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Soft Uni-Abel-Grassmann's Groups

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Abstract. In this paper, we define soft union AG-group (abbreviated as soft uni-AG-group). We also define e-set and α-inclusion of soft uni-AG-groups, normal soft uni-AG-subgroups, conjugate of soft uni-AG-groups and commutators of AG-groups. We investigate various properties of these notions and provide a variety of relevant examples that are produced by a computer package GAP to illustrate these notations.

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1. Introduction

In 1999, soft set theory was proposed by Molodtsov [1] as an alternative approach to fuzzy set and intuitionistic fuzzy set theories. This study of Molodtsov provided a general skeleton to researchers that naturally requires a study on algebraic structures. After that, Maji et al. [2] defined set theoretical operations of soft sets. In 2010, Çağman and Enginoğlu [3] redefined soft set operations in decision making problems. Ali et al. [4], Sezgin and Atagün [5] studied some new operations on soft sets. The first study on algebraic structures of soft sets was made by Aktaş and Çağman [6] in 2007. They defined concept of soft groups. After Aktaş and Çağman's study, studies related to algebraic structures of soft sets increased rapidly. In 2010, Çağman et al. [7] defined concept of soft int-groups with a similar approach to fuzzy group definition of Rosenfeld [8], and obtained some properties of soft int-groups. Kaygısız [9] derived some new results on soft int-groups. Sezgin [10] made some corrections for some problematic case related to soft groups defined by Aktaş and Çağman, and they defined concepts of normalistic soft group and normalistic soft group homomorphism. Sezgin et al. [11] gave definitions of

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soft uni-groups and uni-soft normal subgroup of a group, and investigated their related properties especially with respect to anti-image, $\alpha-$inclusion of a soft set. Sezgin [12] introduced the concept of soft intersection LA-semigroups (Abel-Grassman's groupoids) and studied various ideals in LA-semigroups such as left (right) ideals, bi-ideals, interior ideals and quasi ideals by defining soft intersection product operations. The concept of soft intersection Abel-Grassmann's groups was defined by Ullah et al. [13] and related properties were investigated. There are many studies on algebraic structures of soft sets, some of them are as in following [14–18].

In this study, we introduce the notion of soft uni-AG-groups, and define some new concepts related to soft uni-AG-groups such as e-set and α-inclusion of soft uni-AG-groups, normal soft uni-AG-subgroups, conjugate of soft uni-AG-groups and commutators of AG-groups. We support our definitions with examples to be more understandable. Furthermore, we obtain interrelations of these concepts.

Definition 1. *[1] Let U be the universal set, E be the set of parameters and $P(U)$ be the power set of U. Then a soft set, A is a set of ordered pairs*

$$A = \{(\varepsilon, f_A(\varepsilon)) : \varepsilon \in E\},$$

where f_A is a set valued function from E to $P(U)$ i.e. $f_A : E \to P(U)$. Note that if $f_A(\varepsilon) = \emptyset$, where $\varepsilon \in E$. Then $(\varepsilon, f_A(\varepsilon))$ is not appeared in the set A. The set of all soft sets over U is denoted by $S(U)$.

Definition 2. *[3] Let $A, B \in S(U)$. Then*

1. *If $f_A(\varepsilon) = \emptyset$ for all $\varepsilon \in E$, A is said to be a null soft set, denoted by ϕ.*
2. *If $f_A(\varepsilon) = U$ for all $\varepsilon \in E$, A is said to be absolute soft set, denoted by $\hat{U}$.*
3. *A is soft subset of B, denoted by $A\tilde{\subseteq}B$, if $f_A(\varepsilon) \subseteq f_B(\varepsilon)$ for all $\varepsilon \in E$.*
4. *$A\tilde{=}B$, if $A\tilde{\subseteq}B$ and $B\tilde{\subseteq}A$.*
5. *Soft union of A and B, denoted by $A\tilde{\cup}B$, is a soft set over U and defined by*

$$\begin{aligned} A\tilde{\cup}B &= \{(\varepsilon, (f_A\tilde{\cup}f_B)(\varepsilon)) : \varepsilon \in E\} \\ &= \{(\varepsilon, (f_A(\varepsilon) \cup f_B(\varepsilon))) : \varepsilon \in E\}. \end{aligned}$$

6. *Soft intersection of A and B, denoted by $A\tilde{\cap}B$, is a soft set over U and defined by*

$$\begin{aligned} A\tilde{\cap}B &= \{(\varepsilon, (f_A\tilde{\cap}f_B)(\varepsilon)) : \varepsilon \in E\} \\ &= \{(\varepsilon, (f_A(\varepsilon) \cap f_B(\varepsilon))) : \varepsilon \in E\}. \end{aligned}$$

Definition 3. *[3] Let $A, B \in S(U)$. Then, AND and OR operators of A and B is represented by $A \wedge B$ and $A \vee B$ respectively, defined by*

$$A \wedge B = \left\{ \left((\varepsilon, \varepsilon'), f_{A\wedge B}(\varepsilon, \varepsilon') \right) : \varepsilon, \varepsilon' \in E \right\} \\ = \left\{ \left((\varepsilon, \varepsilon'), f_A(\varepsilon) \cap f_B(\varepsilon') \right) : \varepsilon, \varepsilon' \in E \right\},$$

and

$$A \vee B = \left\{ \left((\varepsilon, \varepsilon'), f_{A\vee B}(\varepsilon, \varepsilon') \right) : \varepsilon, \varepsilon' \in E \right\} \\ = \left\{ \left((\varepsilon, \varepsilon'), f_A(\varepsilon) \cup f_B(\varepsilon') \right) : \varepsilon, \varepsilon' \in E \right\}.$$

In the rest of this paper, G denotes an AG-group and e denotes the left identity of G unless otherwise stated. An AG-group is a non-associative structure, in which commutativity and associativity imply each other and thus AG-group become an abelian group if any one of the property is allowed in AG-group. AG-group is a generalization of abelian group and a special case of quasi-group. An AG-groupoid (or LA-semigroup) is a non-associative groupoid in general, in which the left invertive law: $(ab)c = (cb)a$ holds for all $a, b, c \in G$. An AG-groupoid G is called an AG-group or left almost group (LA-group), if there exists a unique left identity e in G (i.e. $ea = a$ for all $a \in G$), and for all $a \in G$ there exists $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$. Nowadays, many researchers take keen interest to fuzzify AG-groupoids and AG-groups; also they develop soft theory of AG-groupoids and AG-groups [19–23]. An AG-group $(G, *)$ can be easily obtained from an abelian group $(G_1, \cdot)$ by:

$$a * b = a^{-1} \cdot b \text{ or } a * b = b \cdot a^{-1} \ \forall \ a, b \in G_1.$$

It is easy to prove that in an AG-group G the right identity become the two sided identity, and thus G with right identity become an abelian group. AG-group posses the property of cancellativity like groups. A nonempty subset H of G is called an AG-subgroup of G, if H itself is an AG-group under the same binary operation defined on G.

Various comparative properties of AG-groups and groups are explored in [24–26]. The following identities can be easily proved in an AG-group G.

Lemma 1. *[24] Let $e \in G$, and $a, b, c, d \in G$, then*

1. $(ab)(cd) = (ac)(bd)$ (medial law).
2. $a(bc) = b(ac)$.
3. $(ab)(cd) = (db)(ca)$ (paramedial law).
4. $(ab)(cd) = (dc)(ba)$.
5. $(ab)^{-1} = a^{-1}b^{-1}$.

2. Soft Uni-AG-groups

In this section the basic definition of soft union AG-group (soft uni-AG-group) is given, some of the basic results along with suitable examples are provided.

Definition 4. *Let G be an AG-group and $A \in S(U)$ be a soft set. Then, A is called soft uni-AG-group over U if*

1. $f_A(ab) \subseteq f_A(a) \cup f_A(b)\ \forall\ a, b \in G$,
2. $f_A(a^{-1}) = f_A(a)\ \forall\ a \in G$.

The set of all soft uni-AG-group over U is symbolically represented by $S_{\cup AG}(U)$.

Example 1. *Consider a non-associative AG-group $G = \{0, 1, 2\}$ of order 3 with left identity 0, defined in the following table:*

.	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

Let A be a soft set over $U = \{u_1, u_2, \ldots, u_{10}\}$, defined by

$$\begin{aligned} A &= \{(0, f_A(0)), (1, f_A(1)), (2, f_A(2))\} \\ &= \{(0, \{u_1, u_3\}), (1, \{u_1, u_3, u_5\}), (2, \{u_1, u_3, u_5\})\}. \end{aligned}$$

Then, A is a soft uni-AG-group.

Example 2. *Consider a non-associative AG-group $G = \{0, 1, 2, 3\}$ of order 4 with left identity 0 defined by:*

.	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

Let A be a soft set over $U = \mathbb{Z}$, defined by

$$\begin{aligned} A &= \{(0, f_A(0)), (1, f_A(1)), (3, f_A(3)), (4, f_A(4))\} \\ &= \{(0, \{1, 3\}), (1, \{1, 3, 5, 7\}), (2, \{1, 3, 5, 7\}), (3, \{1, 3, 5, 7\})\}. \end{aligned}$$

Then, one can easily show that $A \in S_{\cup AG}(U)$.

Lemma 2. *Let $A \in S_{\cup AG}(U)$. Then, $f_A(e) \subseteq f_A(a)$ for all $a \in G$.*

Proof. Since $A \in S_{\cup AG}(U)$. Then, for all $a \in G$,

$$\begin{aligned} f_A(e) &= f_A(aa^{-1}) \\ &\subseteq f_A(a) \cup f_A(a^{-1}) \\ &= f_A(a) \cup f_A(a) \\ &= f_A(a). \end{aligned}$$

Hence, $f_A(e) \subseteq f_A(a)$ for all $a \in G$.

Lemma 3. *Let $A \in S_{\cup AG}(U)$. Then $f_A(ab) = f_A(ba)$ for all $a, b \in G$.*

Proof. Let $A \in S_{\cup AG}(U)$. Then for all $a, b \in G$,

$$\begin{aligned} f_A(ab) &= f_A((ea)b) \\ &= f_A((ba)e) && \text{(by the left invertive law)} \\ &\subseteq f_A(ba) \cup f_A(e) \\ &= f_A(ba) && \text{(by Lemma 2)} \\ \Rightarrow f_A(ab) &\subseteq f_A(ba). \end{aligned}$$

Similarly, it can be shown that $f_A(ba) \subseteq f_A(ab)$. Hence, $f_A(ab) = f_A(ba)$ for all $a, b \in G$.

Theorem 1. *A soft set A over U is a soft uni-AG-group over U if and only if $f_A(ab^{-1}) \subseteq f_A(a) \cup f_A(b)$ for all $a, b \in G$.*

Proof. Suppose $A \in S_{\cup AG}(U)$. Then, for all $a, b \in G$,

$$\begin{aligned} f_A(ab^{-1}) &\subseteq f_A(a) \cup f_A(b^{-1}) \\ &= f_A(a) \cup f_A(b) \\ \Rightarrow f_A(ab^{-1}) &\subseteq f_A(a) \cup f_A(b). \end{aligned}$$

Conversely, suppose that for all $a, b \in G$,

$$f_A(ab^{-1}) \subseteq f_A(a) \cup f_A(b).$$

Then by choosing $a = e$ we get

$$f_A(b^{-1}) \subseteq f_A(b). \qquad \text{(by Lemma 2)}$$

Thus,

$$f_A(b) = f_A((b^{-1})^{-1}) \subseteq f_A(b^{-1}).$$

Consequently, $f_A(b) = f_A(b^{-1}) \ \forall \ b \in G$. Now,

$$\begin{aligned} f_A(ab) &= f_A(a(b^{-1})^{-1}) \\ &\subseteq f_A(a) \cup f_A(b^{-1}) \\ &= f_A(a) \cup f_A(b). \end{aligned}$$

Hence, $A \in S_{\cup AG}(U)$.

Lemma 4. *Let $A \in S_{\cup AG}(U)$. Then, for all $a, b \in G$, $f_A(ab) = f_A(b)$ if and only if $f_A(a) = f_A(e)$.*

Proof. Let $A \in S_{\cup AG}(U)$ and $f_A(ab) = f_A(b)$ for all $a, b \in G$. By choosing $b = e$ we get

$$\begin{aligned} f_A(ae) &= f_A(e) \\ \Rightarrow f_A(ea) &= f_A(e) \qquad \text{(by Lemma 3)} \\ \Rightarrow f_A(a) &= f_A(e). \end{aligned}$$

Conversely, suppose that $f_A(a) = f_A(e)\ \forall\ a \in G$. Then,

$$\begin{aligned} f_A(ab) &\subseteq f_A(a) \cup f_A(b) \\ &= f_A(e) \cup f_A(b) \\ &= f_A(b) \qquad \text{(by Lemma 2)} \end{aligned}$$

This implies that

$$f_A(ab) \subseteq f_A(b). \tag{1}$$

Also,

$$\begin{aligned} f_A(b) &= f_A(eb) = f_A((a^{-1}a)b) \\ &= f_A\left((ba)a^{-1}\right) \qquad \text{(by the left invertive law)} \\ &\subseteq f_A(ba) \cup f_A(a^{-1}) \\ &= f_A(ab) \cup f_A(a) \qquad \text{(by Lemma 3)} \\ &= f_A(ab) \cup f_A(e) \\ &= f_A(ab). \qquad \text{(by Lemma 2)} \end{aligned}$$

This implies that

$$f_A(b) \subseteq f_A(ab). \tag{2}$$

Consequently, from Equation (1) and (2) we get, $f_A(ab) = f_A(b)$.

Lemma 5. *Let $A \in S_{\cup AG}(U)$. Then $f_A(a) = f_A(b)$, if $f_A(ab^{-1}) = f_A(e)$ for all $a, b \in G$.*

Proof. Let $A \in S_{\cup AG}(U)$ such that $f_A(ab^{-1}) = f_A(e)$. Then, for all $a, b \in G$

$$\begin{aligned} f_A(a) &= f_A(e \cdot a) = f_A((bb^{-1})a) \\ &= f_A\left((ab^{-1})b\right) \qquad \text{(by the left invertive law)} \\ &\subseteq f_A(ab^{-1}) \cup f_A(b) \\ &= f_A(e) \cup f_A(b) \\ &= f_A(b). \qquad \text{(by assumption and Lemma 2)} \end{aligned}$$

Thus

$$f_A(a) \subseteq f_A(b). \tag{3}$$

And

$$\begin{aligned} f_A(b) &= f_A(b^{-1}) = f_A(e \cdot b^{-1}) = f_A((a^{-1}a)b^{-1}) \\ &= f_A\left((b^{-1}a)a^{-1}\right) \qquad \text{(by the left invertive law)} \\ &\subseteq f_A(b^{-1}a) \cup f_A(a^{-1}) \\ &= f_A(ab^{-1}) \cup f_A(a) \qquad \text{(by Lemma 3)} \\ &= f_A(a). \qquad \text{(by Lemma 2)} \end{aligned}$$

Thus

$$f_A(b) \subseteq f_A(a). \tag{4}$$

Hence, $f_A(a) = f_A(b)$ for all $a, b \in G$ using Equations (3) and (4).

Theorem 2. *Let $A, B \in S_{\cup AG}(U)$. Then, $A \vee B \in S_{\cup AG}(U)$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$. Then, by Definition 3 and Theorem 1,

$$\begin{aligned} (f_A \vee f_B)\left((x_1, y_1) \cdot (x_2, y_2)^{-1}\right) &= (f_A \vee f_B)\left((x_1, y_1) \cdot (x_2^{-1}, y_2^{-1})\right) \\ &= (f_A \vee f_B)\left(x_1 x_2^{-1}, y_1 y_2^{-1}\right) \\ &= f_A(x_1 x_2^{-1}) \cup f_B(y_1 y_2^{-1}) \\ &\subseteq \left(f_A(x_1) \cup f_A(x_2^{-1})\right) \cup \left(f_B(y_1) \cup f_B(y_2^{-1})\right) \\ &= (f_A(x_1) \cup f_A(x_2)) \cup (f_B(y_1) \cup f_B(y_2)) \\ &= (f_A(x_1) \cup f_B(y_1)) \cup (f_A(x_2) \cup f_B(y_2)) \\ &= (f_A \vee f_B)(x_1, y_1) \cup (f_A \vee f_B)(x_2, y_2). \end{aligned}$$

Therefore, $A \vee B \in S_{\cup AG}(U)$.

The following counter example shows that $A \wedge B$ of any two soft sets A and B may not be a soft uni-AG-group.

Example 3. *Consider a non-associative AG-group $G = \{0, 1, 2, 3\}$ of order 4 with left identity 0 defined by:*

.	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	3	2	1	0
3	2	3	0	1

Let A and B be any two soft sets over $U = Z_{10}$ as follow:

$$\begin{aligned} A &= \{(0, f_A(0)), (1, f_A(1)), (2, f_A(3)), (3, f_A(4))\} \\ &= \{(0, \{0,1,2\}), (1, \{0,1,2,3,4,5\}), (2, \{0,1,2,3,4,5,6,7\}), (3, \{0,1,2,3,4,5,6,7\})\}. \end{aligned}$$

$$\begin{aligned} B &= \{(0, f_B(0)), (1, f_B(1)), (2, f_B(3)), (3, f_B(4))\} \\ &= \{(0, \{5,6\}), (1, \{5,6,7,8\}), (2, \{5,6,7,8,9,10\}), (3, \{5,6,7,8,9,10\})\}. \end{aligned}$$

It is clear that both $A, B \in S_{\cup AG}(U)$*. Now, take*

$$\begin{aligned} (f_A \wedge f_B)\left((1,1)\cdot(0,2)^{-1}\right) &= (f_A \wedge f_B)((1,1)\cdot(0,3)) \\ & \quad (f_A \wedge f_B)(1\cdot 0, 1\cdot 3) \\ &= (f_A \wedge f_B)(1,2) \\ &= (f_A)(1) \wedge (f_B)(2) = \{5\}, \end{aligned}$$

and

$$(f_A \wedge f_B)(1,1) \cup (f_A \wedge f_B)(0,2) = \phi \cup \phi = \phi,$$

this implies that

$$(f_A \wedge f_B)\left((1,1)\cdot(0,2)^{-1}\right) \subsetneq (f_A \wedge f_B)(1,1) \cap (f_A \wedge f_B)(0,2).$$

Hence, $A \wedge B \notin S_{\cup AG}(U)$*.*

Definition 5. *Let* $A, B \in S_{\cup AG}(U)$ *on AG-groups* G_1 *and* G_2 *respectively. Then, the product of* A *and* B *is denoted by* $A \times B$ *and is defined by*

$$\begin{aligned} A \times B &= \{((a,b), (f_{A\times B})(a,b)) \ \forall\ (a,b) \in G_1 \times G_2\} \\ &= \{((a,b), (f_A(a) \times f_B(b))) \ \forall\ (a,b) \in G_1 \times G_2\}. \end{aligned}$$

Example 4. *Let* $U = Z_{10}$ *be a universal set, and* $G_1 = \{a,b,c,d\}$ *and* $G_2 = \{x,y,z\}$ *are AG-groups of order* 4 *and* 3 *defined in the following tables (i) and (ii) respectively:*

.	a	b	c	d
a	d	a	b	c
b	c	d	a	b
c	b	c	d	a
d	a	b	c	d

(*i*)

.	x	y	z
x	x	y	z
y	z	x	y
z	y	z	x

(*ii*)

Let $A, B \in S_{\cup AG}(U)$ *on AG-groups* G_1 *and* G_2 *respectively defined by:*

$$f_A(a) = \{0,1,2\} = f_A(c),\ f_A(b) = \{0,1\},\ f_A(d) = \{0\},$$

and

$$f_B(x) = \{0\},\ f_B(y) = \{0,1\} = f_B(z).$$

Then

$$\begin{aligned} A\times B &= \{(a,b),\ (f_A(a)\times f_B(b))\ \forall\ (a,b)\in G_1\times G_2\}, \\ &= \{\{(a,x),((0,0),(1,0),(2,0))\},\{(a,y),((0,0),(0,1),(1,0),(1,1),(2,0),(2,1))\}, \\ &\quad \{(a,z),((0,0),(0,1),(1,0),(1,1),(2,0),(2,1))\},\{(b,x),((0,0),(1,0))\}, \\ &\quad \{(b,y),((0,0),(0,1),(1,0),(1,1))\},\{(b,z),((0,0),(0,1),(1,0),(1,1))\}, \\ &\quad \{\{(c,x),((0,0),(1,0),(2,0))\},\{(c,y),((0,0),(0,1),(1,0),(1,1),(2,0),(2,1))\}, \\ &\quad \{(c,z),((0,0),(0,1),(1,0),(1,1),(2,0),(2,1))\},\{(d,x),((0,0))\}, \\ &\quad \{(d,y),((0,0),(0,1))\},\{(d,z),((0,0),(0,1))\}\}. \end{aligned}$$

Theorem 3. *Let $A, B \in S_{\cup AG}(U)$ with respect to AG-groups G_1 and G_2. Then $A\times B \in S_{\cup AG}(U\times U)$.*

Proof. For any $(x_1, y_1), (x_2, y_2) \in G_1 \times G_2$,

$$\begin{aligned} (f_{A\times B})\left((x_1,y_1),(x_2,y_2)^{-1}\right) &= (f_{A\times B})\left((x_1,y_1),(x_2^{-1},y_2^{-1})\right) \\ &= (f_{A\times B})\left((x_1x_2^{-1},y_1y_2^{-1}\right) \\ &= f_A(x_1x_2^{-1})\times f_B(y_1y_2^{-1}) \qquad \text{(by Definition 5)} \\ &\subseteq (f_A(x_1)\cup f_A(x_2))\times(f_B(y_1)\cup f_B(y_2)) \\ &= (f_A(x_1)\times f_B(y_1))\cup(f_A(x_2)\times f_B(y_2)) \\ &= (f_{A\times B})(x_1,y_1)\cup(f_{A\times B})(x_2,y_2). \end{aligned}$$

Hence, $A\times B\in S_{\cup AG}(U\times U)$.

Theorem 4. *Let $A, B \in S_{\cup AG}(U)$, then $A\tilde{\cup}B \in S_{\cup AG}(U)$.*

Proof. Since $A, B \in S_{\cup AG}(U)$. Therefore, $A\tilde{\cup}B \neq \emptyset$. For any $a, b \in A\tilde{\cup}B$, we have

$$\begin{aligned} (f_A\tilde{\cup}f_B)(ab^{-1}) &= f_A(ab^{-1})\cup f_B(ab^{-1}) \qquad \text{(by Definition 2-(v))} \\ &\subseteq (f_A(a)\cup f_A(b))\cup(f_B(a)\cup f_B(b)) \\ &= (f_A(a)\cup f_B(a))\cup(f_A(b)\cup f_B(b)) \\ &= (f_A\tilde{\cup}f_B)(a)\cup(f_A\tilde{\cup}f_B)(b). \end{aligned}$$

Hence, $A\tilde{\cup}B\in S_{\cup AG}(U)$.

The following counter example, depicts that $A\tilde{\cap}B \notin S_{\cup AG}(U)$ for any $A, B \in S_{\cup AG}(U)$.

Example 5. *Let $G = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ be an AG-group of order 9 defined in the following table:*

$\cdot$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	2	0	1	4	5	3	7	8	6
2	1	2	0	5	3	4	8	6	7
3	7	6	8	0	2	1	5	3	4
4	6	8	7	1	0	2	4	5	3
5	8	7	6	2	1	0	3	4	5
6	4	3	5	8	6	7	0	2	1
7	3	5	4	7	8	6	1	0	2
8	5	4	3	6	7	8	2	1	0

Let $A, B \in S_{\cup AG}(Z_{10})$*, defined by*

$$f_A(0) = \emptyset,\ f_A(1) = \{0,1\} = f_A(2),$$
$$f_A(3) = \{0,1,2,3,4,5,6\} = f_A(4) = f_A(5) = f_A(6) = f_A(7) = f_A(8),$$

and

$$f_B(0) = \emptyset,\ f_B(3) = \{0,1,2,3\} = f_B(7),$$
$$f_B(1) = \{0,1,2,3,4,5,6,7,8\} = f_B(2) = f_B(4) = f_B(5) = f_B(6) = f_B(7) = f_B(8),$$

It is clear that

$$(f_A \tilde{\cap} f_B)\left(2 \cdot 3^{-1}\right) = (f_A \tilde{\cap} f_B)(5) = f_A(5) \cap f_B(5) = \{0,1,2,3,4,5,6\}, \tag{5}$$

and

$$\begin{aligned} ((f_A \tilde{\cap} f_B)(2)) \cup ((f_A \tilde{\cap} f_B)(3)) &= (f_A(2) \cap f_B(2)) \cup (f_A(3) \cap f_B(3)) \\ &= \{0,1,2,3\}, \end{aligned}$$

this implies that

$$((f_A \tilde{\cap} f_B)(2)) \cup ((f_A \tilde{\cap} f_B)(3)) = \{0,1,2,3\}. \tag{6}$$

From Equations (5) and (6) it is clear that

$$(f_A \tilde{\cap} f_B)\left(2 \cdot 3^{-1}\right) \nsubseteq ((f_A \tilde{\cap} f_B)(2)) \cup ((f_A \tilde{\cap} f_B)(3)).$$

Hence, $A \tilde{\cap} B \notin S_{\cup AG}(U)$.

Definition 6. *Let* H *be an AG-subgroup of an AG-group* G*. Then a soft subset* B *over* H *is called a soft uni-AG-subgroup of a soft subset* A *over* G *if* B *is a nonempty soft subset of* A*. We denote this by* $B \tilde{\lesssim} A$.

Example 6. *Let $U = Z_{10}$ be the universal set and G be any AG-group of order 9 defined as in Example 5. Define a soft uni-AG-group A as follows:*

$$f_A(0) = \{0,1,2\},\ f_A(3) = \{0,1,2,3,4\} = f_A(7),$$
$$f_A(1) = \{0,1,2,3,4,5,6\} = f_A(2) = f_A(4) = f_A(5) = f_A(6) = f_A(8).$$

Let $H_1 = \{0,3,7\}$ and $H_2 = \{0,1,2\}$ be two AG-subgroups of G. Define soft uni-AG-groups B and C over U, w. r. t. H_1 and H_2 respectively as follow:

$$B = \{(0,\{0,1\}),(3,\{0,1,2\}),(7,\{0,1,2\})\},$$

and

$$C = \{(0,\{0,2\}),(1,\{0,2,4\}),(2,\{0,2,4\})\}.$$

As $B\tilde{\subseteq}A$ and $C\tilde{\subseteq}A$. Therefore, $B\tilde{\leq}A$ and $C\tilde{\leq}A$.

Theorem 5. *Let $B\tilde{\leq}A$ and $C\tilde{\leq}A$. Then, $B\tilde{\cup}C\tilde{\leq}A$.*

Proof. Since, $B\tilde{\leq}A$ and $C\tilde{\leq}A$. Therefore, $B\tilde{\cup}C \neq \emptyset$. Let $x, y \in B\tilde{\cup}C$. Then by Theorem 1

$$\begin{aligned}
(f_B\tilde{\cup}f_C)(xy^{-1}) &= ((f_{B\tilde{\cup}C})(xy^{-1})) \\
&= f_B(xy^{-1}) \cup f_C(xy^{-1}) \\
&\subseteq (f_B(x) \cup f_B(y)) \cup (f_C(x) \cup f_C(y)) \\
&= (f_B(x) \cup f_C(x)) \cup (f_B(y) \cup f_C(y)) \\
&= f_{B\tilde{\cup}C}(x) \cup f_{B\tilde{\cup}C}(y) \\
&= (f_B\tilde{\cup}f_C)(x) \cup (f_B\tilde{\cup}f_C)(y).
\end{aligned}$$

Hence, $B\tilde{\cup}C\tilde{\leq}A$.

Theorem 6. *Let $\{B_i : i \in I\}\tilde{\leq}A$ for all $i \in I$. Then $\tilde{\bigcup}_{i\in I} B_i\tilde{\leq}A$.*

Proof. Since, $\{B_i : i \in I\}\tilde{\leq}A$ for all $i \in I$. Therefore, $\bigcup_{i\in I} B_i \neq \emptyset$. Let $x, y \in \bigcup_{i\in I} B_i$. Then by Theorem 1 we get

$$\begin{aligned}
\left(\tilde{\bigcup_{i\in I}} f_{B_i}\right)(xy^{-1}) &= \left(\left(f_{\tilde{\bigcup}_{i\in I} B_i}\right)(xy^{-1})\right) \\
&= \bigcup_{i\in I}\left(f_{B_i}(xy^{-1}) \ : \ i \in I\right) \\
&\subseteq \bigcup_{i\in I}((f_{B_i}(x) \cup f_{B_i}(y)) \ : \ i \in I) \\
&= \left(\bigcup_{i\in I}(f_{B_i}(x) \ : \ i \in I)\right) \cup \left(\bigcup_{i\in I}(f_{B_i}(y) \ : \ i \in I)\right)
\end{aligned}$$

$$= \left(\left(f_{\underset{i\in I}{\tilde{\cup}}B_i}\right)(x)\right)\cup\left(\left(f_{\underset{i\in I}{\tilde{\cup}}B_i}\right)(y)\right).$$

Hence, $\underset{i\in I}{\tilde{\cup}}B_i\tilde{\leq}A$.

The following counter example clearly shows that $B\tilde{\cap}C\tilde{\nleq}A$.

Example 7. *From, Example 5, we have*

$$(f_B\tilde{\cap}f_C)\left(4\cdot4^{-1}\right)=\left(f_{B\tilde{\cap}C}\right)(4\cdot4)=\left(f_{B\tilde{\cap}C}\right)(0)=f_B(0)\cap f_C(0)=\{0\}, \quad (7)$$

and

$$\begin{aligned}((f_B\tilde{\cap}f_C)(4))\cup((f_B\tilde{\cap}f_C)(4)) &= (f_B(4)\cap f_C(4))\cup(f_B(4)\cap f_C(4))\\ &=\emptyset,\end{aligned}$$

this implies that

$$((f_B\tilde{\cap}f_C)(4))\cup((f_B\tilde{\cap}f_C)(4))=\emptyset. \quad (8)$$

By Equations (7) and (8), we get

$$(f_B\tilde{\cap}f_C)\left(4\cdot4^{-1}\right)\subsetneq((f_B\tilde{\cap}f_C)(4))\cup((f_B\tilde{\cap}f_C)(4)).$$

Hence, $B\tilde{\cap}C\tilde{\nleq}A$.

3. Conjugate Soft Uni-AG-groups

Definition 7. *Let* $A\in S_{\cup AG}(U)$ *and* $x\in G$. *Then* A_x *is called* ***conjugate soft uni-AG-group*** *of* A *(with respect to* x*) denoted by* $A_x\overset{c}{\sim}A$, *and is given by*

$$f_{A_x}(g) \quad = \quad f_A\left((xg)x^{-1}\right), \text{ for all } g\in G.$$

Remark 1. *It is noted that a conjugate soft uni-AG-group may or may not be a soft-uni-AG-group.*

Example 8. *Consider an AG-group* G *of order* 6 *defined by*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	1	2	3	4
2	4	5	0	1	2	3
3	3	4	5	0	1	2
4	2	3	4	5	0	1
5	1	2	3	4	5	0

Let $A \in S_{\cup AG}(\mathbb{Z})$, defined as follows:

$$\begin{aligned} f_A(0) &= \{-1,0,1\}, \\ f_A(2) &= \{-2,-1,0,1,2\} = f_A(4), \\ f_A(1) &= \{-3,-2,-1,0,1,2,3\} = f_A(3) = f_A(5). \end{aligned}$$

The conjugates soft uni-AG-groups of A is given by:

$$\begin{aligned} f_{A_0}(0) = f_{A_3}(0) &= f_A(0) = & \{-1,0,1\}, \\ f_{A_0}(1) = f_{A_3}(1) &= f_A(5) = & \{-3,-2,-1,0,1,2,3\}, \\ f_{A_0}(2) = f_{A_3}(2) &= f_A(4) = & \{-2,-1,0,1,2\}, \\ f_{A_0}(3) = f_{A_3}(3) &= f_A(3) = & \{-3,-2,-1,0,1,2,3\}, \\ f_{A_0}(4) = f_{A_3}(4) &= f_A(2) = & \{-2,-1,0,1,2\}, \\ f_{A_0}(5) = f_{A_3}(5) &= f_A(1) = & \{-3,-2,-1,0,1,2,3\}. \end{aligned}$$

$$\begin{aligned} f_{A_1}(0) = f_{A_4}(0) &= f_A(2) = & \{-2,-1,0,1,2\}, \\ f_{A_1}(1) = f_{A_4}(1) &= f_A(1) = & \{-3,-2,-1,0,1,2,3\}, \\ f_{A_1}(2) = f_{A_4}(2) &= f_A(0) = & \{-1,0,1\}, \\ f_{A_1}(3) = f_{A_4}(3) &= f_A(5) = & \{-3,-2,-1,0,1,2,3\}, \\ f_{A_1}(4) = f_{A_4}(4) &= f_A(4) = & \{-2,-1,0,1,2\}, \\ f_{A_1}(5) = f_{A_4}(5) &= f_A(3) = & \{-3,-2,-1,0,1,2,3\}. \end{aligned}$$

$$\begin{aligned} f_{A_2}(0) = f_{A_5}(0) &= f_A(4) = & \{-2,-1,0,1,2\}, \\ f_{A_2}(1) = f_{A_5}(1) &= f_A(3) = & \{-3,-2,-1,0,1,2,3\}, \\ f_{A_2}(2) = f_{A_5}(2) &= f_A(2) = & \{-2,-1,0,1,2\}, \\ f_{A_2}(3) = f_{A_5}(3) &= f_A(1) = & \{-3,-2,-1,0,1,2,3\}, \\ f_{A_2}(4) = f_{A_5}(4) &= f_A(0) = & \{-1,0,1\}, \\ f_{A_2}(5) = f_{A_5}(5) &= f_A(5) = & \{-3,-2,-1,0,1,2,3\}. \end{aligned}$$

A_1 and A_2 are conjugate soft uni-AG-groups but are not soft uni-AG-groups over $\mathbb{Z}$, as

$$f_{A_1}(2 \cdot 2) = f_{A_1}(0) = \{-2,-1,0,1,2\} \subsetneq f_{A_1}(2) \cup f_{A_1}(2) = \{-1,0,1\},$$

and

$$f_{A_2}(4 \cdot 4) = f_{A_2}(0) = \{-2,-1,0,1,2\} \subsetneq f_{A_2}(4) \cup f_{A_2}(4) = \{-1,0,1\}.$$

Definition 8. *Let $A \in S_{\cup AG}(U)$. Then A is called a **normal soft uni-AG-subgroup** over U if*

$$f_{A_x}(y) = f_A\left((xy)x^{-1}\right) = f_A(y) \;\forall\; x,\, y \in G.$$

In other words A is a normal soft uni-AG-subgroup over U, if A is self-conjugate soft uni-AG-group.

The set of all normal soft uni-AG-subgroups over U is represented by $NS_{\cup AG}(U)$.

Example 9. *Let G be an AG-group of order 6 defined as in Example 8. Let $A \in S_{\cup AG}(\mathbb{Z})$, defined by*

$$\begin{aligned} f_A(0) &= \{-1,0,1\} = f_A(2) = f_A(4), \\ f_A(1) &= \{-2,-1,0,1,2\} = f_A(3) = f_A(5). \end{aligned}$$

The conjugates soft uni-AG-groups of A are given by:

$$\begin{aligned} f_{A_0}(0) = f_{A_3}(0) &= f_A(0) = \{-1,0,1\}, \\ f_{A_0}(1) = f_{A_3}(1) &= f_A(5) = \{-2,-1,0,1,2\}, \\ f_{A_0}(2) = f_{A_3}(2) &= f_A(4) = \{-1,0,1\}, \\ f_{A_0}(3) = f_{A_3}(3) &= f_A(3) = \{-2,-1,0,1,2\}, \\ f_{A_0}(4) = f_{A_3}(4) &= f_A(2) = \{-1,0,1\}, \\ f_{A_0}(5) = f_{A_3}(5) &= f_A(1) = \{-2,-1,0,1,2\}. \end{aligned}$$

$$\begin{aligned} f_{A_1}(0) = f_{A_4}(0) &= f_A(2) = \{-1,0,1\}, \\ f_{A_1}(1) = f_{A_4}(1) &= f_A(1) = \{-2,-1,0,1,2\}, \\ f_{A_1}(2) = f_{A_4}(2) &= f_A(0) = \{-1,0,1\}, \\ f_{A_1}(3) = f_{A_4}(3) &= f_A(5) = \{-2,-1,0,1,2\}, \\ f_{A_1}(4) = f_{A_4}(4) &= f_A(4) = \{-1,0,1\}, \\ f_{A_1}(5) = f_{A_4}(5) &= f_A(3) = \{-2,-1,0,1,2\}. \end{aligned}$$

$$\begin{aligned} f_{A_2}(0) = f_{A_5}(0) &= f_A(4) = \{-1,0,1\}, \\ f_{A_2}(1) = f_{A_5}(1) &= f_A(3) = \{-2,-1,0,1,2\}, \\ f_{A_2}(2) = f_{A_5}(2) &= f_A(2) = \{-1,0,1\}, \\ f_{A_2}(3) = f_{A_5}(3) &= f_A(1) = \{-2,-1,0,1,2\}, \\ f_{A_2}(4) = f_{A_5}(4) &= f_A(0) = \{-1,0,1\}, \\ f_{A_2}(5) = f_{A_5}(5) &= f_A(5) = \{-2,-1,0,1,2\}. \end{aligned}$$

Hence, $A \in NS_{\cup AG}(\mathbb{Z})$, as A is self conjugate soft uni-AG-subgroup.

Lemma 6. *Let $A \in NS_{\cup AG}(U)$. Then for all $x, y \in G$, the following assertions are equivalent:*

1. $f_A\left((xy)x^{-1}\right) = f_A(y)$,
2. $f_A\left((xy)x^{-1}\right) \supseteq f_A(y)$,
3. $f_A\left((xy)x^{-1}\right) \subseteq f_A(y)$.

Proof. $(i) \Rightarrow (ii)$: Obvious.
$(ii) \Rightarrow (iii)$: Assume that (ii) holds. Consider

$$\begin{aligned}
f_A\left((xy)x^{-1}\right) &\subseteq f_A\left(\left(x^{-1}\left((xy)x^{-1}\right)\right)\left((x^{-1})^{-1}\right)\right) \\
&= f_A\left(\left(x^{-1}\left((xy)\,x^{-1}\right)\right)x\right) \\
&= f_A\left(\left(x\left((xy)\,x^{-1}\right)\right)x^{-1}\right) \qquad \text{(by the left invertive law)} \\
&= f_A\left(\left((xy)\left(xx^{-1}\right)\right)x^{-1}\right) \qquad \text{(by Lemma 1-(ii))} \\
&= f_A\left(((xy)\,e)\,x^{-1}\right) \\
&= f_A\left(((ey)\,x)\,x^{-1}\right) \qquad \text{(by the left invertive law)} \\
&= f_A\left((yx)\,x^{-1}\right) \\
&= f_A\left(\left(x^{-1}x\right)y\right) \qquad \text{(by the left invertive law)} \\
&= f_A\left(ey\right) = f_A(y) \\
\Rightarrow f_A\left((xy)x^{-1}\right) &\subseteq f_A(y)\ \forall\ x,\ y \in G.
\end{aligned}$$

$(iii) \Rightarrow (i)$: Assume that (iii) holds. Consider,

$$\begin{aligned}
f_A\left((xy)\,x^{-1}\right) &\supseteq f_A\left(\left(x^{-1}\left((xy)\,x^{-1}\right)\right)\left(\left(x^{-1}\right)^{-1}\right)\right) \\
&= f_A(y),\ \text{as in the proof}\ (ii) \Rightarrow (iii) \\
\Rightarrow f_A\left((xy)\,x^{-1}\right) &\supseteq f_A(y)\ \forall\ x, y \in G.
\end{aligned}$$

Consequently, $f_A\left((xy)x^{-1}\right) \subseteq f_A(y) \subseteq f_A\left((xy)\,x^{-1}\right)$. Hence, $f_A\left((xy)\,x^{-1}\right) = f_A(y)$.

Theorem 7. *Let $A \in S_{\cup AG}(U)$. Then $A \in NS_{\cup AG}(U)$ if and only if $f_A([x,\, y]) \subseteq f_A(x)\ \forall\ x,\ y \in G$, where $[x, y] = xy \cdot y^{-1}x^{-1}$ is a commutator of x and y in AG-group G.*

Proof. Let $A \in NS_{\cup AG}(U)$. Then,

$$\begin{aligned}
f_A([x,\, y]) &= f_A\left((xy)\left(y^{-1}x^{-1}\right)\right) \qquad \text{(by Definition of commutator in } G) \\
&= f_A\left(\left(y^{-1}x^{-1}\right)(xy)\right) \qquad \text{(by Lemma 3)} \\
&= f_A\left((yx)\left(x^{-1}y^{-1}\right)\right) \qquad \text{(by Lemma 1-(iv))} \\
&= f_A\left(x^{-1}\left((yx)\,y^{-1}\right)\right) \qquad \text{(by Lemma 1-(ii))} \\
&\subseteq f_A\left(x^{-1}\right) \cup f_A\left((yx)\,y^{-1}\right) \\
&= f_A(x) \cup f_A(x) \qquad \text{(as } A \in NS_{\cup AG}(U)) \\
&= f_A(x).
\end{aligned}$$

Hence, $f_A([x, y]) \subseteq f_A(x)\ \forall\ x,\ y \in G$.

Conversely, assume that $f_A([x,\, y]) \subseteq f_A(x)\ \forall\ x,\ y \in G$. Then, for any $z \in G$,

$$\begin{aligned}
f_A\left((xz)\,x^{-1}\right) &= f_A\left(e\left((xz)\,x^{-1}\right)\right) \\
&= f_A\left(\left(zz^{-1}\right)\left((xz)\,x^{-1}\right)\right) \\
&= f_A\left(\left(\left((xz)\,x^{-1}\right)z^{-1}\right)z\right) \qquad \text{(by the left invertive law)}
\end{aligned}$$

$$\begin{aligned}
&= f_A\left(\left(\left(z^{-1}x^{-1}\right)(xz)\right)z\right) && \text{(by the left invertive law)}\\
&= f_A\left(\left((zx)\left(x^{-1}z^{-1}\right)\right)z\right) && \text{(by Lemma 1-(iv))}\\
&= f_A\left([z,\, x]z\right)\\
&\subseteq f_A([z,\, x]) \cup f_A(z)\\
&\subseteq f_A(z) \cup f_A(z) = f_A(z).
\end{aligned}$$

This implies that $f_A\left((xz)\,x^{-1}\right) \subseteq f_A(z)\ \forall\ x \in G$. Now by Theorem 6, we have $f_A\left((xz)\,x^{-1}\right) = f_A(z)\ \forall\ x \in G$. Hence, $A \in NS_{\cup AG}(U)$.

Proposition 1. *Let $A \in S_{\cup AG}(U)$. Then $f_A([x,\, y]) = f_A(e)\ \forall\ x,\, y \in G$ if and only if $A \in NS_{\cup AG}(U)$.*

Proof. $A \in NS_{\cup AG}(U)$, if and only if

$$\begin{aligned}
f_A\left((yx)\,y^{-1}\right) &= f_A(x)\ \forall\ x,\, y \in G\\
\Leftrightarrow\quad f_A\left(e\left((yx)\,y^{-1}\right)\right) &= f_A(x)\\
\Leftrightarrow f_A\left(\left(xx^{-1}\right)\left((yx)\,y^{-1}\right)\right) &= f_A(x)\\
\Leftrightarrow f_A\left(\left(\left((yx)\,y^{-1}\right)x^{-1}\right)x\right) &= f_A(x) && \text{(by the left invertive law)}\\
\Leftrightarrow f_A\left(\left(\left(x^{-1}y^{-1}\right)(yx)\right)x\right) &= f_A(x) && \text{(by the left invertive law)}\\
\Leftrightarrow f_A\left(\left((xy)\left(y^{-1}x^{-1}\right)\right)x\right) &= f_A(x) && \text{(by Lemma 1-(iv))}\\
\Leftrightarrow f_A\left(([x,\, y])\,x\right) &= f_A(x)\\
\Leftrightarrow f_A([x,\, y]) &= f_A(e). && \text{(by Lemma 4)}
\end{aligned}$$

Hence, $A \in NS_{\cup AG}(G)$ if and only if $f_A([x,\, y]) = f_A(e)\ \forall\ x,\, y \in G$.

4. α-inclusion of Soft Uni-AG-groups

Definition 9. *Let $A \in S_{\cup AG}(U)$. Then, e-set of A is denoted by $A_{\tilde{e}}$ and defined as*

$$A_{\tilde{e}} = \{x \in G\ :\ f_A(x) = f_A(e)\}\,.$$

Example 10. *In Example 2, $A_{\tilde{e}} = \{0\}$.*

Theorem 8. *Let $A \in S_{\cup AG}(U)$. Then, $A_{\tilde{e}}$ is an AG-subgroup of G.*

Proof. By definition of $A_{\tilde{e}}$, it is obvious that $A_{\tilde{e}} \neq \emptyset$. Let $x, y \in A_{\tilde{e}}$. Then, $f_A(x) = f_A(e) = f_A(y)$. Consider,

$$\begin{aligned}
f_A(xy^{-1}) &\subseteq f_A(x) \cup f_A(y)\\
&= f_A(e) \cup f_A(e)\\
&= f_A(e),
\end{aligned}$$

also by Theorem 2, $f_A(e) \subseteq f_A(xy^{-1})\ \forall\ x, y \in G$. Consequently, $f_A(xy^{-1}) = f_A(e)$. This implies that $xy^{-1} \in A_{\tilde{e}}$. Hence $A_{\tilde{e}}$ is an AG-subgroup of G.

Definition 10. *Let* $A \in S_{\cup AG}(U)$ *and* $\alpha \in P(U)$. *Then* α*-inclusion of* A*, is denoted by* $A_{\tilde{\alpha}}$*, and defined by*

$$A_{\tilde{\alpha}} = \{x \in G : f_A(x) \subseteq \alpha\},$$

while the set

$$A_{\tilde{\alpha}^+} = \{x \in G : f_A(x) \subset \alpha\},$$

is called the strong α*-inclusion of* A.

Note that if $\alpha = U$. Then $A_{\tilde{\alpha}} = \{ x \in G : f_A(x) \neq U \}$, and is called support of A, and is denoted by $\mathrm{supp}(A)$.

Example 11. *Let* $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ *be the universal set and* $G = \{0, 1, 2, 3, 4, 5\}$ *be an AG-group of order* 6 *defined as in Example 8. If we define soft uni-AG-group A over U by*

$$\begin{array}{rcl} f_A(0) & = & \{u_1, u_2, u_3\}, \\ f_A(2) & = & \{u_1, u_2, u_3, u_4, u_5\} = f_A(4), \\ f_A(1) & = & \{u_1, u_2, u_3, u_4, u_5, u_6\} = f_A(3) = f_A(5) \end{array}$$

Let $\alpha = \{u_1, u_2, u_3, u_4, u_5\}$*, then* $A_{\tilde{\alpha}} = \{0, 2, 4\}$ *and* $A_{\tilde{\alpha}^+} = \{0\}$.

Corollary 1. *Let* $B \tilde{\leq} A$ *and* $C \tilde{\leq} A$*. Then, the following assertions hold;*

1. *If* $B \tilde{\subseteq} C$, $\alpha \in P(U)$*. Then* $C_{\tilde{\alpha}} \subseteq B_{\tilde{\alpha}}$,
2. *Let* $\alpha_2 \subseteq \alpha_1$*, with* $\alpha_1, \alpha_2 \in P(U)$*. Then* $B_{\tilde{\alpha}_2} \subseteq B_{\tilde{\alpha}_1}$,
3. $B \tilde{=} C \Leftrightarrow B_{\tilde{\alpha}} = C_{\tilde{\alpha}}$*, for all* $\alpha \in P(U)$.

Proof. Let $B \tilde{\leq} A$ and $C \tilde{\leq} A$.

1. Let $x \in C_{\tilde{\alpha}}$, then, $f_C(x) \subseteq \alpha$. Since $B \tilde{\subseteq} C$, $\alpha \in P(U)$. This implies that $f_B(x) \subseteq f_C(x) \subseteq \alpha \Rightarrow f_B(x) \subseteq \alpha \Rightarrow x \in B_{\tilde{\alpha}}$. Hence $C_{\tilde{\alpha}} \subseteq B_{\tilde{\alpha}}$.
2. Let $\alpha_2 \subseteq \alpha_1$, $\alpha_1, \alpha_2 \in P(U)$, and $x \in B_{\tilde{\alpha}_2}$. Then $f_B(x) \subseteq \alpha_2$. Since, $\alpha_2 \subseteq \alpha_1$ implies that $f_B(x) \subseteq \alpha_1 \Rightarrow x \in B_{\tilde{\alpha}_1}$. Therefore, $B_{\tilde{\alpha}_2} \subseteq B_{\tilde{\alpha}_1}$.

Proof. The proof is straight forward.

Theorem 9. *Let* B, C *are any two soft sets of* G *over* U *and* $\alpha \in P(U)$*. Then,*

1. $B_{\tilde{\alpha}} \cup C_{\tilde{\alpha}} \subseteq (B \tilde{\cup} C)_{\tilde{\alpha}}$,
2. $B_{\tilde{\alpha}} \cap C_{\tilde{\alpha}} = (B \tilde{\cap} C)_{\tilde{\alpha}}$.

Theorem 10. *Let* $\{B_i : i \in I\}$ *be the family of soft sets of* G *over* U*. Then, for any* $\alpha \in P(U)$

1. $\bigcup_{i\in I}(B_{i\tilde{\alpha}}) \subseteq \left(\tilde{\bigcup}_{i\in I} B_i\right)_{\tilde{\alpha}}$,

2. $\bigcap_{i\in I}(B_{i\tilde{\alpha}}) = \left(\tilde{\bigcap}_{i\in I} B_i\right)_{\tilde{\alpha}}$.

Theorem 11. *Let G be an AG-group and $\alpha \in P(U)$. Then $A \in S_{\cup AG}(U)$ if and only if $A_{\tilde{\alpha}}$ is a subgroup of G, where $A_{\tilde{\alpha}} \neq \emptyset$.*

Proof. Let $A \in S_{\cup AG}(U)$ and $A_{\tilde{\alpha}} \neq \emptyset$. Suppose that $x, y \in A_{\tilde{\alpha}}$, then $f_A(x) \subseteq \alpha$ and $f_A(y) \subseteq \alpha$. Therefore,

$$f_A(xy^{-1}) \quad \subseteq \quad f_A(x) \cup f_A(y) \subseteq \alpha.$$

This implies that, $xy^{-1} \in A_{\tilde{\alpha}}$. Hence, $A_{\tilde{\alpha}}$ is a subgroup of G.

Conversely, suppose that $A_{\tilde{\alpha}}$ is a subgroup of G for any $A_{\tilde{\alpha}} \neq \emptyset$. Let $x, y \in G$ such that $f_A(x) = \beta$ and $f_A(y) = \gamma$ and let $\delta = \beta \cup \gamma$. Then $x, y \in A_{\tilde{\delta}}$ and $A_{\tilde{\delta}} \leq G$ by hypothesis. So $xy^{-1} \in A_{\tilde{\delta}}$. Therefore, $f_A(xy^{-1}) \subseteq \delta = \beta \cup \gamma = f_A(x) \cup f_A(y)$. Hence, $A \in S_{\cup AG}(U)$.

Theorem 12. *Let $A \in NS_{\cup AG}(U)$. Then, $A_{\tilde{e}}$ is a normal AG-subgroup of G.*

Proof. By Theorem 8, $A_{\tilde{e}} \leq G$. Let $x \in A_{\tilde{e}}$ and $g \in G$. Then, by Definition 8, we get

$$f_A(gx \cdot g^{-1}) = f_A(x) = f_A(e) \text{ this implies that } gx \cdot g^{-1} \in A_{\tilde{e}}.$$

Hence, $A_{\tilde{e}}$ is a normal AG-subgroup of G.

5. Conclusion

In this paper, the concepts of “soft uni-groups” are extended to soft uni-AG-groups. The notion of conjugates soft uni-AG-groups, normal soft uni-AG-groups, e-set and α-inclusion of soft uni-AG-groups are presented and investigated. In future, these concepts can further be generalized to bipolar soft uni-AG-groups, soft uni-LA-rings and soft uni-LA-near-rings. Moreover, the study of isomorphism theorems may also be a nice work in this area.

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