Lie algebras with $BCL$ algebras

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Abstract. The subject matter of this work is hoping for a new relationship between the Lie algebras and the algebra of logic, which will constitute an important part of our study of “pure” algebra theory. $BCL$ algebras as a class of logical algebras can be generated by a Lie algebra. The opposite is also true that when special conditions occur. The aim of this paper is to prove several theorems on Lie algebras with $BCL$ algebras. I introduce the notion of a “pseudo-association” which I propose as the adjoint notion of $BCL$ algebra in the abelian group.

2010 Mathematics Subject Classifications: 17B60, 03G25

Key Words and Phrases: Lie algebras, $BCL$ algebras, abelian Lie algebras, pseudo-association $BCL$ algebra

1. Introduction

Lie algebras comprise a significant part of Lie group theory (see [1]) and are being vibrantly studied. On the other hand, Lie algebras and their representations are used extensively in physics, notably in quantum mechanics and particle physics. But it is significant that our results show that the Lie algebra and logical algebra are closely linked. Sure, $BCL$ algebras as a class of logical algebras were introduced by Liu in 2011 [2]. The last results was discovered and developed in [3-15]. From set theory perspective, $BCL$ algebras are the algebraic formulations of the set difference together with its properties.

In the paper, I just want to prove that the connectivity theorems but that I have suspected for a long time, which is the relationship between the Lie algebras and the $BCL$ algebras. More importantly, we developed the theory that Lie algebras do have a preferred direction that causes us to the study of logic issues so we can capture new method. Meanwhile, let the theory of $BCL$ algebras becomes strong enough.

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2. Basic definitions

In this section, we list two definitions from the literature that will be used in the sequel.

**Definition 2.1** A Lie algebra over a field $k$ is a vector space $g$ over $k$ together with a $k$-bilinear map

$$\{\ , \ : g \times g \to g$$

(called the bracket) such that

(\text{Lie 1}) \quad [x, x] = 0 \text{ for all } x \in g

(\text{Lie 2}) \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \text{ for all } x, y, z \in g.

A homomorphism of Lie algebras is a $k$-linear map $\alpha: g \to g'$ such that

$$\alpha([x, y]) = [\alpha(x), \alpha(y)] \text{ for all } x, y, z \in g.$$

Condition (\text{Lie 2}) is called the Jacobi identity. Note that (\text{Lie 1}) applied to $[x + y, x + y]$ shows that the Lie bracket is skew-symmetric.

$$[x, y] = -[y, x] \text{ for all } x, y, z \in g.$$

**Definition 2.2** ([2], Definition 2.1.) A $BCL$ algebra is a triple $(A; \rightarrow, 0)$, where $A$ is a nonempty set, $\rightarrow$ is a binary operation on $A$, the following three axioms hold for any $x$, $y$, $z \in A$.

(\text{BCL 1}) \quad x \rightarrow x = 0.

(\text{BCL 2}) \quad x \rightarrow y = 0 \text{ and } y \rightarrow x = 0 \text{ imply } x = y.

(\text{BCL 3}) \quad (((x \rightarrow y) \rightarrow z) \rightarrow ((x \rightarrow z) \rightarrow y)) \rightarrow ((z \rightarrow y) \rightarrow x) = 0.

3. Results

**Theorem 3.1** Let $L$ be Lie algebras. Define

$$x \rightarrow y = [x, y] - [y, x],$$

$$(x \rightarrow y) \rightarrow z = [[x, y], z],$$

and

$$0 \rightarrow x = 0 = [x, x].$$

Then $L$ be $BCL$ algebras.

**Proof.** Let $x, y, z \in L$. Then
(1) \[ x \to x = [x, x] - [x, x] = 0. \]

(2) \[ [x, z] = 0 = x \to y \text{ and } [y, y] = 0 = y \to x \text{ imply } x = y. \]

(3) \[ [[[x, y], z], [[x, z], y]], [[z, y], x]] = 0. \]

Clearly, proving (1) and (2).

Now we need to prove (3), we define \[ [x, y] = x + y. \]

Then

\[ [[z, y], x] = [x, [z, y]] \]
\[ \subseteq [z, [y, x]] + [y, [x, z]] \]
\[ = [[x, y], z] + [[x, z], y] \]
\[ \subseteq [[x, y], z], [[x, z], y]], \]

and (3) is proved. We see that \( L \) be \( BCL \) algebras.

**Theorem 3.2** Let \( x, y, z \in P \) be \( BCL \) algebras. Then \( P \) is abelian Lie algebra iff \( x = y = z. \)

**Proof.** Assume that \( P \) is abelian Lie algebra, since \( x, y, z \in P \), we have \[ [x, y] = 0 = [y, z]. \]

Therefore, \( x = y = z. \)

Conversely, assume \( x = y = z. \) To prove that this algebra is a \( BCL \) algebra. Let \( x, y, z \in P. \) By Theorem 2.1. Then

(4) \[ [x, x] = 0 = x \to x. \]

(5) \[ [x, x] = 0 = x \to y \text{ and } [y, y] = 0 = y \to x \text{ imply } x = y. \]

(6) \[ [0, [0, x]] = [[0, x], 0] = (0 \to x) \to 0 = 0 \to 0 = 0. \]

This completes the proof.

**Definition 3.1** Let \( (G, +) \) be an abelian, \( (G; -, 0) \) be an adjoint \( BCL \) algebras and
(G; →, 0) be a pseudo-association BCL algebras. Suppose the following conditions hold:

(GPA 1) \( x - (0 - y) = x + y. \)

(GPA 2) \( x - y = x \rightarrow y. \)

Then adjoint group of \((G; -, 0)\) is abelian \((G, +)\), and adjoint BCL algebras of abelian \((G, +)\) is \((G; \rightarrow, 0)\).

**Theorem 3.3** Let \( P \) be a pseudo-association BCL algebra. The bracket

\[ [x, y] = x \rightarrow y - (y \rightarrow x), \text{ for all } x, y \in P. \]

Then \( P \) be a Lie algebras about the bracket \([, ]\) and we use notation \( P_L \), for the Lie algebra is generated by the pseudo-association BCL algebra.

**Proof.** By definition of the bracket, \([x, x] = 0\) trivially hold. To prove bilinear, sine \( x_1, x_2, y \in P_L \), and \( \lambda_1, \lambda_2 \in P_L \), we have

\[
\begin{align*}
[\lambda_1 x_1 + \lambda_2 x_2, y] & = ((\lambda_1 x_1 + \lambda_2 x_2) \rightarrow y) - (y \rightarrow (\lambda_1 x_1 + \lambda_2 x_2)) \\
& = (\lambda_1(x_1 \rightarrow y) + \lambda_2(x_2 \rightarrow y)) - (\lambda_1(y \rightarrow x_1) + \lambda_2(y \rightarrow x_2)) \\
& = \lambda_1[x_1, y] + \lambda_2[x_2, y]
\end{align*}
\]

To prove Jacobi identity, sine \( x, y, z \in P_L \), we have

\[
\begin{align*}
[\lambda_1 x_1 + \lambda_2 x_2, y] & = ((\lambda_1 x_1 + \lambda_2 x_2) \rightarrow y) - (y \rightarrow (\lambda_1 x_1 + \lambda_2 x_2)) \\
& = (\lambda_1(x_1 \rightarrow y) + \lambda_2(x_2 \rightarrow y)) - (\lambda_1(y \rightarrow x_1) + \lambda_2(y \rightarrow x_2)) \\
& = \lambda_1[x_1, y] + \lambda_2[x_2, y]
\end{align*}
\]

Therefore, the sum of three brackets, i.e., (7), (8) and (9) satisfying the Jacobi identity

\[
[\lambda_1 x_1 + \lambda_2 x_2, y] + [\lambda_2 x_2, y] + [\lambda_1 x_1, y] = 0.
\]
References


