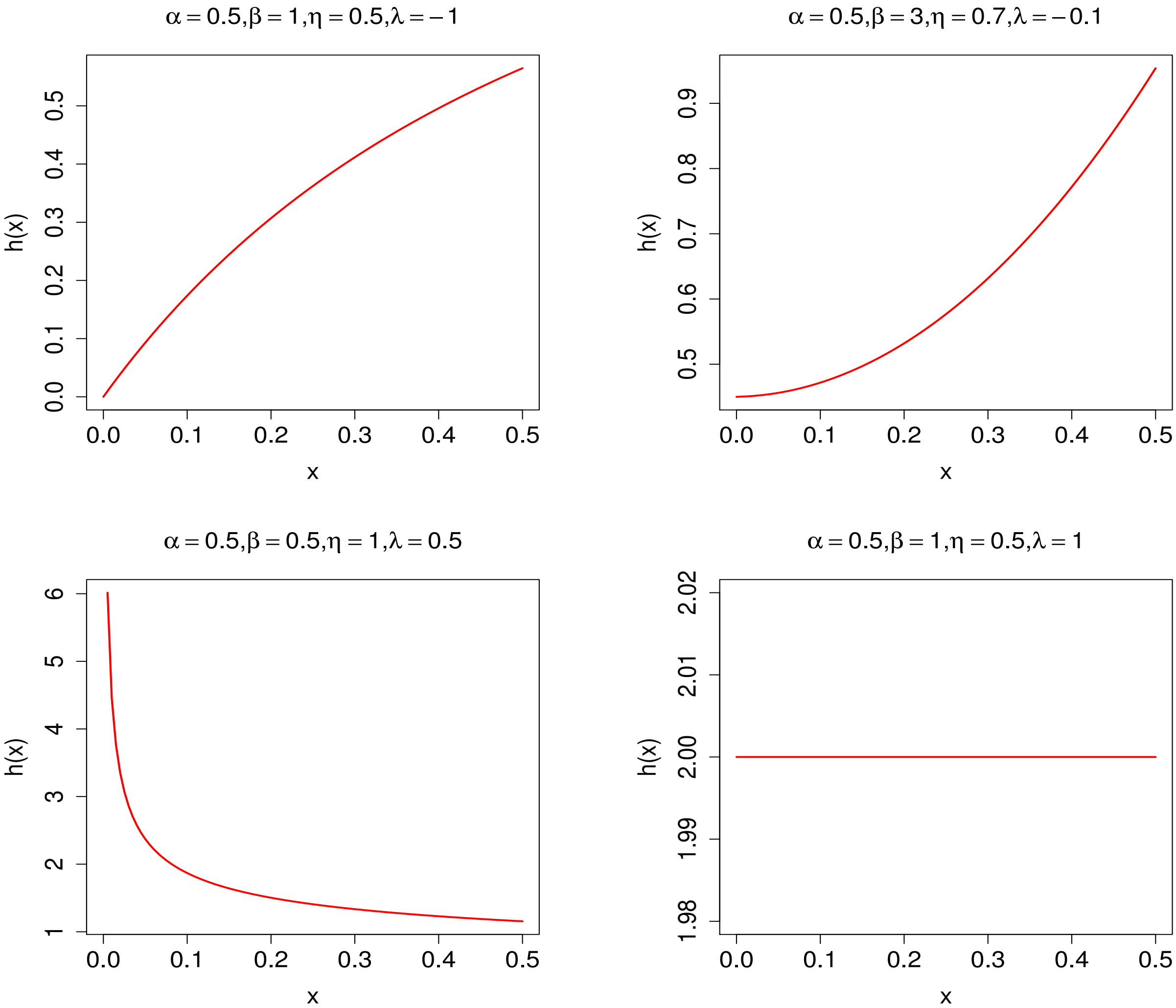

Figure 2: Plots of the TMW hf for some parameter values.

Theorem 1.

If X has the TMW$(x; \alpha, \beta, \eta, \lambda)$ with $|\lambda| \leq 1$, then the k^{th} moment of X, $\acute{\mu}_k$ is given as follows

$$\acute{\mu}_k = \sum_{i=0}^{\infty} \frac{(-1)^i \eta^i}{i!} \left[(1-\lambda) \left(\frac{\alpha \Gamma(i\beta + k + 1)}{\alpha^{i\beta+k+1}} + \frac{\beta \eta \Gamma(\beta(i+1) + k)}{\alpha^{\beta(i+1)+k}} \right) \right]$$

$$+2\lambda \sum_{i=0}^{\infty} \frac{(-1)^i (2\eta)^i}{i!} \left[\left(\frac{\alpha \Gamma(i\beta + k + 1)}{(2\alpha)^{i\beta+k+1}} + \frac{\beta \eta \Gamma(\beta(i+1) + k)}{(2\alpha)^{\beta(i+1)+k}} \right) \right].$$

Proof: By defination

$$\acute{\mu}_k = \int_0^\infty x^k \left(\alpha + \eta\beta x^{\beta-1}\right) \exp\left\{-\alpha x - \eta x^\beta\right\} \left\{1 - \lambda + 2\lambda \exp\left\{-\alpha x - \eta x^\beta\right\}\right\} dx.$$

The above expression reduces to

$$\acute{\mu}_k = (1-\lambda) \int_0^\infty x^k \left(\alpha + \eta\beta x^{\beta-1}\right) \exp\left\{-\alpha x - \eta x^\beta\right\} dx$$

$$+2\lambda \int_0^\infty x^k \left(\alpha + \eta\beta x^{\beta-1}\right) \exp\left\{-2\alpha x - 2\eta x^\beta\right\} dx.$$

The above integral reduces to

$$\acute{\mu}_k = (1-\lambda)\alpha \sum_{i=0}^\infty \frac{(-1)^i \eta^i}{i!} \int_0^\infty x^{k+i\beta} \exp(-\alpha x) dx$$

$$+(1-\lambda)\beta\eta \sum_{i=0}^\infty \frac{(-1)^i \eta^i}{i!} \int_0^\infty x^{k+i\beta+\beta-1} \exp(-\alpha x) dx$$

$$+2\alpha\lambda \sum_{i=0}^\infty \frac{(-1)^i (2\eta)^i}{i!} \int_0^\infty x^{k+i\beta} \exp(-2\alpha x) dx$$

$$+2\lambda\beta\eta \sum_{i=0}^\infty \frac{(-1)^i (2\eta)^i}{i!} \int_0^\infty x^{k+i\beta+\beta-1} \exp(-2\alpha x) dx$$

Hence, it follows that

$$\acute{\mu}_k = \sum_{i=0}^\infty \frac{(-1)^i \eta^i}{i!} \left[(1-\lambda)\left(\frac{\alpha\Gamma(i\beta+k+1)}{\alpha^{i\beta+k+1}} + \frac{\beta\eta\Gamma(\beta(i+1)+k)}{\alpha^{\beta(i+1)+k}}\right)\right]$$

$$+2\lambda \sum_{i=0}^\infty \frac{(-1)^i (2\eta)^i}{i!} \left[\left(\frac{\alpha\Gamma(i\beta+k+1)}{(2\alpha)^{i\beta+k+1}} + \frac{\beta\eta\Gamma(\beta(i+1)+k)}{(2\alpha)^{\beta(i+1)+k}}\right)\right]. \tag{9}$$

The important features and characteristics of the TMW distribution can be studied through moments. The mean, variance, coefficient of variation, skewness and kurtosis measures can be calculated using well-known relationships. The computations of the moments are performed using the R language for some selected choices of parameters are displayed in Tables 1 and 2.

Theorem 2.

If X has the TMW$(x; \alpha, \beta, \eta, \lambda)$ with $|\lambda| \leq 1$, then the incomplete moment is given as follows

$$\acute{\mu}_{(k,x)}(z) = \sum_{i=0}^\infty \frac{(-1)^i \eta^i}{i!} \left[(1-\lambda)\left(\frac{\alpha\gamma(i\beta+k+1, \alpha z)}{\alpha^{i\beta+k+1}} + \frac{\beta\eta\gamma(\beta(i+1)+k, \alpha z)}{\alpha^{\beta(i+1)+k}}\right)\right]$$

$$+2\lambda \sum_{i=0}^{\infty} \frac{(-1)^i (2\eta)^i}{i!} \left[\left(\frac{\alpha\gamma(i\beta+k+1, 2\alpha z)}{(2\alpha)^{i\beta+k+1}} + \frac{\beta\eta\gamma(\beta(i+1)+k), 2\alpha z}{(2\alpha)^{\beta(i+1)+k}} \right) \right].$$

Proof: By defination

$$\acute{\mu}_{(k,x)}(z) = \int_0^z x^k \left(\alpha + \eta\beta x^{\beta-1}\right) \exp\left\{-\alpha x - \eta x^\beta\right\} \left\{1 - \lambda + 2\lambda \exp\left\{-\alpha x - \eta x^\beta\right\}\right\} dx.$$

The above expression reduces to

$$\acute{\mu}_{(k,x)}(z) = (1-\lambda) \int_0^z x^k \left(\alpha + \eta\beta x^{\beta-1}\right) \exp\left\{-\alpha x - \eta x^\beta\right\} dx$$

$$+2\lambda \int_0^z x^k \left(\alpha + \eta\beta x^{\beta-1}\right) \exp\left\{-2\alpha x - 2\eta x^\beta\right\} dx.$$

The above integral reduces to

$$\acute{\mu}_{(k,x)}(z) = (1-\lambda)\alpha \sum_{i=0}^{\infty} \frac{(-1)^i \eta^i}{i!} \int_0^z x^{k+i\beta} \exp(-\alpha x) dx$$

$$+(1-\lambda)\beta\eta \sum_{i=0}^{\infty} \frac{(-1)^i \eta^i}{i!} \int_0^z x^{k+i\beta+\beta-1} \exp(-\alpha x) dx$$

$$+2\alpha\lambda \sum_{i=0}^{\infty} \frac{(-1)^i (2\eta)^i}{i!} \int_0^z x^{k+i\beta} \exp(-2\alpha x) dx$$

$$+2\lambda\beta\eta \sum_{i=0}^{\infty} \frac{(-1)^i (2\eta)^i}{i!} \int_0^z x^{k+i\beta+\beta-1} \exp(-2\alpha x) dx$$

Hence, it follows that

$$\acute{\mu}_{(k,x)}(z) = \sum_{i=0}^{\infty} \frac{(-1)^i \eta^i}{i!} \left[(1-\lambda) \left(\frac{\alpha\gamma(i\beta+k+1, \alpha z)}{\alpha^{i\beta+k+1}} + \frac{\beta\eta\gamma(\beta(i+1)+k, \alpha z)}{\alpha^{\beta(i+1)+k}} \right) \right]$$

$$+2\lambda \sum_{i=0}^{\infty} \frac{(-1)^i (2\eta)^i}{i!} \left[\left(\frac{\alpha\gamma(i\beta+k+1, 2\alpha z)}{(2\alpha)^{i\beta+k+1}} + \frac{\beta\eta\gamma(\beta(i+1)+k), 2\alpha z}{(2\alpha)^{\beta(i+1)+k}} \right) \right]. \quad (10)$$

The incomplete moments are useful for finding the mean deviations, mean residual life, the Bonferroni and Lorenz curves. These curves are very useful in econometrics, reliability engineering, actuarial sciences and medical sciences.

Table 1: *Moments values of the* TMW *distribution*

(α, β, η)	λ	$\acute{\mu}_1$	$\acute{\mu}_2$	$\acute{\mu}_3$	$\acute{\mu}_4$
$1, 0.75, 0.5$	-1	1.0201	1.7267	4.1720	13.267
	-0.5	0.8429	1.3488	3.1871	10.037
	0.5	0.4885	0.5930	1.2173	3.5744
	1	0.3113	0.2151	0.2324	0.3433
$1, 1.5, 0.75$	-1	0.8422	0.9755	1.4235	2.4906
	-0.5	0.7130	0.7776	1.1034	1.9028
	0.5	0.4545	0.3818	0.4631	0.7270
	1	0.3252	0.1839	0.1430	0.1392
$2, 2, 1$	-1	0.5471	0.4056	0.3703	0.3960
	-0.5	0.4630	0.3238	0.2877	0.3033
	0.5	0.2948	0.1603	0.1226	0.1179
	1	0.2106	0.0786	0.0400	0.0252
$2, 3, 2$	-1	0.5106	0.3259	0.2393	0.1944
	-0.5	0.4364	0.2637	0.1885	0.1507
	0.5	0.2880	0.1394	0.0869	0.0634
	1	0.2139	0.0772	0.0360	0.0197

3. Entropies

The Rényi [11] introduced the entropy denoted as, $I_R(\rho)$, for X is a measure of variation of uncertainty and is defined as

$$I_R(\rho) = \frac{1}{1-\rho} \log \left\{ \int_0^\infty f(x)^\rho dx \right\}, \tag{11}$$

where $\rho > 0$ and $\rho \neq 1$. The integral in $I_R(\rho)$ for the TMW$(x; \alpha, \beta, \eta, \lambda)$ can be defined by substituting (5) in (11) as

$$I_R(\rho) = \frac{1}{1-\rho} \log \left\{ \int_0^\infty \frac{\left(\alpha + \eta\beta x^{\beta-1}\right)^\rho \exp\left\{-\alpha\rho x - \eta\rho x^\beta\right\}}{\left\{1-\lambda+2\lambda\exp\left\{-\alpha x - \eta x^\beta\right\}\right\}^{-\rho}} dx \right\},$$

the above integral reduces to

$$I_R(\rho) = \frac{1}{1-\rho} \log \left\{ \sum_{i,j=0}^{\infty} u_{\alpha,\beta,\eta,\rho,\lambda,i,j} \int_0^\infty x^{j(\beta-1)} \exp\Big\{-\alpha x(i+\rho) - \eta x^\beta (i+\rho)\Big\}\, dx \right\},$$

where

$$u_{\alpha,\beta,\eta,\rho,\lambda,i,j} = \alpha^\rho \binom{\rho}{i} \binom{\rho}{j} \left(\frac{\beta\eta}{\alpha}\right)^j \left(\frac{2\lambda}{1-\lambda}\right)^i (1-\lambda)^\rho.$$

Table 2: *Moments based measures of the* TMW *distribution*

(α, β, η)	λ	*Mean*	*Var*	*CV*	*CS*	*CK*
$1, 0.75, 0.5$	-1	1.0201	0.6861	0.8119	1.7786	8.0213
	-0.5	0.8429	0.6383	0.9478	1.9101	8.6557
	0.5	0.4885	0.3544	1.2186	2.7561	14.9233
	1	0.3113	0.1182	1.1043	2.2605	10.7958
$1, 1.5, 0.75$	-1	0.8422	0.2662	0.6126	1.1179	4.7603
	-0.5	0.7130	0.2692	0.7277	1.1814	4.8621
	0.5	0.4545	0.1752	0.9210	1.7762	7.4997
	1	0.3252	0.0781	0.8596	1.4818	5.9482
$2, 2, 1$	-1	0.5471	0.1063	0.5958	0.9265	4.0087
	-0.5	0.4630	0.1094	0.7145	1.0068	4.0996
	0.5	0.2948	0.0734	0.9189	1.6129	6.3601
	1	0.2106	0.0342	0.8787	1.4234	5.5582
$2, 3, 2$	-1	0.5106	0.0652	0.5000	0.3801	2.7151
	-0.5	0.4364	0.0732	0.6202	0.4783	2.6403
	0.5	0.2880	0.0564	0.8250	1.0611	3.7732
	1	0.2139	0.0314	0.8290	1.0820	3.8540

Finally we obtain the TMW Rényi entropy as

$$\begin{aligned} I_R(\rho) &= \frac{\rho}{1-\rho}\log\alpha + \frac{\rho}{1-\rho}\log(1-\lambda) + \frac{1}{1-\rho} \\ &\log\left\{\sum_{i,j=0}^{\infty}\sum_{m=0}^{\infty}\binom{\rho}{i}\binom{\rho}{j}\left(\frac{\beta\eta}{\alpha}\right)^j\left(\frac{2\lambda}{1-\lambda}\right)^i\frac{(-1)^m}{m!}V_{j,\beta,m}\right\}, \end{aligned}$$

where

$$V_{j,\beta,m} = \frac{\eta^m(i+\rho)^m}{(\alpha(i+\rho))^{\beta(m+j)-j+1}}.\Gamma\left(\beta(m+j)-j+1\right)$$

The q-entropy was introduced by Havrda and Charvat [4], and is defined as

$$I_H(q) = \frac{1}{q-1}\left\{1-\int_0^{\infty} f(x)^q dx\right\}, \tag{12}$$

where $q > 0$ and $q \neq 1$. Suppose X has the TMW distribution then by substituting (5) in (12), we obtain

$$I_H(q) = \frac{1}{q-1}\left\{1-\int_0^{\infty}\frac{\left(\alpha+\eta\beta x^{\beta-1}\right)^q\exp\left\{-\alpha q x-\eta q x^{\beta}\right\}}{\left\{1-\lambda+2\lambda\exp\left\{-\alpha x-\eta x^{\beta}\right\}\right\}^{-q}}dx\right\},$$

the above integral yields the TMW q-entropy as

$$I_H(q) = \frac{1}{q-1}\left\{1-\sum_{i,j=0}^{\infty}\sum_{m=0}^{\infty}\frac{z_{\alpha,\beta,\eta,\lambda,i,j,m}\eta^m(i+q)^m}{(\alpha(i+q))^{\beta(m+j)-j+1}}.\Gamma\left(\beta(m+j)-j+1\right)\right\},$$

where

$$z_{\alpha,\beta,\eta,\lambda,i,j,m} = \alpha^q\begin{pmatrix} q \\ i \end{pmatrix}\begin{pmatrix} q \\ j \end{pmatrix}\left(\frac{\beta\eta}{\alpha}\right)^j\left(\frac{2\lambda}{1-\lambda}\right)^i(-1)^m(1-\lambda)^q.$$

Table 3 lists the values of Rényi entropy and q-entropy of the TMW distribution for some selected values of parameters.

Table 3: *Rényi entropy and q-entropy for some selected values of parameters*

(α,β,η)	λ	$I_R(\rho=0.5)$	$I_H(q=0.5)$	$I_R(\rho=1.5)$	$I_H(q=1.5)$
$1, 0.75, 0.5$	-1	0.5299	1.6810	0.2985	0.5816
	-0.5	0.4754	1.4574	0.1233	0.2646
	0.5	0.2791	0.7580	-0.3692	-1.0594
	1	0.0159	0.0370	-0.5986	-1.9840
$1, 1.5, 0.75$	-1	0.3326	0.9332	0.1100	0.2380
	-0.5	0.2863	0.7810	0.0168	0.0384
	0.5	0.1054	0.2580	-0.3217	-0.8966
	1	-0.1408	-0.2992	-0.5065	-1.5834
$2, 2, 1$	-1	0.0912	0.2214	-0.0912	-0.2214
	-0.5	0.0588	0.1400	-0.1396	-0.3486
	0.5	-0.0926	-0.2022	-0.3927	-1.1432
	1	-0.3010	-0.5858	-0.5509	-1.7712
$2, 3, 2$	-1	0.0078	0.0180	-0.1043	-0.2552
	-0.5	0.0010	0.0024	-0.1016	-0.2482
	0.5	-0.0960	-0.2092	-0.2570	-0.6886
	1	-0.2394	-0.4818	-0.3842	-1.1128

4. Parameter Estimation

Consider the random samples $x_1, x_2, ..., x_n$ consisting of n observations from the transmuted modified Weibull distribution with parameter vector $\Theta = (\alpha, \beta, \eta, \lambda)$ then the log-likelihood function $\ell(\Theta)$=lnL of (5) is given by

$$\ell(\Theta) = \sum_{i=1}^{n}\ln(\alpha+\beta\eta x_i^{\beta-1}) - \alpha\sum_{i=1}^{n}x_i - \eta\sum_{i=1}^{n}x_i^{\beta} + \sum_{i=1}^{n}\ln(1-\lambda+2\lambda\exp(-\alpha x_i-\eta x_i^{\beta})) \quad (13)$$

By setting the first partial derivatives of (13) with respect to α, β, η and λ then equating it to zero, we obtain the components of score vector $U(\Theta)$ are given by

$$\frac{\partial \ell(\Theta)}{\partial \alpha} = \sum_{i=1}^{n} (\alpha + \beta\eta x_i^{\beta-1})^{-1} - \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{2\lambda x_i \exp(-\alpha x_i - \eta x_i^{\beta})}{(1-\lambda+2\lambda\exp(-\alpha x_i - \eta x_i^{\beta}))},$$

$$\frac{\partial \ell(\Theta)}{\partial \beta} = \sum_{i=1}^{n} \frac{x_i^{\beta-1}(1+\beta\ln(x_i))}{(\alpha+\beta\eta x_i^{\beta-1})} - \sum_{i=1}^{n} x_i^{\beta}\ln(x_i) - \sum_{i=1}^{n} \frac{2\lambda\exp(-\alpha x_i - \eta x_i^{\beta}) x_i^{\beta}\ln(x_i)}{(1-\lambda+2\lambda\exp(-\alpha x_i - \eta x_i^{\beta}))},$$

$$\frac{\partial \ell(\Theta)}{\partial \eta} = \sum_{i=1}^{n} \frac{\beta x_i^{\beta-1}}{(\alpha+\beta\eta x_i^{\beta-1})} - \sum_{i=1}^{n} x_i^{\beta} - \sum_{i=1}^{n} \frac{2\lambda x_i \exp(-\alpha x_i - \eta x_i^{\beta}) x_i^{\beta}}{(1-\lambda+2\lambda\exp(-\alpha x_i - \eta x_i^{\beta}))},$$

and

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \sum_{i=1}^{n} \frac{2\exp(-\alpha x_i - \eta x_i^{\beta}) - 1}{(1-\lambda+2\lambda\exp(-\alpha x_i - \eta x_i^{\beta}))}.$$

respectively, by solving the non-linear system of equations simultaneously we can obtain the parameters of the TMWD. The asymptotic variance covariance matrix of MLEs for the parameter vector $\Theta = (\alpha, \beta, \eta, \lambda)^T$ can be considered as the multivariate normal with the variance covariance matrix and its inverse of the expected information matrix is given by

$$\left((\hat{\alpha}-\alpha), (\hat{\beta}-\beta), (\hat{\eta}-\eta), (\hat{\lambda}-\lambda)\right) \sim N_4\left\{0, K(\Theta)^{-1}\right\},$$

where $K(\Theta)^{-1}$ is the variance covariance matrix of the unknown parameters. The multivariate normal distribution can be used to obtain an approximate $100(1-\gamma)\%$ confidence intervals for the parameters α, β, η and λ can be determined as

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V_{11}}}, \qquad \hat{\beta} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V_{22}}}, \qquad \hat{\eta} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V_{33}}}, \qquad \hat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V_{44}}},$$

where $Z_{\frac{\gamma}{2}}$ is the upper γ^{th} percentile of the standard normal distribution.

5. Application

In this section, we present analysis for illustrative proposes using the nicotine cigarettes data. The data set consist of 396 observations of nicotine content in milligrams in cigarettes of several brands of cigarettes in 1995. The data have been obtained from the Federal trade commission (FTC), which is an independent agency of the US government, whose main mission is the promotion of consumer protection. The report entitled " Tar, Nicotine and Carbon Monoxide of the Smoke of 1249 varieties of domestic cigarettes for the Year 1995"

and the data is freely available at $http://w.w.w.ftc.gov/reports/$.
We fitted the (transmuted modified Weibull) TMW, Generalized Power Weibull (GPW), modified Weibull (MW) and Weibull (W) distributions by the method of maximum likelihood. The required numerical evaluations are implemented using R language. The MLEs for the four fitted models are displayed in Table 4. Table 4 gives the MLEs of the unknown parameters (with their corresponding standard errors) with AIC goodness of-fit measure. The goodness of fit measure shows that the transmuted modified Weibull distribution provides a better fit among four fitted models. We also evaluate the performance of these models by using the Cramr-von Mises test (W) and Anderson-Darling (A) goodness of-fit measures are listed in Table 5. The lowest values of Cramr-von Mises test (W) and Anderson-Darling (A) goodness of-fit statistics shows the better fit among these considered models in this paper. Therefore the transmuted modified Weibull distribution could be chosen as the best model for fitting survival data.

Table 4: *MLEs of the Parameters for the nicotine cigarettes data, the Corresponding SE (given in parentheses)with the AIC measures*

$Model$	α	η	β	λ	AIC
TMW	0.0465	1.0964	3.2884	0.5511	140.015
	(0.0253)	(0.2036)	(0.1656)	(0.2526)	
GPW	0.8930	0.9101	2.7583	-	143.950
	(0.1261)	(0.2410)	(0.2311)		
MW	0.0548	1.5208	3.0107	-	140.467
	(0.0370)	(0.0883)	(0.1503)		
W	-	1.5844	2.83432	-	142.085
		(0.0796)	(0.1108)		

Table 5: *Goodness of-fit statistics*

$Model$	W	A
TMW	0.6680	3.5621
GPW	0.7484	3.9707
MW	0.7199	3.7843
W	0.7407	3.9543

6. Conclusion

This paper discussed the performance of the transmuted modified Weibull distribution. Some statistical properties have been derived and discussed with application to survival data. We have compared the transmuted modified Weibull distribution with Generalized

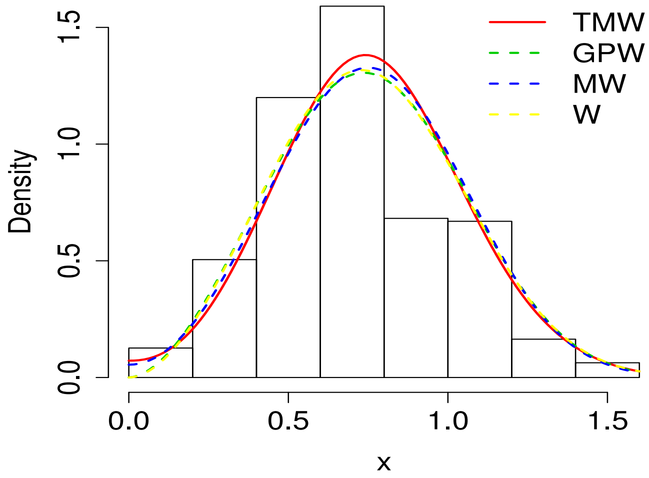

Figure 3: Fitted models for the nicotine cigarettes data.

Power Weibull (GPW), modified Weibull (MW) and Weibull (W) distributions by the method of maximum likelihood. It is observed that based on three goodness of fit measures the transmuted modified Weibull distribution provides the better fit than the other three lifetime distributions. The TMW distribution has increasing, decreasing and constant hazard function for survival data. We have calculated the values of raw moments and also calculated the mean, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis. The application of nicotine cigarettes data shown that the transmuted modified Weibull distribution fits the data very well than the other three distributions.

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