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On approximation properties of generalised (p,q)-Bernstein operators

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Abstract. In this study, a (p,q)–analogue of Bernstein operators is introduced and approximation properties of (p,q)–Bernstein operators are investigated. Some basic theorems are proved. The rate of approximation by modulus of continuity is estimated.

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1. Introduction

In 1912, for a function $f(x)$ defined on the closed interval $[0,1]$, the expression

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \tag{1}$$

was called the Bernstein polynomial of order n of the function $f(x)$ in [26]. Later, the various generalizations of Bernstein polynomials (1) were investigated in [2], [3], [10]-[13]. In recent years, the development of q-calculus has allowed to be made of new generalizations of approximation theory. Firstly, Lupaş [4] introduced the q-analogue of the Bernstein operators and investigated its approximation properties in 1987. After then, the various applications of q-Bernstein operators were handled by Phillips [7], [8]. The approximation properties of q-generalization of other operators were studied in [1], [5], [9], [23], [24], [27], [28] .

Recently, Mursaleen et al. applied (p,q)-in calculus approximation theory and introduced the (p,q)-analogue of Bernstein operators and other operators [12], [14]–[22].

In [3], Izgi introduced a class of new type Bernstein polynomials and investigated its approximation properties:

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$$F_{n,a,b}(f;x)=\sum_{k=0}^{n} f\left(\frac{k(n+a)}{n(n+b)}\right) q_{n,k,a,b}(x), \quad 0\leq x\leq \frac{n+a}{n+b}, \tag{2}$$

where $a,b\in\mathbb{N}$, $\quad 0\leq a\leq b$,

$$q_{n,k,a,b}(x)=\left(\frac{n+b}{n+a}\right)^{n}\binom{n}{k}x^{k}\left(\frac{n+a}{n+b}-x\right)^{n-k}. \tag{3}$$

The aim of this paper is to introduce (p,q)-analogue of generalized Bernstein operator (2) and is to study approximation properties for (p,q)-Bernstein operator.

Now we remember certain notations of (p,q)-calculus.

For any $p>0$ and $q>0$, the (p,q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+p^{n-3}q^{2}+...+pq^{n-2}+q^{n-1}=\begin{cases}\frac{p^{n}-q^{n}}{p-q}, & \text{when } p\neq q\neq 1\\ n\ p^{n-1}, & \text{when } p=q\neq 1\\ [n]_{q}, & \text{when } p=1\\ n, & \text{when } p=q=1\end{cases}$$

where $[n]_q$ denotes the q-integers and $n=0,1,2,\cdots$.

Let $p,q>0$ be given. We define a (p,q)-factorial, $[n]_{p,q}!$ of $k\in\mathbb{N}$, as

$$[n]_{p,q}!=\begin{cases}[1]_{p,q}[2]_{p,q}...[n]_{p,q}, & n\in\mathbb{N}\\ 1, & n=0.\end{cases} \tag{4}$$

The (p,q)-binomial coefficient $\begin{bmatrix}n\\ r\end{bmatrix}_{p,q}$ by

$$\begin{bmatrix}n\\ r\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-r]_{p,q}![r]_{p,q}!}. \tag{5}$$

We recall that (p,q)-derivative operator $\mathcal{D}_{p,q}$ is given by

$$\mathcal{D}_{p,q}f(x)=\frac{f(px)-f(qx)}{(p-q)x}, \quad x\neq 0; \quad \mathcal{D}_{p,q}f(0)=\lim_{x\to 0}\mathcal{D}_{p,q}f(x). \tag{6}$$

For any polynomial $f(x)$ of degree $N$and any number c, we have the following (p,q)-Taylor expansion:

$$f(x)=\sum_{j=0}^{N}\left(D_{p,q}^{j}f\right)(c)\frac{(n-c)_{p,q}^{j}}{[j]_{p,q}!} \tag{7}$$

where $(x-c)_{p,q}^{n}$ is (p,q)-analogue of $(x-c)^{n}$ and

$$(x-c)_{p,q}^{n}=\begin{cases}1, & n=0,\\ (x-c)(px-qc)...(p^{n-1}x-q^{n-1}c), & n\geq 1\end{cases} \tag{8}$$

Now, we give construction of our operators and some properties of them. For $f \in C\left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$,

$$F_{n,a,b}^{p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} f\left(\frac{[k]_{p,q}[n+a]_{p,q}}{p^{k-n}[n]_{p,q}[n+b]_{p,q}}\right) q_{n,k,a,b}^{p,q}(x), \quad 0 \leq x \leq \frac{[n+a]_{p,q}}{[n+b]_{p,q}}, \tag{9}$$

where $a, b \in \mathbb{N}, \quad 0 \leq a \leq b$,

$$q_{n,k,a,b}^{p,q}(x) = \left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^n \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} \left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}} p^s - q^s x\right). \tag{10}$$

2. Main Results

Lemma 1. *For $\forall x \in \left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$ and $\forall n \in \mathbb{N}$, (p,q)-Bernstein operators (9) are satisfied the following equalities:*

i. $F_{n,a,b}^{p,q}(1;x) = 1$,

ii. $F_{n,a,b}^{p,q}(t;x) = x$,

iii. $F_{n,a,b}^{p,q}(t^2;x) = \frac{p^{n-1}[n+a]_{p,q}}{[n]_{p,q}[n+b]_{p,q}} x + \frac{q[n-1]_{p,q}}{[n]_{p,q}} x^2$.

Proof.

$$\begin{aligned} F_{n,a,b}^{p,q}(1;x) =& \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} q_{n,k,a,b}^{p,q}(x) \\ =& \left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} \left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}} p^s - q^s x\right) \\ =& 1 \end{aligned}$$

$$\begin{aligned} F_{n,a,b}^{p,q}(t;x) &= \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{[k]_{p,q}[n+a]_{p,q}}{p^{k-n}[n]_{p,q}[n+b]_{p,q}} q_{n,k,a,b}^{p,q}(x) \\ &= \frac{1}{p^{\frac{n(n-3)}{2}}} \left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^{n-1} \sum_{k=1}^{n} \frac{[k]_{p,q}}{[n]_{p,q}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-3)}{2}} x^k \prod_{s=0}^{n-k-1} \left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}} p^s - q^s x\right) \\ &= \frac{1}{p^{\frac{n(n-3)}{2}}} \left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^{n-1} \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} p^{\frac{k(k-3)}{2}} x^k \prod_{s=0}^{n-k-1} \left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}} p^s - q^s x\right) \\ &= \frac{x}{p^{\frac{(n-1)(n-2)}{2}}} \left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^{n-1} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{\frac{(k+1)(k-2)}{2}} x^k \prod_{s=0}^{n-k-2} \left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}} p^s - q^s x\right) \end{aligned}$$

$$= x$$

$$
\begin{aligned}
F_{n,a,b}^{p,q}(t^2;x) &= \frac{1}{p^{\frac{n(n-1)}{2}}}\sum_{k=0}^{n}\left(\frac{[k]_{p,q}[n+a]_{p,q}}{p^{k-n}[n]_{p,q}[n+b]_{p,q}}\right)^2 q_{n,k,a,b}^{p,q}(x)\\
&= \frac{1}{p^{\frac{n(n-5)}{2}}}\left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^{n-2}\sum_{k=0}^{n-1}\frac{[k+1]_{p,q}}{[n]_{p,q}}\begin{bmatrix} n-1\\ k\end{bmatrix}_{p,q} p^{\frac{(k+1)(k-4)}{2}}x^{k+1}\\
&\times\prod_{s=0}^{n-k-2}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s x\right)\\
&= \frac{1}{p^{\frac{n(n-5)}{2}}[n]_{p,q}}\left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^{n-2}\sum_{k=0}^{n-1}\begin{bmatrix} n-1\\ k\end{bmatrix}_{p,q} p^{\frac{(k+1)(k-4)}{2}}x^{k+1}(p^k+q[k]_{p,q})\\
&\times\prod_{s=0}^{n-k-2}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s x\right)\\
&= \frac{1}{p^{\frac{n(n-5)}{2}}[n]_{p,q}}\left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^{n-2}\left\{\sum_{k=0}^{n-1}\begin{bmatrix} n-1\\ k\end{bmatrix}_{p,q} p^{\frac{k^2-k-4}{2}}x^{k+1}\right.\\
&\times\prod_{s=0}^{n-k-2}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s x\right)+q[n-1]_{p,q}\sum_{k=0}^{n-2}\begin{bmatrix} n-2\\ k\end{bmatrix}_{p,q} p^{\frac{(k-2)(k-3)}{2}}x^{k+2}\\
&\left.\times\prod_{s=0}^{n-k-3}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s x\right)\right\}\\
&= \frac{p^{n-1}x}{[n]_{p,q}}\frac{1}{p^{\frac{(n-1)(n-2)}{2}}}\left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^{n-2}\sum_{k=0}^{n-1}\begin{bmatrix} n-1\\ k\end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}}x^{k}\\
&\times\prod_{s=0}^{n-k-2}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s x\right)+\frac{q[n-1]_{p,q}x^2}{[n]_{p,q}}\frac{1}{p^{\frac{(n-2)(n-3)}{2}}}\left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^{n-2}\\
&\times\sum_{k=0}^{n-2}\begin{bmatrix} n-2\\ k\end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}}x^{k}\prod_{s=0}^{n-k-3}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s x\right)\\
&= \frac{p^{n-1}[n+a]_{p,q}}{[n]_{p,q}[n+b]_{p,q}}x+\frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2
\end{aligned}
$$

Theorem 1. *Let* $0 < q_n < p_n \le 1$ *and*

$$\lim_{n\to\infty} p_n = 1, \quad \lim_{n\to\infty} q_n = 1.$$

If $\forall f \in C\left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$, *then*

$$\lim_{n\to\infty} F_{n,a,b}^{p_n,q_n}(f;x) = f(x)$$

is uniformly on $\left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$.

Proof. The proof is based on Korovkin theorem, so it is enough to prove the conditions

$$\lim_{n\to\infty} \left\| F_{n,a,b}^{p_n,q_n}(t^m;x) - x^m \right\| = 0, \quad m = 0,1,2$$

uniformly on $\left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$. From Lemma 1, we get

$$\lim_{n\to\infty} \left\| F_{n,a,b}^{p_n,q_n}(1;x) - 1 \right\| = 0,$$

$$\lim_{n\to\infty} \left\| F_{n,a,b}^{p_n,q_n}(t;x) - x \right\| = 0.$$

Now, we show that

$$\lim_{n\to\infty} \left\| F_{n,a,b}^{p_n,q_n}(t^2;x) - x^2 \right\| = 0.$$

From Lemma 1, we obtain

$$\max_{x\in\left[0, \frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}}\right]} |F_{n,a,b}^{p_n,q_n}(t^2;x) - x^2|$$

$$= \max_{x\in\left[0, \frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}}\right]} \left| \frac{p_n^{n-1}[n+a]_{p,q}}{[n]_{p,q}[n+b]_{p,q}} x + \frac{q_n x^2 [n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - x^2 \right|$$

$$\leq \left| \left(\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}} \right)^2 \frac{p_n^{n-1}}{[n]_{p_n,q_n}} \right| + \left| \left(\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}} \right)^2 \left(\frac{q_n[n-1]_{p_n,q_n}}{[n]_{p_n,q_n}} - 1 \right) \right|$$

$$= \left(\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}} \right)^2 \frac{2p_n^{n-1}}{[n]_{p_n,q_n}}.$$

Then, we get

$$\left\| F_{n,a,b}^{p_n,q_n}(t^2;x) - x^2 \right\| \leq \left(\frac{[n+a]_{p_n,q_n}}{[n+b]_{p_n,q_n}} \right)^2 \frac{2p_n^{n-1}}{[n]_{p_n,q_n}}.$$

Thus, we have

$$\lim_{n\to\infty} \max_{x\in\left[0, \frac{[n+a]}{[n+b]}\right]} |F_{n,a,b}^{p_n,q_n}(t^m;x) - x^m| = 0, \quad m = 0,1,2.$$

In accordance with the Bohman- Korovkin theorem [25], we obtained the desired result.

Lemma 2. *Let $k-th$ degree moment for the polynomials (9) defined by*

$$T_{n,k}^{p,q}(x) = F_{n,a,b}^{p,q}\left((t-x)^k;x\right), \quad k = 0,1,2. \tag{11}$$

Then we have $T_{n,0}^{p,q}(x)=1, \quad T_{n,1}^{p,q}(x)=0$ *and*

$$T_{n,2}^{p,q}(x)=\frac{p^{n-1}[n+a]_{p,q}}{[n]_{p,q}[n+b]_{p,q}}x+\left(\frac{q[n-1]_{p,q}}{[n]_{p,q}}-1\right)x^2. \tag{12}$$

Moreover, let the sequence $\{p_n\},\{q_n\}$ *satisfying* $0<q_n<p_n\leq 1$ *such that* $p_n\to 1, \quad q_n\to 1$ *and* $p_n^n\to\alpha, \quad q_n^n\to\beta$ *as* $n\to\infty$*, where* $0\leq\alpha,\beta<1$*. Then*

$$\lim_{n\to\infty}[n]_{p_n,q_n}F_{n,a,b}^{p_n,q_n}\left((t-x)^2;x\right)=\lambda x-\alpha x^2 \tag{13}$$

is uniformly on $\left[0,\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$*, where* $0<\lambda\leq 1$*.*

Proof. It is clear that $T_{n,0}^{p,q}(x)=1$ and $T_{n,1}^{p,q}(x)=0$ hold. From (11), we obtain

$$\begin{aligned}
T_{n,2}^{p,q}(x) &= F_{n,a,b}^{p,q}\left((t-x)^2;x\right)\\
&=\sum_{k=0}^{n}\left(\frac{[k]_{p,q}[n+a]_{p,q}}{p^{k-n}[n]_{p,q}[n+b]_{p,q}}-x\right)^2 q_{n,k,a,b}^{p,q}(x)\\
&=\sum_{k=0}^{n}\left(\frac{[k]_{p,q}[n+a]_{p,q}}{p^{2k-2n}[n][n+b]}\right)^2 q_{n,k,a,b}^{p,q}(x)\\
&-2x\sum_{k=0}^{n}\frac{[k]_{p,q}[n+a]_{p,q}}{p^{k-n}[n]_{p,q}[n+b]_{p,q}}q_{n,k,a,b}^{p,q}(x)+x^2\sum_{k=0}^{n}q_{n,k,a,b}^{p,q}(x)\\
&=\frac{p^{n-1}[n+a]_{p,q}}{[n]_{p,q}[n+b]_{p,q}}x+\frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2-2x^2+x^2\\
&=\frac{p^{n-1}[n+a]_{p,q}}{[n]_{p,q}[n+b]_{p,q}}x+\left(\frac{q[n-1]_{p,q}}{[n]_{p,q}}-1\right)x^2.
\end{aligned}$$

Using the equality (12) we have

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n}F_{n,a,b}^{p_n,q_n}\left((t-x)^2;x\right)\\
&=\lim_{n\to\infty}\left(\frac{p_n^{n-1}[n+a]_{p,q}}{[n+b]_{p,q}}x+\left(q_n[n-1]_{p_n,q_n}-[n]_{p_n,q_n}\right)x^2\right)\\
&=\lim_{n\to\infty}\left(\frac{p_n^{n-1}[n+a]_{p,q}}{[n+b]_{p,q}}x\right)+\lim_{n\to\infty}\left(q_n\frac{p_n^{n-1}-q_n^{n-1}}{p_n-q_n}-\frac{p_n^n-q_n^n}{p_n-q_n}\right)x^2\\
&=\lambda x+\lim_{n\to\infty}\frac{p_n^{n-1}(q_n-p_n)}{p_n-q_n}x^2\\
&=\lambda x-\alpha x^2.
\end{aligned}$$

Lemma 3.

$$F_{n,a,b}(f;0)=f(0) \quad and \quad F^{p,q}_{n,a,b}\left(f;\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)=f\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right). \tag{14}$$

Proof. Taking $x=0$ into equation (9), we get

$$\begin{aligned}
F^{p,q}_{n,a,b}(f;0) &= \frac{1}{p^{\frac{(n-1)n}{2}}}\left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^n f\left(\frac{[0]_{p,q}[n+a]_{p,q}}{p^{0-n}[n]_{p,q}[n+b]_{p,q}}\right)\begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} \\
&\quad \times p^{\frac{0(0-1)}{2}}x^0\prod_{s=0}^{n-1}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s0\right)+0+0+... \\
&= \frac{1}{p^{\frac{(n-1)n}{2}}}\left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^n f(0)\prod_{s=0}^{n-1}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s\right) \\
&= \frac{1}{p^{\frac{(n-1)n}{2}}}f(0)\prod_{s=0}^{n-1}p^s \\
&= f(0).
\end{aligned}$$

Similarly, taking $x=\frac{[n+a]_{p,q}}{[n+b]_{p,q}}$ into equation (9), we get

$$\begin{aligned}
F^{p,q}_{n,a,b}\left(f;\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right) &= \frac{1}{p^{\frac{(n-1)n}{2}}}\left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^n f\left(\frac{[0]_{p,q}[n+a]_{p,q}}{p^{0-n}[n]_{p,q}[n+b]_{p,q}}\right)\begin{bmatrix} n \\ 0 \end{bmatrix}_{p,q} p^{\frac{0(0-1)}{2}}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)^0 \\
&\quad \times\prod_{s=0}^{n-1}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)+... \\
&\quad +\frac{1}{p^{\frac{(n-1)n}{2}}}\left(\frac{[n+b]_{p,q}}{[n+a]_{p,q}}\right)^n f\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)\begin{bmatrix} n \\ n \end{bmatrix}_{p,q} p^{\frac{n(n-1)}{2}}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)^n \\
&\quad \times\prod_{s=0}^{-1}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)
\end{aligned}$$

in above expansion, the terms corresponding to $k=0,1,...,n-1$, becomes zero, because for $k=0$, we find $\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}-\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)$ as the first factor of each product. It is accepted $\prod_{s=0}^{-1}\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}p^s-q^s\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)=1$ so we get

$$F^{p,q}_{n,a,b}\left(f;\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)\right)=f\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right).$$

Theorem 2. *If* $f \in C\left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$, *then the following inequality holds.*

$$|F_{n,a,b}^{p,q}(f;x) - f(x)| \leq \left(1 + \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right) \omega\left(f; \sqrt{\frac{2p^{n-1}}{[n]_{p,q}}}\right) \tag{15}$$

Proof. From the well-known properties of modulus of continuity we have

$$|f(t) - f(x)| \leq \left(1 + \frac{|t-x|}{\delta_n}\right) \omega\left(f; \delta_n\right),$$

where δ_n is any sequences of positive numbers. Since the polynomials $F_{n,a,b}^{p,q}(f;x)$ also linear positive operators, we have

$$|F_{n,a,b}^{p,q}(f;x) - f(x)| \leq \left(1 + \frac{1}{\delta_n}\sqrt{F_{n,a,b}^{p,q}\left((t-x)^2; x\right)}\right) \omega\left(f; \delta_n\right).$$

Use Cauchy-Schwartz inequality and Lemma 1 , then we obtain

$$\begin{aligned} |F_{n,a,b}(f;x) - f(x)| &\leq \left(1 + \frac{1}{\delta_n}\sqrt{\left(\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right)^2 \frac{2p^{n-1}}{[n]_{p,q}}}\right) \omega\left(f; \delta_n\right) \\ &= \left(1 + \frac{1}{\delta_n}\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\sqrt{\frac{2p^{n-1}}{[n]_{p,q}}}\right) \omega\left(f; \delta_n\right). \end{aligned}$$

Put $\delta_n = \sqrt{\frac{2p^{n-1}}{[n]_{p,q}}}$, then we get inequality (15).

Theorem 3. *(Voronovskaya Type Theorem) Let the sequence* $\{p_n\}, \{q_n\}$ *satisfying* $0 < q_n < p_n \leq 1$ *such that* $p_n \to 1$, $\quad q_n \to 1$ *and* $p_n^n \to \alpha$, $\quad q_n^n \to \beta$ *as* $n \to \infty$, *where* $0 \leq \alpha, \beta < 1$. *For* $\forall f \in C^2\left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$, *we have*

$$\lim_{n\to\infty} [n]_{p_n,q_n}\left(F_{n,a,b}^{p_n,q_n}(f;x) - f(x)\right) = \frac{x(\lambda - \alpha x)}{[2]_{p,q}} D_{p,q}^2\left(f(x)\right), \quad 0 < \lambda \leq 1. \tag{16}$$

Proof. Let $f \in C^2\left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$, that is $f, \mathcal{D}_{p,q}(f), \mathcal{D}_{p,q}^2(f) \in C\left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$. Define

$$\psi(t,x) = \begin{cases} \frac{f(t) - f(x) - (t-x)\mathcal{D}_{p,q}(f) - \frac{1}{[2]_{p,q}}(t-x)_{p,q}^2 \mathcal{D}_{p,q}^2(f)}{(t-x)_{p,q}^2}, & t \neq x \\ 0, & t = x. \end{cases}$$

Then, it is clear that $\psi(x,x) = 0$ and $\psi(.,x) \in C\left[0, \frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$. Hence, from Taylor's theorem we have

$$f(t) = f(x) + (t-x)\mathcal{D}_{p,q}(f) + \frac{1}{[2]_{p,q}}(t-x)_{p,q}^2 \mathcal{D}_{p,q}^2(f) + (y-x)_q^2 \psi(y,x).$$

Form Lemma 2,

$$[n]_{p_n,q_n}\left(F^{p_n,q_n}_{n,a,b}\,(f;x)-f(x)\right)=[n]_{p_n,q_n}F^{p_n,q_n}_{n,a,b}\,((t-x);x)\,\mathcal{D}_{p_n,q_n}(f)$$

$$+\frac{[n]_{p_n,q_n}}{[2]_{p_n,q_n}}F^{p_n,q_n}_{n,a,b}\,\left((t-x)^2;x\right)\mathcal{D}^2_{p_n,q_n}(f)$$

$$+[n]_{p_n,q_n}F^{p_n,q_n}_{n,a,b}\,\left((t-x)^2\psi(t,x);x\right). \tag{17}$$

If we apply the Cauchy-Schwartz inequality for the last term on the right hand side of (17), we conclude that

$$[n]_{p_n,q_n}F^{p_n,q_n}_{n,a,b}\,\left((t-x)^2\psi(t,x);x\right)\leq\left([n]^2_{p_n,q_n}F^{p_n,q_n}_{n,a,b}\,\left((t-x)^4;x\right)\right)^{\frac{1}{2}}$$

$$\times\left(F^{p_n,q_n}_{n,a,b}\,\left(\psi^2(t,x);x\right)\right)^{\frac{1}{2}}. \tag{18}$$

Let $\eta(t,x):=\psi^2(t,x)$. So, we get $\eta(x,x)=0$ and $\eta(.,x)\in C^2\left[0,\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$. From Theorem 1, we have

$$\lim_{n\to\infty}F^{p_n,q_n}_{n,a,b}\,\left(\psi^2(t,x);x\right)=\lim_{n\to\infty}F^{p_n,q_n}_{n,a,b}\,(\eta(t,x);x)=\eta(x,x)=0\,. \tag{19}$$

Then taking limit as $n\to\infty$ in (17) and using (18), (19) and Lemma 2

$$\lim_{n\to\infty}[n]_{p_n,q_n}\left(F^{p_n,q_n}_{n,a,b}\,(f;x)-f(x)\right)=\frac{\lambda x-\alpha x^2}{[2]_{p,q}}D^2_{p,q}(f)$$

uniformly with respect to $x\in\left[0,\frac{[n+a]_{p,q}}{[n+b]_{p,q}}\right]$.

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