EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 2, 2018, 390-399
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

On the elementary solution for the partial differential operator $\circledcirc_c^k$ related to the wave equation

Sudprathai Bupasiri

Department of Mathematics, Sakon Nakhon Rajabhat University, Sakon Nakhon 47000, Thailand

Abstract. In this article, we study an elementary solution of the operator $\circledcirc_c^k$, iterated k-times and is defined by

$$\circledcirc_c^k = \left(\left(\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + m^2 \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k$$

where $p + q = n$, k is a nonnegative integer, c is a positive real number, m is a nonnegative real number and n is the dimension of $\mathbb{R}^n$. In this work we study an elementary solution of the operator $\circledcirc_c^k$. After that, we apply such an elementary solution to solve the solution of the equation $\circledcirc_c^k u(x) = f(x)$, where f is generalized function and $u(x)$ is unknown function for $x \in \mathbb{R}^n$.

2010 Mathematics Subject Classifications: 46F10
Key Words and Phrases: Elementary solution, Dirac-delta distribution, Temper distribution

1. Introduction

Trione [10] has showed that the generalized function $R^H_{2k,1}(x)$ defined by (13) is the unique elementary solution of the operator $\Box_1^k$, that is $\Box_1^k R^H_{2k,1}(x) = \delta$ where $x \in \mathbb{R}^n$, with n-dimensional Euclidean space. Also, Tellez ([7], p.147-149) has proved that $R^H_{2k,1}(x)$ exists only if n is an odd with p odd and q even, or only n is an even with p odd and q odd. Later, Bupasiri [9] has showed that the solution of the convolution form $u(x) = (-1)^k R^e_{2k,c}(x) * R^H_{2k,c}(x)$ is an elementary solution of the $\Diamond_c^k u(x) = \delta$, where the operator $\Diamond_c^k$ is defined by

$$\Diamond_c^k = \left(\frac{1}{c^4} \left(\sum_{r=1}^{p} \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k, \tag{1}$$

where $p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, c is a positive real number and k is a nonnegative integer. Otherwise, the operator $\Diamond_c^k$ can be expressed in the form

Email address: sudprathai@gmail.com

$\diamondsuit_c^k = \square_c^k \triangle_c^k = \triangle_c^k \square_c^k$, where $\square_c^k$ is the operator related to the ultra-hyperbolic operator iterated k-times, defined by

$$\square_c^k = \left(\frac{1}{c^2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right) - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{2}$$

and $\triangle_c^k$ is the operator related to the Laplace operator iterate k-times, defined by

$$\triangle_c^k = \left(\frac{1}{c^2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots \frac{\partial^2}{\partial x_p^2} \right) + \frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right)^k. \tag{3}$$

Next, Tellez [8] has studied the convolution product of $W_\alpha(u,m) * W_\beta(u,m)$. Now in this paper, the operator $\circledcirc_c^k$ can be expressed in the form

$$\begin{aligned} \circledcirc_c^k &= \left(\left(\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + m^2 \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k \\ &= \left(\left(\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) + m^2 \right)^k \left(\frac{1}{c^2} \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} + m^2 \right)^k. \end{aligned} \tag{4}$$

Thus equation (4) can be written as

$$\circledcirc_c^k = \left(\square_c + m^2\right)^k \left(\triangle_c + m^2\right)^k = \left(\triangle_c + m^2\right)^k \left(\square_c + m^2\right)^k, \tag{5}$$

where $\left(\triangle_c + m^2\right)^k$ is the operator related to the Helmholtz operator iterated k-times which is denoted by

$$\left(\triangle_c + m^2\right)^k = \left(\frac{1}{c^2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots \frac{\partial^2}{\partial x_p^2} \right) + \left(\frac{\partial^2}{\partial x_{p+1}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right) + m^2 \right)^k \tag{6}$$

and $\left(\square_c + m^2\right)^k$ is the operator related to the Klein-Gordon operator iterated k-times which is denoted by

$$\left(\square_c + m^2\right)^k = \left(\frac{1}{c^2} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} \right) - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2} \right) + m^2 \right)^k, \tag{7}$$

$p + q = n$ and from (4) with $q = 0$, $c = 1$ and $k = 1$, we obtain

$$\circledcirc_1 = \left(\triangle_p + m^2\right)^2 \tag{8}$$

where

$$\left(\triangle_p + m^2\right) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} + m^2 \right). \tag{9}$$

By putting $p = 1, m = 0$, $c = 1$ and $x_1 = t$ (time) in (7) then we obtain the wave operator

$$\Box_1 = \frac{\partial^2}{\partial x_t^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} \tag{10}$$

and from (8) with $q = 0, m = 0$, $c = 1$ and $k = 1$, we obtain Laplace operator iterated 2-times of p-dimension

$$\circledcirc_1 = \triangle_p^2. \tag{11}$$

In this paper, we study an elementary solution for the operator $\circledcirc_c^k$, that is

$$\circledcirc_c^k G(x) = \delta,$$

where $G(x)$ is an elementary solution , δ is the Dirac - delta distribution, k is a nonnegative integer, c is a positive real number and m is a nonnegative real number.

We then also apply such an elementary solution to solve the solution of the equation $\circledcirc_c^k u(x) = f(x)$, where $f(x)$ is a given generalized function and $u(x)$ is an unknown function for $x \in \mathbb{R}^n$.

2. Preliminaries

Definition 1. *Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n - dimensional space $\mathbb{R}^n$,*

$$u = c^2 \left(x_1^2 + x_2^2 + \cdots + x_p^2\right) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{12}$$

where c is a positive real number, $p + q = n$. Define $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ which designates the interior of the forward cone and $\overline{\Gamma}_+$ designates its closure and the following functions introduce by Nozaki ([12], p.72) that

$$R^H_{\alpha,c}(x) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{13}$$

$R^H_{\alpha,1}(x)$ is called the ultra-hyperbolic kernel of Marcel Riesz. Here α is a complex parameter and n the dimension of the space. The constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{2+\alpha-n}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \Gamma(\alpha)}{\Gamma\left(\frac{2+\alpha-p}{2}\right) \Gamma\left(\frac{p-\alpha}{2}\right)} \tag{14}$$

and p is the number of positive terms of

$$u = c^2 \left(x_1^2 + x_2^2 + \cdots + x_p^2\right) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \qquad p + q = n$$

and let supp $R^H_{\alpha,c}(x) \subset \overline{\Gamma}_+$. Now $R^H_{\alpha,c}(x)$ is an ordinary function if Re $(\alpha, c) \geq n$ and is a distribution of α if Re $(\alpha, c) < n$.

Now, if $p = 1$ *then* (13) *reduces to the function* $M_{\alpha,c}(u)$ *say, and defined by*

$$M_{\alpha,c}(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & \text{if } x \in \Gamma_+ \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{15}$$

where $u = c^2x_1^2 - x_2^2 - \cdots - x_n^2$ *and* $H_n(\alpha) = \pi^{\frac{(n-1)}{2}} 2^{\alpha-1}\Gamma(\frac{\alpha-n+2}{2})$. *The function* $M_{\alpha,1}(u)$ *is called the hyperbolic kernel of Marcel Riesz.*

Definition 2. *Let* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *and*

$$v = c^2\left(x_1^2 + x_2^2 + \cdots + x_p^2\right) + x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2, \qquad p + q = n. \tag{16}$$

For any complex number β , *we define the function*

$$R^e_{\beta,c}(v) = 2^{-\beta}\pi^{-n/2}\Gamma\left(\frac{n-\beta}{2}\right)\frac{v^{(\beta-n)/2}}{\Gamma(\beta/2)}. \tag{17}$$

The function $R^e_{\beta,1}(v)$ *is called the elliptic kernel of Marcel Riesz. It is an ordinary function if* $Re(\beta, c) \geq n$ *and a distribution of* β *if* $Re(\beta, c) < n$.

Lemma 1. *Given the equation* $\triangle_c^k u(x) = \delta$ *for* $x \in \mathbb{R}^n$, *where* $\triangle_c^k$ *is the operator related to the Laplace operator iterated* k*-times defined by* (3). *Then* $u(x) = (-1)^k R^e_{2k,c}(v)$ *is an elementary solution of the operator* $\triangle_c^k$ *,with* $\beta = 2k$.

Proof. See [2].

Lemma 2. *If* $\Box_c^k u(x) = \delta$ *for* $x \in \Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0$ *and* $u > 0\}$, *where* $\Box_c^k$ *is the operator related to the ultra-hyperbolic operator iterated* k*-times defined by* (2). *Then* $u(x) = R^H_{2k,c}(u)$ *is the unique elementary solution of the operator* $\Box_c^k$, *with* $\alpha = 2k$.

Proof. See [10].

Lemma 3. *Given the equation* $\left(\Box_c + m^2\right)^k u(x) = \delta$ *for* $x \in \mathbb{R}^n$, *where* $\left(\Box_c + m^2\right)^k$ *is the operator related to the Klein-Gordon operator iterated* k*-times defined by equation* (7), δ *is the Dirac-delta distribution,* k *is a nonnegative integer and* m *is a nonnegative real number, then* $u(x) = W_{2k,c}(u, m)$ *is an elementary solution of the operator* $\left(\Box_c + m^2\right)^k$, *where*

$$W_{2k,c}(u, m) = \sum_{r=0}^{\infty} \binom{-k}{r} m^{2r} R^H_{2k+2r,c}(u), \tag{18}$$

$R^H_{2k,c}(u)$ *is defined by* (13).

Proof. See [6].

Lemma 4. *Let $\Box_c$ be the operator related to the ultra-hyperbolic operator, defined by* (2) *and δ is the Dirac delta distribution for $x \in \mathbb{R}^n$, then*

$$\left(\Box_c + m^2\right)^k \delta = W_{-2k,c}(u, m),$$

where $W_{-2k,c}(u, m)$ is the inverse of $W_{2k,c}(u, m)$ in the convolution algebra.

Proof. Let

$$V(x) = \left(\Box_c + m^2\right)^k \delta,$$

convolving both sides by $W_{2k,c}(u, m)$, then

$$\begin{aligned} W_{2k,c}(u, m) * V(x) &= W_{2k,c}(u, m) * \left(\Box_c + m^2\right)^k \delta \\ &= \left(\Box_c + m^2\right)^k W_{2k,c}(u, m) * \delta \\ &= \delta. \end{aligned} \tag{19}$$

Since $W_{2k,c}(u, m)$ is lie in S', where S' is a space of tempered distribution, choose $S' \subset D'_R$, where D'_R is the right-side distribution which is a subspace of D' of distribution. Thus $W_{2k,c}(u, m) \in D'_R$, it follow that $W_{2k,c}(u, m)$ is an element of convolution algebra, thus by ([1], p.150-151), we have that the equation (19) has a unique solution

$$V(x) = W_{-2k,c}(u, m) * \delta = W_{-2k,c}(u, m). \tag{20}$$

That complete the proof.

Lemma 5. *Given the equation $\left(\triangle_c + m^2\right)^k u(x) = \delta$ for $x \in \mathbb{R}^n$, where $\left(\triangle_c + m^2\right)^k$ is the operator related to the Helmholtz operator iterated k-times defined by equation* (6)*, δ is the Dirac-delta distribution, k is a nonnegative integer, then $u(x) = Y_{2k,c}(v, m)$ is an elementary solution of the operator $\left(\triangle_c + m^2\right)^k$, where*

$$Y_{2k,c}(v, m) = \sum_{r=0}^{\infty} \binom{-k}{r} m^{2r} (-1)^{k+r} R^e_{2k+2r,c}(v), \tag{21}$$

$R^e_{2k,c}(v)$ is defined by (17).

Proof. See [6].

Lemma 6. *Let $\triangle_c$ be the operator related to the Laplace operator, defined by* (3) *and δ is the Dirac delta distribution for $x \in \mathbb{R}^n$, then*

$$\left(\triangle_c + m^2\right)^k \delta = Y_{-2k,c}(v, m),$$

where $Y_{-2k,c}(v, m)$ is the inverse of $Y_{2k,c}(v, m)$ in the convolution algebra.

Proof. The proof of this lemma similar lemma 4.

Lemma 7. *The convolution* $W_{2k,c}(u,m) * Y_{2k,c}(v,m)$ *exists and is a tempered distribution where* $W_{2k,c}(u,m)$ *and* $Y_{2k,c}(v,m)$ *be defined by* (18) *and* (21)*, respectively.*

Proof. From (18) and (21), we have

$$\begin{aligned} W_{2k,c}(u,m) * Y_{2k,c}(v,m) &= \left(\sum_{r=0}^{\infty}\binom{-k}{r}m^{2r}R^{H}_{2k+2r,c}(u)\right) \\ &\quad * \left(\sum_{r=0}^{\infty}\binom{-k}{r}m^{2r}(-1)^{k+r}R^{e}_{2k+2r,c}(v)\right) \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{-k}{r}\binom{-k}{s}m^{2r+2s}(-1)^{k+r}R^{e}_{2k+2r,c}(v) * R^{H}_{2k+2s,c}(u). \end{aligned}$$

Since the function $R^{e}_{2k+2r,c}(v)$ and $R^{H}_{2k+2s,c}(u)$ are tempered distributions, see([3], p.34, [5], p.302 and [4], p.97) and the convolution of functions

$$(-1)^{k+r}R^{H}_{2k+2r,c}(u) * R^{e}_{2k+2s,c}(v)$$

exists and is also a tempered distribution, see ([11], p.152). Thus, $W_{2k,c}(u,m) * Y_{2k,c}(v,m)$ exists and also is a tempered distribution.

3. Main results

Theorem 1. *Given the equation*

$$\circledcirc_c^k G(x) = \delta \tag{22}$$

for $x \in \mathbb{R}^n$*, where* $\circledcirc_c^k$ *is the operator related to the Helmhotz operator and Klein-Gordon operator iterated* k*-times defined by* (4)*, then*

$$G(x) = W_{2k,c}(u,m) * Y_{2k,c}(v,m) \tag{23}$$

is an elementary solution of (22)*, where* $W_{2k,c}(u,m)$ *and* $Y_{2k,c}(v,m)$ *are defined by* (18) *and* (21)*, respectively,* k *is a nonnegative integer and* m *is a nonnegative real number. Moreover, from* (23) *we obtain*

$$W_{-2k,c}(u,m) * G(x) = Y_{2k,c}(v,m) \tag{24}$$

as the elementary solution of the operator $(\triangle_c + m^2)^k$ *related to the Helmholtz operator iterated* k*-times defined by* (6) *and in particular, for* $q = 0$ *and* $c = 1$ *then* $\circledcirc_c^k$ *reduces to the Helmhotz operator* $\left(\triangle_p + m^2\right)^{2k}$ *of* p*-dimension iterated* $2k$*-times and is defined by* (9)*, where*

$$\triangle_p = \frac{1}{c^2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right),$$

thus (22) *becomes*

$$\left(\triangle_p + m^2\right)^{2k} G(x) = \delta, \tag{25}$$

we obtain

$$G(x) = Y_{4k,1}(v,m) \tag{26}$$

is an elementary solution of (25) *and from* (23). *Moreover,*

$$Y_{-2k,c}(u,m) * G(x) = W_{2k,c}(u,m) \tag{27}$$

is an elementary solution of operator related to the Klein-Gordon operator. In particular, we obtain

$$(-1)^k R^e_{-2,1}(v) * G(x) = M_{2,1}(u)$$

is an elementary solution of the wave operator defined by (10) *where* $u = t^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2$. *Also, for* $m = 0$, $q = 0$ *and* $c = 1$ *then* (25) *becomes*

$$\triangle_p^{2k} G(x) = \delta \tag{28}$$

where $\triangle_p^{2k}$ *is the Laplacian of* p*-dimension iterated* $2k$*-times. We have*

$$G(x) = R^e_{4k,1}(v)$$

is an elementary solution of (28) *where*

$$v = c^2\left(x_1^2 + x_2^2 + \cdots + x_p^2\right).$$

Proof. From (5) and (22) we have

$$\circledcirc_c^k G(x) = \left(\left(\Box_c + m^2\right)^k \left(\triangle_c + m^2\right)^k\right) G(x) = \delta.$$

Convolving both sides of the above equation by the convolution $W_{2k,c}(u,m) * Y_{2k,c}(v,m)$ and the properties of convolution with derivatives, we obtain

$$\begin{aligned}&\left(\Box_c + m^2\right)^k W_{2k,c}(u,m) * \left(\triangle_c + m^2\right)^k Y_{2k,c}(v,m) * G(x)\\&= W_{2k,c}(u,m) * Y_{2k,c}(v,m) * \delta.\end{aligned} \tag{29}$$

Thus

$$G(x) = \delta * \delta * G(x) = W_{2k,c}(u,m) * Y_{2k,c}(v,m) \tag{30}$$

by Lemma 3 and 5. Now from (23) and by Lemma 3 and Lemma 4 and properties of inverses in the convolution algebra, we obtain

$$W_{-2k,c}(u,m) * G(x) = \delta * Y_{2k,c}(v,m) = Y_{2k,c}(v,m)$$

is an elementary solution of operator related to the Helmhotz operator iterated k-times defined by (6). In particular, for $q = 0$ and $c = 1$ then (22) becomes

$$\left(\triangle_p + m^2\right)^{2k} G(x) = \delta \tag{31}$$

where $\left(\triangle_p + m^2\right)^{2k}$ is the Helmholtz operator of p-dimension, iterated $2k$-times and is defined by (9). By Lemma 5, we have

$$G(x) = Y_{4k,1}(v, m) \tag{32}$$

is an elementary solution of (31). Moreover, from (23) and by Lemma 6 and Lemma 5 and properties of inverses in the convolution algebra, we obtain

$$Y_{-2k,c}(u, m) * G(x) = W_{2k,c}(u, m) * \delta = W_{2k,c}(u, m)$$

is an elementary solution of operator related to the Klein-Gordon operator. In particular, by putting $p = 1, q = n-1, k = 1, x_1 = t$, $c = 1$ and $m = 0$ in (23) and (27), $W_{2,1}(u, m = 0) = R^H_{2,1}(u)$ reduces to $M_{2,1}(u)$ where $M_{2,1}(u)$ is defined by (15) with $\alpha = 2$. Thus we obtain

$$(-1)^k R^e_{-2,1}(v) * G(x) = M_{2,1}(u)$$

is an elementary solution of the wave operator defined by (10) where $u = t^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2$. Also, for $m = 0$, $c = 1$ and $q = 0$ then (25) becomes

$$\triangle_p^{2k} G(x) = \delta \tag{33}$$

where $\triangle_p^{2k}$ is the Laplacian of p-dimension iterated $2k$-times. By Lemma 1, we have

$$G(x) = (-1)^{2k} R^e_{4k,1}(v) = R^e_{4k,1}(v)$$

is an elementary solution of (33) where

$$v = c^2(x_1^2 + x_2^2 + \cdots + x_p^2).$$

On the other hand, we can also find $G(x)$ from (23), since $q = 0$, $c = 1$ and $m = 0$, we have $W_{2k,1}(u, m = 0) = R^H_{2k,1}(u)$ reduces to $(-1)^k R^e_{2k,1}(v)$, where $v = c^2(x_1^2 + x_2^2 + \cdots + x_p^2)$. Thus, by (23) for $q = 0$, $c = 1$ and $m = 0$, we obtain

$$\begin{aligned} G(x) &= (-1)^k R^e_{2k,1}(v) * (-1)^k R^e_{2k,1}(v) \\ &= (-1)^{2k} R^e_{2k+2k,1}(v) \\ &= R^e_{4k,1}(v) \quad \text{by W.F. Donoghue ([11],p 158).} \end{aligned}$$

That complete the proofs.

Theorem 2. *Given the equation*

$$\circledcirc_c^k u(x) = f(x), \tag{34}$$

where f is a given generalized function and $u(x)$ is an unknown function, we obtain

$$u(x) = G(x) * f(x)$$

is a solution of the equation (34)*, where $G(x)$ is an elementary solution for $\circledcirc_c^k$ operator.*

Proof. Convolving both sides of (34) by $G(x)$, where $G(x)$ is an elementary solution of $\circledcirc_c^k$ in Theorem 1, we obtain

$$G(x) * \circledcirc_c^k u(x) = G(x) * f(x)$$

or,

$$\circledcirc_c^k G(x) * u(x) = G(x) * f(x)$$

applying the Theorem 1 , we have

$$\delta * u(x) = G(x) * f(x).$$

Therefore,

$$u(x) = G(x) * f(x).$$

Acknowledgements

The author would like to thank the referee for his suggestions which enhanced the presentation of the paper. The author was supported by Sakon Nakhon Rajabhat University .

References

[1] A. H. Zemanian, *Distribution theory and transform analysis*, New York, McGraw-Hill, 1964.

[2] A. Kananthai, *On the solutions of the n-dimensional diamond operator*, Appl. Math. Comput. **88** (1997), 27–37.

[3] A. Kananthai, *On the convolution equation related to the diamond kernel of Marcel Riesz*, J. Comp. Appl. Math. **100** (1998), 33–39.

[4] A. Kananthai, *On the convolution of the diamond kernel of Marcel Riesz*, Appl. Math. Comput. **114** (2000), 95–101.

[5] A. Kananthai, *On the convolution equation related to the N-dimensional ultra-hyperbolic operator*, J. Comp. Appl. Math. **115** (2000), 301–308.

[6] J. Tariboon, and A. Kananthai, *On the Green function of the $(\oplus + m^2)$ operator*, Integral Transform and Special Functions **18** (2007), 297–304.

[7] M. A. Tellez, *The distributional hankel transform of Marcel Riesz's ultra-hyperbolic kernel*, Studies in Applied Mathematics **93** (1994) , 133–162.

[8] M. A. Tellez, *The convolution product of $W_\alpha(u,m) * W_\beta(u,m)$*, Mathematic **38** (195-196) ,105–111.

[9] S. Bupasiri, *On the solution of the n-dimentional operator related to the diamond operator*, FJMS. **45** (2010), 69–80.

[10] S. E. Trione, *On Marcel Riesz's ultra-hyperbolic kernel*, Trabajos de Mathematica **116** (1987).

[11] W. F. Donoghue, *Distribution and Fourier transform*, New York, Academic Press, 1969.

[12] Y. Nozaki, *On Reimann-Liouvlle integral of ultra-hyperbolic type*, Kodai Mathemaical Seminar Report **6** (1964), 69–87.