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On $\mathcal{M}$-Principally Injective and Projective $\mathcal{S}$-acts

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Abstract. In this paper, we have introduced the notions of $\mathcal{M}$-cyclic $\mathcal{S}$-acts, $\mathcal{M}$-principally projective and injective $\mathcal{S}$-acts, semi-projective $\mathcal{S}$-acts and co-cyclic $\mathcal{S}$-acts, where $\mathcal{M}$ is a right $\mathcal{S}$-act. Several interesting properties, characterizations relations between newly defined structures have been investigated.

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1. Introduction

Group action has played a very significant role in the development of the theory of groups. Similarly representation of the Semi group $\mathcal{S}$ by transformation of a set, i.e. acts plays an essential role in Semigroup theory from the beginning (as can be seen from the title of A. K. Suschkewitsch's dissertation "The action of generalized group theory" (in Russian), 1992). A representation of a semigroup $\mathcal{S}$ by a transformation of a set defines an $\mathcal{S}$-act just as the representation of a ring R by endomorphisms of an Abelian group defines an R-module.

Probably first time, the definition of $\mathcal{S}$-acts appeared in two papers of the H. J. Hoehnke with a different name in connection with the consideration of radicals of a semigroup. Acts over the semigroups appeared and were used in various applications like algebra automata theory, mathematical linguistics, graph theory, system theory, information theory, the theory of communications and electronic circuits, databases and other fields of theoretical computer science. In Ring theory, nowadays it is impossible to imagine the main directions without homological methods using categories of modules. Similarly, it is important for monoids to consider an associated category of $\mathcal{S}$-acts Projective and injective $\mathcal{S}$-acts, injective envelopes, projective covers were mainly developed by P. Berthiaume [11], C. S. Johnson & F. Mc. Morris [3], B.M Schein [10]. The monoids over which all $\mathcal{S}$-acts are (injective) projective were investigated by L.Skronjakov. The most notable difference between the theory of $\mathcal{S}$-acts and the theory of modules was discovered by T .G Mustfin, J.

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Fountain and V. Gould, that there exist $\mathcal{S}$-acts which have unstable theories whereas, all complete theories of modules are stable. Principally injective $\mathcal{S}$-acts were first considered by J. Luedeman [8], F. Mc. Morris , and S. K. Sim [13], P. Berthiaume [11], introduced the notion of weak injectivity. M. Shabir and J. Ahsan in [2] have characterized the monoids by P-injective $\mathcal{S}$-acts and a normal system. The novelity of this work is not only that new sturctures has been defined and several of their nontrivial and interesting properties has been studied but also that several of connections of our defined structures has been eastablsihed with existing sturctures like injective, projective, Co-hereditary S-acts and PP-semigroups. For example see Theorems 9, 10, 11, 14 and Corollary 4.
In this paper, we have introduced the notions of $\mathcal{M}$-cyclic $\mathcal{S}$-acts, $\mathcal{M}$-principally projective and injective $\mathcal{S}$-acts and semi-projective $\mathcal{S}$-acts, where $\mathcal{M}$ is a right $\mathcal{S}$-act. The paper consists of 4 sections. We are giving a section-wise description of the paper. First section is running introduction. In section 2, we have defined $\mathcal{M}$-principally injective $\mathcal{S}$-acts and established the connection between the $\mathcal{M}$-cyclic sub-acts and the ideals of endomorphism monoid of a quasi-principally injective $\mathcal{S}$-act $\mathcal{M}$. Further a relation between the kernel congruence induced by endomorphisms on quasi-principally injective $\mathcal{S}$-act $\mathcal{M}$ and the ideals of its endomorphism monoid. After that, some of the properties of $\mathcal{M}$-principally injective $\mathcal{S}$-acts have also been investigated. In section 2, we have defined $\mathcal{M}$-principally projective $\mathcal{S}$-acts and investigated some interesting results and connection with $\mathcal{M}$-principally injective $\mathcal{S}$-acts. A note on Co-hereditary $\mathcal{S}$-acts is included at the end of the section. In section 4, Semi-projective $\mathcal{S}$-acts have been defined and connections with its endomorphism monoid are investigated and significant results are proved.

2. $\mathcal{M}$-principally injective $\mathcal{S}$-acts and their properties in terms of $\mathcal{M}$-cyclic sub-acts

Throughout the paper $\mathcal{S}$ will denote the semigroup with fixed element θ and $\mathbb{E}$ will denote the endomorphism monoid of a right $\mathcal{S}$-act $\mathcal{M}$, i.e. $\mathbb{E} = End_{\mathcal{S}}(\mathcal{M})$. Moreover, by $\ker\gamma$ we mean the usual kernel congruence on $\mathcal{M}$ induced by $\gamma \in \mathbb{E}$.

Definition 1. *Let $\mathcal{N}$ be a sub-act of a right $\mathcal{S}$-act $\mathcal{M}$. Then $\mathcal{N}$ is called an $\mathcal{M}$-**cyclic sub-act** of $\mathcal{M}$ if $\mathcal{N} \cong \mathcal{M}/\rho$ for some congruence ρ on $\mathcal{M}$.*

Lemma 1. *Let $\mathcal{M}$ be a right $\mathcal{S}$-act and $\mathcal{N}$ be a sub-act of $\mathcal{M}$. Then the following are equivalent.*

1) $\mathcal{N} \cong \mathcal{M}/\rho$.
2) $\mathcal{N} = \alpha(\mathcal{M})$, for some $\alpha \in \mathbb{E}$.

Proof. 1)$\Rightarrow$ 2) Let $\pi : \mathcal{M} \to \mathcal{M}/\rho$ be a natural epimorphism and $\gamma : \mathcal{M}/\rho \to \mathcal{N}$ be an isomorphism then it follows that $\gamma\pi : \mathcal{M} \to \mathcal{N}$ is an epimorphism and $\gamma\pi(\mathcal{M}) = \mathcal{N}$.

2)$\Rightarrow$1) Suppose $\mathcal{N} = \alpha(\mathcal{M})$ for some $\alpha \in \mathbb{E}$. Let $K = \ker\alpha$. Define $\Phi : \mathcal{N} = \alpha(\mathcal{M}) \to \mathcal{M}/K$ by $\Phi(\alpha(m)) = [m]_K$. Indeed Φ is an $\mathcal{S}$-isomorphism.

Thus by Lemma (1) we conclude that $\alpha(\mathcal{M})$, where $\alpha \in \mathbb{E}$, can also be used as an alternate notion for $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$.

Definition 2. *Let $\mathcal{M}$ be a right $\mathcal{S}$-act. A right $\mathcal{S}$-act $\mathcal{N}$ is called* ***$\mathcal{M}$-principally injective*** *if every $\mathcal{S}$-homomorphism from an $\mathcal{M}$-cyclic sub-act to $\mathcal{N}$ can be extended to an $\mathcal{S}$-homomorphism from $\mathcal{M}$ to $\mathcal{N}$.*

Definition 3. *A right $\mathcal{S}$-act $\mathcal{M}$ is called* ***quasi-principally*** *injective if it is $\mathcal{M}$-principally injective.*

Definition 4. *A semigroup $\mathcal{S}$ with a fixed element θ is called s**elf principally injective** if $\mathcal{S}$ is $\mathcal{S}$-principally injective.*

Theorem 1. *Let $\mathcal{M}$ be a quasi-principally injective $\mathcal{S}$-act and $\alpha, \beta \in \mathbb{E}$.*
1) If $\alpha(\mathcal{M})$ can be embedded into $\beta(\mathcal{M})$ then $\mathbb{E}\alpha$ is the $\mathbb{E}$-homomorphic image of $\mathbb{E}\beta$.
2) If $\beta(\mathcal{M})$ is the $\mathbb{E}$-homomorphic image of $\alpha(\mathcal{M})$ then $\mathbb{E}\beta$ can be embedded into $\mathbb{E}\alpha$.
3) If $\alpha(\mathcal{M}) \cong \beta(\mathcal{M})$ then $\mathbb{E}\alpha \cong \mathbb{E}\beta$.
Here $\mathbb{E}\alpha = \{u\alpha : u \in \mathbb{E}\}$ and $\mathbb{E}\beta = \{u\beta : u \in \mathbb{E}\}$.

Proof. 1) Suppose $f : \alpha(\mathcal{M}) \to \beta(\mathcal{M})$ is an $\mathcal{S}$-monomorphism. Consider the inclusions $i : \alpha(\mathcal{M}) \to \mathcal{M}$, $j : \beta(\mathcal{M}) \to \mathcal{M}$ and maps $\alpha' : \mathcal{M} \to \alpha(\mathcal{M})$, $\beta' : \mathcal{M} \to \beta(\mathcal{M})$ such that $\alpha = i\alpha'$ and $\beta = j\beta'$. Now since, $\mathcal{M}$ is quasi-principally injective so there exists an $\mathcal{S}$-homomorphism $g : \mathcal{M} \to \mathcal{M}$ such that, $jf = gi$, where g extends f. Moreover, since $g(\alpha(m)) = \beta(m') \in \beta(\mathcal{M})$, for some $m' \in \mathcal{M}$, therefore $g\alpha(\mathcal{M}) \subseteq \beta(\mathcal{M})$.
Define $\Phi : \mathbb{E}\beta \to \mathbb{E}\alpha$ by $\Phi(u\beta) = ug\alpha$. We claim that Φ is $\mathbb{E}$-epimorphism. In order to see that Φ is well-defined, take $u\beta, v\beta \in \mathbb{E}_\beta$ such that $u\beta = v\beta$. Suppose contrary that $\Phi(u\beta) \neq \Phi(v\beta)$ i.e. $ug\alpha \neq vg\alpha$, so there exists $m \in \mathcal{M}$ such that $ug\alpha(m) \neq vg\alpha(m)$, but $g(m) = \beta(m')$ for some $m' \in \mathcal{M}$. Thus $u\beta(m') \neq v\beta(m')$ this leads to a contradiction. Now for any $u\beta \in \mathbb{E}\beta$,

$$\Phi(\gamma(u\beta)) = \Phi((\gamma u)\beta) = (\gamma u)g\alpha = \gamma(ug\alpha) = \gamma\Phi(u\beta),$$

for all $\gamma \in \mathbb{E}$. So Φ is an $\mathbb{E}$-homomorphism. To see surjectivity of Φ, let $u\alpha \in \mathbb{E}\alpha$, where $u \in \mathcal{M}$. Consider $ui : \alpha(\mathcal{M}) \to \mathcal{M}$, since $\mathcal{M}$ is quasi-principally injective so there exists $\gamma : \mathcal{M} \to \mathcal{M}$ such that $\gamma jf = ui$. Consider

$$\Phi(\gamma\beta) = \gamma g\alpha = \gamma gi\alpha' = \gamma jf\alpha' = ui\alpha' = u\alpha.$$

Thus $\Phi : \mathbb{E}\beta \to \mathbb{E}\alpha$ is an $\mathbb{E}$-epimorphism.
2) Let us keep the same notation as in part 1) of proof. Let $f : \alpha(\mathcal{M}) \to \beta(\mathcal{M})$ be an $\mathcal{S}$-epimorphism and we also have $jf = gi$. Define $\Phi : \mathbb{E}\beta \to \mathbb{E}\alpha$ by $\Phi(u\beta) = ug\alpha$. The map is well-defined and an $\mathbb{E}$-homomorphism as is in 1). Moreover, $\beta(\mathcal{M}) = g\alpha(\mathcal{M})$. To show injectivity of Φ, we let $u\beta, v\beta \in \mathbb{E}\beta$ such that $u\beta \neq v\beta$ so $u\beta(m') \neq v\beta(m')$ for some $m' \in \mathcal{M}$. Now $g\alpha(m) = \beta(m')$ for some $m \in \mathcal{M}$, which implies that $ug\alpha(m) \neq vg\alpha(m)$ and therefore $ug\alpha \neq vg\alpha$ hence $\Phi(u\beta) \neq \Phi(v\beta)$.
3) A direct consequence of 1) and 2).

An immediate corollary of last theorem is following.

Corollary 1. *Let $\mathcal{S}$ be a right self-principally injective semigroup and $a, b \in \mathcal{S}$ then following are equivalent:*

1) If $b\mathcal{S}$ embeds in $a\mathcal{S}$ then $\mathcal{S}b$ is a homomorphic image of $\mathcal{S}a$.

2) If $a\mathcal{S}$ is a homomorphic image of $b\mathcal{S}$ then $\mathcal{S}b$ can be embedded in $\mathcal{S}a$.

3) If $b\mathcal{S} \cong a\mathcal{S}$ then $\mathcal{S}b \cong \mathcal{S}a$.

Theorem 2. *If $\mathcal{M}$ be a right $\mathcal{S}$-act then, for all $\alpha, \beta \in \mathbb{E}$, following are equivalent.*

1) $\mathcal{M}$ is quasi-principally injective.

2) If $Ann_{\mathbb{E}}(\ker\alpha) = \{\beta \in \mathbb{E} : \ker\alpha \subseteq \ker\beta\}$ then $Ann_{\mathbb{E}}(\ker\alpha) = \mathbb{E}\alpha$.

3) If $\ker\beta \subseteq \ker\alpha$ then $\mathbb{E}\alpha \subseteq \mathbb{E}\beta$.

Proof. 1)⇒2) Let $u\alpha \in \mathbb{E}\alpha$, where $u \in \mathbb{E}$. To show that $u\alpha \in Ann_{\mathbb{E}}(\ker\alpha)$, we need to show that $\ker\alpha \subseteq \ker u\alpha$. Let $(x, y) \in \ker\alpha$, we have $\alpha(x) = \alpha(y)$ so $u\alpha(x) = u\alpha(y)$, which implies $(x, y) \in \ker u\alpha$, it follows that $\ker\alpha \subseteq \ker u\alpha$ and so $u\alpha \in Ann_{\mathbb{E}}(\ker\alpha)$. For reverse inclusion, let $\beta \in Ann_{\mathbb{E}}(\ker\alpha)$ so $\ker\alpha \subseteq \ker\beta$. Consider $\alpha' : \mathcal{M} \to \alpha(\mathcal{M})$ and $\beta' : \mathcal{M} \to \beta(\mathcal{M})$induced by α and β. If $i : \alpha(\mathcal{M}) \to \mathcal{M}$ and $j : \beta(\mathcal{M}) \to \mathcal{M}$ are inclusions then $\alpha = i\alpha'$and $\beta = j\beta'$. Since α' is an epimorphism so there exists an $\mathcal{S}$-homomorphism $\Phi : \alpha(\mathcal{M}) \to \beta(\mathcal{M})$ such that $\Phi\alpha' = \beta'$. Since $\mathcal{M}$ is quasi-principally injective so there exists $\psi \in \mathbb{E}$ such that $\psi i = j\Phi$. Finally consider,

$$\psi\alpha = \psi(i\alpha') = (\psi i)\alpha' = (j\Phi)\alpha' = j(\Phi\alpha') = j\beta' = \beta$$

This shows that $\beta \in \mathbb{E}\alpha$. Thus $Ann_{\mathbb{E}}(\ker\alpha) = \mathbb{E}\alpha$.

2)⇒3) Suppose $\ker\beta \subseteq \ker\alpha$. By hypothesis it is sufficient to show that $Ann_{\mathbb{E}}(\ker\alpha) \subseteq Ann_{\mathbb{E}}(\ker\beta)$. Let $u \in Ann_{\mathbb{E}}(\ker\alpha)$ so $\ker\alpha \subseteq \ker u$ then $\ker\beta \subseteq \ker u$ and therefore $u \in Ann_{\mathbb{E}}(\ker\beta)$. Thus $Ann_{\mathbb{E}}(\ker\alpha) \subseteq Ann_{\mathbb{E}}(\ker\beta)$.

3)⇒1) Suppose $\Phi : \alpha(\mathcal{M}) \to \mathcal{M}$ is an $\mathcal{S}$-homomorphism. Now $\Phi\alpha' \in \mathbb{E}$. We can easily see $\ker\alpha \subseteq \ker\Phi\alpha'$. So by hypothesis $\mathbb{E}\Phi\alpha' \subseteq \mathbb{E}\alpha$. Since $\Phi\alpha' \in \mathbb{E}\Phi\alpha' \subseteq \mathbb{E}\alpha$ so $\Phi\alpha' = u\alpha$ for some $u \in \mathbb{E}$, which implies the desired result.

If $Ann^r_{\mathcal{S}}(a) = \{s \in \mathcal{S} : as = \theta\}$ is a right annihilator and $Ann^l_{\mathcal{S}}(a) = \{s \in \mathcal{S} : sa = \theta\}$ a left annihilator then we can have following corollary from Theorem 2.

Corollary 2. *The following are equivalent for a semigroup $\mathcal{S}$.*

a) $\mathcal{S}$ is right self-principally injective.

b) $Ann^l_{\mathcal{S}}(a) = \mathcal{S}a$.

c) $Ann^r_{\mathcal{S}}(b) \subseteq Ann^r_{\mathcal{S}}(a)$ implies that $\mathcal{S}b \subseteq \mathcal{S}a$.

We denote the set of all homomorphisms from a right $\mathcal{S}$-act $\mathcal{M}$ to a right $\mathcal{S}$-act $\mathcal{N}$, by $Hom_{\mathcal{S}}(\mathcal{M}, \mathcal{N})$. Moreover, $Hom_{\mathcal{S}}(\mathcal{M}, \mathcal{N})$ is a right $\mathbb{E}$-act by action $(u, \alpha) \to u\alpha$, the usual composition of functions.

Lemma 2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two right $\mathcal{S}$-acts then $\mathcal{N}$ is $\mathcal{M}$-principally injective iff for all $\alpha \in \mathbb{E}$, $Hom_{\mathcal{S}}(\mathcal{M}, \mathcal{N})\alpha = \{\beta \in Hom_{\mathcal{S}}(\mathcal{M}, \mathcal{N})$: $\ker\alpha \subseteq \ker\beta\}$.*

Proof. *Let* $K = \beta \in Hom_{\mathcal{S}}(\mathcal{M},\mathcal{N}) : \ker\alpha \subseteq \ker\beta\}$. *Let* $\beta \in K$ *then* $ker\alpha \subseteq \ker\beta$. *Define* $\Phi : \alpha(\mathcal{M}) \to \mathcal{N}$ *by* $\Phi(\alpha(m)) = \beta(m)$. *Clearly* Φ *is well-defined and an* $\mathcal{S}$-*homomorphism. Since* $\mathcal{N}$ *is* $\mathcal{M}$-*principally injective so there exists* $\gamma : \mathcal{M} \to \mathcal{N}$ *such that* $\gamma i = \Phi$ *i.e.* γ *extends*Φ, *where* $i : \alpha(\mathcal{M}) \to \mathcal{M}$*is the inclusion. Consider,*

$$\beta(m) = \Phi(\alpha(m)) = \gamma i(\alpha(m)) = \gamma(\alpha(m)) = \gamma\alpha(m)$$

for all $m \in \mathcal{M}$, *hence* $\beta = \gamma\alpha \in Hom_{\mathcal{S}}(\mathcal{M},\mathcal{N})\alpha$. *For the reverse inclusion let* $u\alpha \in Hom_{\mathcal{S}}(\mathcal{M},\mathcal{N})\alpha$. *We claim that* $\ker\alpha \subseteq \ker u\alpha$., *which is sufficient to show that* $u\alpha \in K$. *To do so, let* $(x,y) \in \ker\alpha$ *so* $\alpha(x) = \alpha(y)$ *and therefore* $u\alpha(x) = u\alpha(y)$ *which shows* $(x,y) \in \ker u\alpha$, *hence* $\ker\alpha \subseteq \ker u\alpha$. *Hence* $Hom_{\mathcal{S}}(\mathcal{M},\mathcal{N})\alpha = K$.
For the converse, let $\Phi : \alpha(\mathcal{M}) \to \mathcal{N}$ *be a* $\mathcal{S}$-*homomorphism. Consider the map* $\Phi\alpha \in Hom_{\mathcal{S}}(\mathcal{M},\mathcal{N})$. *Clearly,* $\ker\alpha \subseteq \ker\Phi\alpha$. *Therefore* $\Phi\alpha \in K = Hom_{\mathcal{S}}(\mathcal{M},\mathcal{N})\alpha$ *so* $\Phi\alpha = u\alpha$, *for some* $u \in Hom_{\mathcal{S}}(\mathcal{M},\mathcal{N})$. *Hence* $\mathcal{N}$ *is* $\mathcal{M}$-*principally injective.*

Lemma 3. *Every* X-*cyclic sub-act of* X *is an* $\mathcal{M}$-*cyclic sub-act of* $\mathcal{M}$, *for every* $\mathcal{M}$-*cyclic sub-act* X *of* $\mathcal{M}$.

Proof. Suppose $\mathcal{N}$ is an X-cyclic sub-act of X so $\mathcal{N} = \alpha(X)$ for some $\alpha \in End_{\mathcal{S}}(X)$.Now since X is $\mathcal{M}$-cyclic so $X = \gamma(\mathcal{M})$ for some $\gamma \in \mathbb{E}$, hence $\mathcal{N} = \alpha\gamma(\mathcal{M})$ so $\mathcal{N}$ is $\mathcal{M}$-cyclic.

Theorem 3. *Let* $\mathcal{N}$ *and* $\mathcal{M}$ *be right* $\mathcal{S}$-*acts then* $\mathcal{N}$ *is* $\mathcal{M}$-*principally injective iff* $\mathcal{N}$ *is* $X-$*principally injective for every* $\mathcal{M}$-*cyclic sub-act* X *of* $\mathcal{M}$.

Proof. *Let* $\mathcal{N}$ *be* $\mathcal{M}$-*principally injective and* X *be an* $\mathcal{M}$-*cyclic sub-act of* $\mathcal{M}$, *so* $X = \gamma(\mathcal{M})$, *for some* $\gamma \in \mathbb{E}$. *Let* $\Phi : \alpha(X) \to \mathcal{N}$ *where* $\alpha \in End_{\mathcal{S}}(X)$ *so* $\alpha(X) = \alpha(\gamma(\mathcal{M})) = \alpha\gamma(\mathcal{M})$, *since* $\mathcal{N}$ *is* $\mathcal{M}$-*principally injective so* $\widehat{\Phi} : \mathcal{M} \to \mathcal{N}$ *extends* Φ. *Now* $\widehat{\Phi}/_X = \psi$, *clearly* $\psi : X \to \mathcal{N}$ *extends* Φ, *hence* $\mathcal{N}$ *is* X-*cyclic.*
Conversely assume as mentioned in the statement. Since $i(\mathcal{M}) = \mathcal{M}$, *where* i *is an identity* $\mathcal{S}$-*homomorphism on* $\mathcal{M}$, *so* $\mathcal{M}$, *itself is an* $\mathcal{M}$-*cyclic sub-act of* $\mathcal{M}$. *Thus* $\mathcal{N}$ *is* $\mathcal{M}$-*principally injective.*

The direct product of $\mathcal{S}$-acts is defined in [4]. If $\mathcal{N} = \bigoplus_{i\in I}\mathcal{N}_i$ is the direct sum of right $\mathcal{S}$-acts $\mathcal{N}_i$, for each $i \in I$. Moreover, we treat $\lambda_i : \mathcal{N}_i \to \mathcal{N}$ as a natural $\mathcal{S}$-injection and $\pi_j : \mathcal{N} \to \mathcal{N}_j$ as a natural $\mathcal{S}$-projection such that $\pi_j\lambda_j = i_{\mathcal{N}_j}$, where $i_{\mathcal{N}_j}$ is the identity $\mathcal{S}$-homomorphism on $\mathcal{N}_j$.

Theorem 4. $\mathcal{N} = \bigoplus_{i\in I}\mathcal{N}_i$ *is* $\mathcal{M}$-*principally injective iff each* $\mathcal{N}_i$ *is* $\mathcal{M}$-*principally injective, for all* $i \in I$.

Proof. Suppose $\Phi_j : \alpha(\mathcal{M}) \to \mathcal{N}_j$ is an $\mathcal{S}$-homomorphism for each $j \in I$. Now since $\mathcal{N}$ is $\mathcal{M}$-principally injective so we have $\gamma : \mathcal{M} \to \mathcal{N}$ such that $\gamma i = \lambda_j\Phi_j$, where $i : \alpha(\mathcal{M}) \to \mathcal{M}$ is inclusion. Using $\pi_j\gamma : \mathcal{M} \to \mathcal{N}_j$ it follows that,

$$(\pi_j\gamma)i = \pi_j(\gamma i) = \pi_j(\lambda_j\Phi_j) = (\pi_j\lambda_j)\Phi_j = \Phi_j.$$

Conversely assume that $\Phi : \alpha(\mathcal{M}) \to \mathcal{N}$ is an $\mathcal{S}$-homomorphism. Now $\pi_j \Phi : \alpha(\mathcal{M}) \to \mathcal{N}_j$ is also an $\mathcal{S}$-homomorphism. Since each $\mathcal{N}_j$ is $\mathcal{M}$-principally injective, so we have $\gamma_j : \mathcal{M} \to \mathcal{N}_j$ such that $\gamma_j i = \pi_j \Phi$. Define $\gamma : \mathcal{M} \to \mathcal{N}$ by $\gamma(t) = (\gamma_j(t))$, for $t \in \alpha(\mathcal{M})$. Clearly γ is an $\mathcal{S}$-homomorphism and

$$\gamma(i(t)) = (\gamma_j(i(t))) = (\gamma_j i(t)) = (\pi_j \Phi(t)) = \Phi(t),$$

for all $t \in \alpha(\mathcal{M})$. Thus γ extends Φ, so $\mathcal{N}$ is $\mathcal{M}$-principally injective.

Corollary 3. *$\mathcal{N} = \underset{i \in I}{\oplus} \mathcal{N}_i$ is quasi-principally injective iff each $\mathcal{N}_i$ is quasi-principally injective, for all $i \in I$.*

3. $\mathcal{M}$-principally projective $\mathcal{S}$-acts

In this section we have defined $\mathcal{M}$-principally injective $\mathcal{S}$-acts and characterized them in terms of $\mathcal{M}$-cyclic sub-acts. Further we have also investigated few basic and important properties of the mentioned structure.

3.1. $\mathcal{M}$-principally and quasi-principally Projective $\mathcal{S}$-act

Definition 5. *A right $\mathcal{S}$-act $\mathcal{N}$ is called $\mathcal{M}$-**principally projective** if every $\mathcal{S}$-homomorphism from $\mathcal{N}$ to an $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ can be lifted to an $\mathcal{S}$-homomorphism from $\mathcal{N}$ to $\mathcal{M}$.*

Lemma 4. *Let $\mathcal{M}$ and $\mathcal{N}$ be right $\mathcal{S}$-acts then N is $\mathcal{M}$-principally projective iff $Hom_{\mathcal{S}}(\mathcal{N}, \alpha(\mathcal{M})) = \alpha Hom_{\mathcal{S}}(\mathcal{N}, \mathcal{M})$, for all $\alpha \in \mathbb{E}$.*

Proof. Assume that $\mathcal{N}$ is $\mathcal{M}$-principally projective. Let $\alpha \in \mathbb{E}$ and $\alpha u \in \alpha Hom_{\mathcal{S}}(\mathcal{N}, \mathcal{M})$ where $u \in Hom_{\mathcal{S}}(\mathcal{N}, \mathcal{M})$. Since $\alpha : \mathcal{M} \to \alpha(\mathcal{M})$ so $\alpha u : \mathcal{N} \to \alpha(\mathcal{M})$ this means $\alpha u \in Hom_{\mathcal{S}}(\mathcal{N}, \alpha(\mathcal{M}))$ which gives $Hom_{\mathcal{S}}(\mathcal{N}, \alpha(\mathcal{M})) \supseteq \alpha Hom_{\mathcal{S}}(\mathcal{N}, \mathcal{M})$. For the converse inclusion, let $\Phi \in Hom_{\mathcal{S}}(\mathcal{N}, \alpha(\mathcal{M}))$. Since $\mathcal{N}$ is $\mathcal{M}$ principally projective so we have an $\mathcal{S}$-homomorphism $\gamma : \mathcal{N} \to \mathcal{M}$ such that $\Phi = \alpha\gamma$ which implies $\Phi \in \alpha Hom_{\mathcal{S}}(\mathcal{N}, \mathcal{M})$. Thus equality holds.
Conversely assume that $Hom_{\mathcal{S}}(\mathcal{N}, \alpha(\mathcal{M})) = \alpha Hom_{\mathcal{S}}(\mathcal{N}, \mathcal{M})$ for all $\alpha \in \mathbb{E}$. Let $\Phi : \mathcal{N} \to \alpha(\mathcal{M})$ so by hypothesis $\Phi = \alpha u$ for some $u \in Hom_{\mathcal{S}}(\mathcal{N}, \mathcal{M})$. Hence $\mathcal{N}$ is $\mathcal{M}$-principally projective.

Theorem 5. *If $K \cong \mathcal{N}$, $\mathcal{M} \cong \mathcal{M}'$ and $\mathcal{N}$ is $\mathcal{M}$-principally projective then K is $\mathcal{M}$-principally projective and $\mathcal{N}$ is $\mathcal{M}'$-principally projective.*

Proof. Straight forward.

Lemma 5. *Let $\mathcal{M}$ and $\mathcal{N}$ be two right $\mathcal{S}$-acts then $\mathcal{N}$ is $\mathcal{M}$-principally projective iff $\mathcal{N}$ is X-principally projective for every $\mathcal{M}$-cyclic sub-act X of $\mathcal{M}$.*

Proof. Suppose $\mathcal{N}$ is X-principally projective. As X X is a $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ so $X = \alpha(\mathcal{M})$, for some α. Moreover, $\Phi : \mathcal{N} \to \beta(X)$ be an $\mathcal{S}$-homomorphism, where $\beta \in End(X)$. Now $\beta(X) = \beta\alpha(\mathcal{M})$ so $\beta(X)$ is $\mathcal{M}$-cyclic and since $\mathcal{N}$ is $\mathcal{M}$-principally projective so there exists $\overline{\Phi} : \mathcal{N} \to \mathcal{M}$ such that $(\beta\alpha)\overline{\Phi} = \Phi$. Now consider $\alpha\overline{\Phi} : \mathcal{N} \to X$ which clearly lifts Φ i.e. $\beta(\alpha\Phi) = (\beta\alpha)\overline{\Phi} = \Phi$. Hence $\mathcal{N}$ is X-principally projective. Converse is trivially follows by taking particular $X = \mathcal{M}$.

Theorem 6. *Let $\mathcal{N}$ be an $\mathcal{M}$-principally projective $\mathcal{S}$-act. If ϕ is idempotent(i.e. $\phi^2 = \phi$) then the homomorphic image $\phi(\mathcal{N})$ is also $\mathcal{M}$-principally projective.*

Proof. Let $\gamma : \phi(\mathcal{N}) \to \alpha(\mathcal{M})$ be an $\mathcal{S}$-homomorphism. Consider $\gamma\phi : \mathcal{N} \to \alpha(\mathcal{M})$, is an $\mathcal{S}$-homomorphism, since $\mathcal{N}$ is $\mathcal{M}$-principally projective so there exists $\theta : \mathcal{N} \to \mathcal{M}$ lifting $\gamma\phi$ i.e. $\alpha\theta = \gamma\phi$. Note that $\theta i : \phi(\mathcal{N}) \to \mathcal{M}$, where $i : \phi(\mathcal{N}) \to \mathcal{N}$ is inclusion. It follows that for all $\phi(n) \in \phi(\mathcal{N})$ we have,

$$(\alpha(\theta i))(\phi(n)) = (\alpha\theta)i(\phi(n)) = \alpha\theta(\phi(n)) = \gamma\phi(\phi(n)) = \gamma(\phi(n)),$$

which simply implies that $\alpha(\theta i) = \gamma$. Hence $\phi(\mathcal{N})$ is $\mathcal{M}$-principally projective.

Theorem 7. *Let $\mathcal{M}$ and $\mathcal{N}$ be right $\mathcal{S}$-acts, then $\mathcal{M}$ is $\mathcal{N}$-principally projective and every $\mathcal{N}$-cyclic sub-act of $\mathcal{N}$ is $\mathcal{M}$-principally injective iff $\mathcal{N}$ is $\mathcal{M}$-principally injective and every $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{N}$-principally projective.*

Proof. Assume that $\mathcal{M}$ is $\mathcal{N}$-principally projective and each $\mathcal{N}$-cyclic sub-act of $\mathcal{N}$ is $\mathcal{M}$-principally injective. Since $i(\mathcal{N}) = \mathcal{N}$, where i is the identity on $\mathcal{N}$, so $\mathcal{N}$ is itself $\mathcal{N}$-cyclic sub-act of $\mathcal{N}$ and hence $\mathcal{M}$-principally injective. Let $\alpha(\mathcal{M})$ be an $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$, where $\alpha \in \mathbb{E}$. To show $\alpha(\mathcal{M})$ is $\mathcal{N}$-principally projective, we let $\gamma : \alpha(\mathcal{M}) \to \beta(\mathcal{N})$ be an $\mathcal{S}$-homomorphism, where β is an endomorphism on $\mathcal{N}$. Since $\beta(\mathcal{N})$ is $\mathcal{M}$-principally injective so there exists $\theta : \mathcal{M} \to \beta(\mathcal{N})$ such that $\theta i = \gamma$, where $i : \alpha(\mathcal{M}) \to \mathcal{M}$ is inclusion. Now since $\mathcal{M}$ is $\mathcal{N}$-principally projective so there exists $\phi : \mathcal{M} \to \mathcal{N}$ such that $\beta\phi = \theta$. Now consider $\phi i : \alpha(\mathcal{M}) \to \mathcal{N}$ which lifts γ, i.e. $\beta(\phi i) = (\beta\phi)i = \theta i = \gamma$. Hence $\alpha(\mathcal{M})$ is $\mathcal{N}$-principally projective.
Conversely assume that $\mathcal{N}$ is $\mathcal{M}$-principally injective and every $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{N}$-principally projective. Trivially $\mathcal{M}$ is $\mathcal{N}$-principally projective. Let $\beta(\mathcal{N})$ be $\mathcal{N}$-cyclic sub-act of $\mathcal{N}$ and $\gamma : \alpha(\mathcal{M}) \to \beta(\mathcal{N})$ be an $\mathcal{S}$-homomorphism. Since $\alpha(\mathcal{M})$ is $\mathcal{N}$-principally projective so there exists $\theta : \alpha(\mathcal{M}) \to \mathcal{N}$ such that $\beta\theta = \gamma$. Now since $\mathcal{N}$ is $\mathcal{M}$-principally injective so there exists $\phi : \mathcal{M} \to \mathcal{N}$ such that $\phi i = \theta$. Consider $\mathcal{S}$-homomorphism $\beta\phi : \alpha(\mathcal{M}) \to \mathcal{N}$ which extends γ i.e. $(\beta\phi)i = \beta(\phi i) = \beta\theta = \gamma$. Hence $\beta(\mathcal{N})$ is $\mathcal{M}$-principally injective.

Theorem 8. *$\mathcal{N} = \bigoplus_{i \in I} \mathcal{N}_i$ is $\mathcal{M}$-principally projective where each $\mathcal{N}_i$ is $\mathcal{M}$-principally projective for all $i \in I$.*

Proof.
Follows on the same lines of Theorem 4.

Definition 6. *For right S-acts $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M}$ is called $\mathcal{N}$-Projective or projective relative to $\mathcal{N}$, if every right S-act C, every homomorphism $f : \mathcal{M} \to \mathcal{C}$ can be lifted w.r.t every $g : \mathcal{N} \to \mathcal{C}$ i.e. there exists a homomorphism $h : \mathcal{M} \to \mathcal{N}$ such that $f = gh$.*
Moreover, $\mathcal{M}$ is called $\mathcal{N}$-injective right S-act, if for any Sub-act $\mathcal{C}$ of $\mathcal{N}$, and any homormorphism $f : \mathcal{C} \to \mathcal{M}$ there exists a homormophism $g : \mathcal{N} \to \mathcal{M}$ extending f.

Theorem 9. *Let $\mathcal{M}$ and $\mathcal{N}$ be two right $\mathcal{S}$-acts, then $\mathcal{M}$ is $\mathcal{N}$-injective and each sub-act of $\mathcal{N}$ is $\mathcal{M}$-principally projective iff $\mathcal{N}$ is $\mathcal{M}$-principally projective and each $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{N}$-injective.*

Proof. Assume that $\mathcal{M}$ is $\mathcal{N}$-injective and each sub-act of $\mathcal{N}$ is $\mathcal{M}$-principally projective. Since $\mathcal{N}$ is a sub-act of itself and $\mathcal{M}$-principally projective. Let $\alpha(\mathcal{M})$ be an $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$, where $\alpha \in \mathbb{E}$. To show $\alpha(\mathcal{M})$ is $\mathcal{N}$-injective, we let $\beta : \mathcal{N}' \to \alpha(\mathcal{M})$ be an $\mathcal{S}$-homomorphism, where $\mathcal{N}'$ is a sub-act of $\mathcal{N}$. Since each sub-act of $\mathcal{N}$ is $\mathcal{M}$-principally projective so $\mathcal{N}'$ is also $\mathcal{M}$-principally projective. Therefore there exists $\gamma : \mathcal{N}' \to \mathcal{M}$ such that $\alpha\gamma = \beta$. Since $\mathcal{M}$ is $\mathcal{N}$-injective so there exists $\phi : \mathcal{N} \to \mathcal{M}$ such that $\phi i = \gamma$. Now consider $\alpha\phi : \mathcal{N} \to \alpha(\mathcal{M})$ and $(\alpha\phi)i = \alpha(\phi i) = \alpha\gamma = \beta$. Hence $\alpha(\mathcal{M})$ is $\mathcal{N}$-injective. Conversely assume that $\mathcal{N}$ is $\mathcal{M}$-principally projective and each $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{N}$-injective. $\mathcal{M}$ itself is an $\mathcal{M}$-cyclic sub-act and $\mathcal{N}$-injective. Let $\mathcal{N}'$ be a sub-act of $\mathcal{N}$ and $\beta : \mathcal{N}' \to \alpha(\mathcal{M})$ be an $\mathcal{S}$-homomorphism. Since $\alpha(\mathcal{M})$ is $\mathcal{N}$-injective so there exists $\gamma : \mathcal{N} \to \alpha(\mathcal{M})$ such that $\gamma i = \beta$, where $i : \mathcal{N}' \to \mathcal{N}$ is inclusion. Since $\mathcal{N}$ is $\mathcal{M}$-principally projective so we have $\phi : \mathcal{N} \to \mathcal{M}$ such that $\alpha\phi = \gamma$. Now keep in view $\phi i : \mathcal{N}' \to \mathcal{M}$ and $\alpha(\phi i) = (\alpha\phi)i = \gamma i = \beta$. Hence $\mathcal{N}'$ is $\mathcal{M}$-principally projective.

Theorem 10. *The following statements are equivalent for a projective right $\mathcal{S}$-act $\mathcal{M}$.*
1) Every $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is projective.
2) Every factor $\mathcal{S}$-act of an $\mathcal{M}$-principally injective $\mathcal{S}$-act is $\mathcal{M}$-principally injective.
3) Every factor $\mathcal{S}$-act of an injective $\mathcal{S}$-act is $\mathcal{M}$-principally injective.

Proof. 1)⇒2) Let $\mathcal{N}$ be an $\mathcal{M}$-principally injective $\mathcal{S}$-act and ρ be a congruence on $\mathcal{N}$. Let $\phi : \alpha(\mathcal{M}) \to \mathcal{N}/\rho$ be an $\mathcal{S}$-homomorphism, where $\alpha \in \mathbb{E}$. Since $\alpha(\mathcal{M})$ is projective so there exists $\overline{\phi} : \alpha(\mathcal{M}) \to \mathcal{N}$ such that $\pi\overline{\phi} = \phi$, where $\pi : \mathcal{N} \to \mathcal{N}/\rho$ is a canonical epimorphism. Now as $\mathcal{N}$ is $\mathcal{M}$-principally injective so there exists $\gamma : \mathcal{M} \to \mathcal{N}$ which extends ϕ. Consider $\pi\gamma : \mathcal{M} \to \mathcal{N}/\rho$ which extends ϕ. Hence $\mathcal{N}/\rho$ is $\mathcal{M}$-principally injective.

2)⇒3) Let $\mathcal{N}$ be an injective right $\mathcal{S}$-act and ρ be congruence on $\mathcal{N}$. Let $\phi : \alpha(\mathcal{M}) \to \mathcal{N}$ be an $\mathcal{S}$-homomorphism and $\mathcal{N}$ be injective, so there exists $\gamma : \mathcal{M} \to \mathcal{N}$ extending ϕ. By hypothesis $\mathcal{N}$ is $\mathcal{M}$-principally injective. Thus $\mathcal{N}/\rho$ is $\mathcal{M}$-principally injective.

3)⇒1) Let $\alpha(\mathcal{M})$ be an $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$, $h : A \to B$ be an $\mathcal{S}$-epimorphism and $\beta : \alpha(\mathcal{M}) \to B$ be an $\mathcal{S}$-homomorphism. by corollary 2.1.2 of [1] there exists an injective $\mathcal{S}$-act Q in which A can be embedded. Now define $\phi : B = h(A) \to A/K$ by $\phi(h(a)) = [a]_K$, where $K = \ker h$, the kernel congruence on A induced by h. Clearly ϕ is an $\mathcal{S}$-isomorphism. Let $K' = K \cup \triangle_Q$ be a congruence on Q, where $\triangle_Q$ is a diagonal congruence on Q. Moreover, A/K is a sub-act of Q/K'. Thus $\beta : \alpha(\mathcal{M}) \to B$ can also be viewed as $\beta : \alpha(\mathcal{M}) \to Q/K'$. By hypothesis Q/K' is $\mathcal{M}$-principally injective, so there exists $\overline{\beta} : \mathcal{M} \to Q/K'$ which extends β. Since $\mathcal{M}$ is projective so there exists $\gamma : \mathcal{M} \to Q$ such that $\pi\gamma = \overline{\beta}$, where $\pi : Q \to Q/K'$ is a canonical epimorphism i.e. γ lifts $\overline{\beta}$. Clearly $\gamma(\alpha(\mathcal{M}))$ lies in the domain of h, so we must have $\gamma(\alpha(\mathcal{M})) \subseteq A$. Thus we have lifted β. This completes the proof of the theorem.

Corollary 4. *The following statements are equivalent for a semigroup $\mathcal{S}$.*
1) $\mathcal{S}$ is a right PP-Semigroup.
2) Every factor $\mathcal{S}$-act of $\mathcal{PM}$-injective $\mathcal{S}$-act is PM-injective.
3) Every factor $\mathcal{S}$-act of an injective $\mathcal{S}$-act is PM-injective.

Theorem 11. *Let $\mathcal{N}$ be an $\mathcal{M}$-principally projective $\mathcal{S}$-act. If $\mathcal{M}_0$ is either an $\mathcal{S}$-homomorphic image or an $\mathcal{S}$-sub-act of $\mathcal{M}$, then $\mathcal{N}$ is $\mathcal{M}_0$-principally projective.*

Proof. If $\mathcal{N}$ is a homomorphic image of $\mathcal{M}_0$ then the result follows directly from the Lemma 5. Now assume that $\mathcal{M}_0$ is $\mathcal{S}$- sub-act of $\mathcal{M}$. To show that $\mathcal{N}$ is $\mathcal{M}_0$-principally projective, we let $\phi : \mathcal{N} \to \alpha(\mathcal{M}_0)$ be an $\mathcal{S}$-homomorphism, where $\alpha \in End(\mathcal{M}_0)$. By Lemma 1, it is clear that $\alpha(\mathcal{M}_0) \simeq \mathcal{M}_0/\ker\alpha$. Consider $\pi_0 : \mathcal{M}_0 \to \mathcal{M}_0/\ker\alpha$ be the natural epimorphism. Let ρ be the congruence on $\mathcal{M}$ defined by $\rho = \ker\alpha \cup \triangle_{\mathcal{M}}$, where $\triangle_{\mathcal{M}}$ is a diagonal congruence on $\mathcal{M}$. Let $\pi : \mathcal{M} \to \mathcal{M}/\rho \simeq \gamma(\mathcal{M})$ for some endomorphism γ on $\mathcal{M}$. We can take another view of ϕ, notice that $\alpha(\mathcal{M}_0) \simeq \mathcal{M}_0/\ker\alpha \subset \mathcal{M}/\rho \simeq \gamma(\mathcal{M})$ thus we can treat ϕ as an $\mathcal{S}$-homomorphism $\phi : \mathcal{N} \to \gamma(\mathcal{M})$, also surely π extends π_0. Since $\mathcal{N}$ is $\mathcal{M}$-principally projective so there exists $\beta : \mathcal{N} \to \mathcal{M}$ such that $\pi\beta = \phi$. But

$$\pi(\beta(\mathcal{N})) = \pi\beta(\mathcal{N}) = \phi(\mathcal{N}) \subset \alpha(\mathcal{M}_0) \simeq \mathcal{M}_0/\ker\alpha,$$

which clearly shows that $\beta(\mathcal{N}) \subset \mathcal{M}_0$. Thus we can treat β from $\mathcal{N}$ to $\mathcal{M}_0$. Since π and π_0 agrees at $\mathcal{M}_0$, so

$$\pi_0\beta(n) = \pi_0(\beta(n)) = \pi(\beta(n)) = \pi\beta(n) = \phi(n)$$

for all $n \in \mathcal{N}$, which implies that $\pi_0\beta = \phi$. Hence $\mathcal{N}$ is $\mathcal{M}_0$-principally projective.

Theorem 12. *Let $\mathcal{M}$ be a projective $\mathcal{S}$-act and $\mathbb{E} = \mathbb{E}(\mathcal{M})$ be an injective hull of $\mathcal{M}$. If $\mathbb{E}$ is completely $\mathcal{M}$-principally injective i.e. each factor $\mathcal{S}$-act of $\mathbb{E}$ is $\mathcal{M}$-principally injective, then each $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{M}$-principally projective.*

Proof. Let $\alpha(\mathcal{M})$ be an $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$, $\alpha \in End_{\mathcal{S}}(\mathcal{M})$ and $i : \alpha(\mathcal{M}) \to \mathcal{M}$ be an inclusion map. Consider $\pi : \mathbb{E} \to \mathbb{E}/\rho$, where ρ is congruence on $\mathbb{E}$. Let $\gamma : \alpha(\mathcal{M}) \to \beta(\mathbb{E})$, where β is an endomorphism on $\mathbb{E}$. Clearly $\beta(\mathbb{E}) \simeq \mathbb{E}/\ker\beta$. Since $\mathbb{E}$ is completely $\mathcal{M}$-principally injective which implies $\mathbb{E}/\ker\beta$ is $\mathcal{M}$-principally injective. Therefore, there exists $\phi : \mathcal{M} \to \mathbb{E}/\ker\beta \simeq \beta(\mathbb{E})$, such that $\phi i = \gamma$. Since $\mathcal{M}$ is projective so there exists $\psi : \mathcal{M} \to \mathbb{E}$ such that $\beta\psi = \phi$. Let $\theta := \psi i : \alpha(\mathcal{M}) \to \mathbb{E}$ so $\beta\theta = \beta\psi i = \phi i = \gamma$. Thus $\alpha(\mathcal{M})$ is $\mathbb{E}$-principally projective. Now since $\mathcal{M}$ can be embedded in $\mathbb{E}$ so we can treat $\mathcal{M}$ as the sub-act of $\mathbb{E}$ and so by Theorem 11, it follows that $\alpha(\mathcal{M})$ is $\mathcal{M}$-principally projective.

Theorem 13. *a) Let $\mathcal{M}$ be a right $\mathcal{S}$-act, if every $\mathcal{M}$-cyclic sub-act $\alpha(\mathcal{M})$ (where $\alpha \in End_{\mathcal{S}}(\mathcal{M})$) of $\mathcal{M}$ is $\mathcal{A}$-principally projective and $\mathcal{A}$ is $\mathcal{M}$-principally injective then every $\mathcal{A}$-cyclic sub-act $\beta(\mathcal{A})$ (where $\beta \in End_{\mathcal{S}}(A)$) of $\mathcal{A}$ is $\mathcal{M}$-principally injective.*
b) Let $i : \beta(A) \to A$ (where $\beta \in End_{\mathcal{S}}(A)$) be an inclusion map and $\mathcal{M}$ be an $\mathcal{S}$-act. If every $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{A}$-principally injective and $\mathcal{A}$ is $\mathcal{M}$-principally projective then $\beta(\mathcal{A})$ is $\mathcal{M}$-principally projective.

Proof. a) Let $i : \alpha(\mathcal{M}) \to \mathcal{M}$ be an inclusion map and let $\gamma : \alpha(\mathcal{M}) \to \beta(\mathcal{A})$ be an $\mathcal{S}$-homomorphism, where $\beta \in End_{\mathcal{S}}(A)$. Since $\alpha(\mathcal{M})$ is A-principally projective so there exists $h : \alpha(\mathcal{M}) \to A$ such that $\beta h = \gamma$. Since $\mathcal{A}$ is $\mathcal{M}$-principally injective so there exists $\lambda : \mathcal{M} \to \mathcal{A}$ such that $\lambda i = h$. Let $\mu = \beta\lambda : \mathcal{M} \to \beta(\mathcal{A})$, since $\mu i = \beta\lambda i = \beta h = \gamma$. Hence $\beta(\mathcal{A})$ is $\mathcal{M}$-principally injective.
b) To show that $\beta(\mathcal{A})$ is $\mathcal{M}$-principally projective, we let $\gamma : \beta(A) \to \alpha(\mathcal{M})$ be an $\mathcal{S}$-homomorphism, where $\alpha \in \mathbb{E}$. Keeping γ in view, since $\alpha(\mathcal{M})$ is $\mathcal{A}$-principally injective so there exists $\phi : \mathcal{A} \to \alpha(\mathcal{M})$ such that $\phi i = \gamma$. Since $\mathcal{A}$ is $\mathcal{M}$-principally projective so there exists $\theta : A \to \mathcal{M}$ such that $\alpha\theta = \phi$. Consider $\theta i : \beta(A) \to \mathcal{M}$ and $\alpha(\theta i) = (\alpha\theta)i = \phi i = \gamma$. Hence $\beta(\mathcal{A})$ is $\mathcal{M}$-principally projective.

3.2. A Note on Co-hereditary $\mathcal{S}$-acts

Remark 1. *Usually co-hereditary $\mathcal{S}$-acts are defined as those $\mathcal{S}$-acts whose every proper factor $\mathcal{S}$-act is injective. Keeping in view Lemma 1 we can redefine the Co-hereditary $\mathcal{S}$-acts as following.*

Definition 7. *A right $\mathcal{S}$-act $\mathcal{M}$ is called **Co-hereditary** $\mathcal{S}$-act, if each $\mathcal{M}$-cyclic sub-act is injective.*

Definition 8. *A right $\mathcal{S}$-act $\mathcal{M}$ is $\mathcal{N}$**PI-Co-hereditary** if every $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{N}$-principally injective.*

Definition 9. *A right $\mathcal{S}$-act is quasi **PI-Co-hereditary** if every $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{M}$-principally injective.*

Remark 2. *Co-hereditary $\to$ $\mathcal{N}$PI-co-hereditary $\to$PI- co-hereditary.*

Theorem 14. *If an $\mathcal{S}$-act $\mathcal{M}$ is API-co-hereditary, then every $\mathcal{A}$-cyclic sub-act of an $\mathcal{M}$-principally projective $\mathcal{S}$-act $\mathcal{A}$ is $\mathcal{M}$-principally projective.*

Proof. Let $\alpha(A)$ be an $\mathcal{A}$-cyclic sub-act of $\mathcal{A}$. We show it is $\mathcal{M}$-principally projective. By definition every $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{A}$-principally injective. Since $\mathcal{M}$ is itself $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ so is $\mathcal{A}$-principally injective Thus by Theorem 13 again, $\alpha(\mathcal{A})$ is $\mathcal{M}$-principally projective.

Corollary 5. *If an $\mathcal{S}$-act $\mathcal{M}$ is PI-cohereditary and $\mathcal{M}$-principally projective, then every $\mathcal{M}$-cyclic sub-act of $\mathcal{M}$ is $\mathcal{M}$-principally projective.*

4. Semi-projective $\mathcal{S}$-acts

Definition 10. *A right $\mathcal{S}$-act $\mathcal{M}$ is called a **Semi-projective** $\mathcal{S}$-act if $\alpha\mathbb{E} = Hom_{\mathcal{S}}(\mathcal{M}, \alpha(\mathcal{M}))$ for all $\alpha \in \mathbb{E}$.*

Theorem 15. *Let $\mathcal{M}$ be an $\mathcal{S}$-act then the following conditions are equivalent.*
1) $\mathcal{M}$ is quasi-principally projective.
2) $\mathcal{M}$ is semi-projective.
3) For $\alpha, \beta \in \mathbb{E}$ if $\alpha(\mathcal{M}) \subseteq \beta(\mathcal{M})$ then $\alpha\mathbb{E} \subseteq \beta\mathbb{E}$.

Proof. 1)⇒2) follows directly from Lemma 4, for $\mathcal{N} = \mathcal{M}$.
2)⇒3) let $\alpha(\mathcal{M}) \subseteq \beta(\mathcal{M})$ then for $u \in Hom_{\mathcal{S}}(\mathcal{M}, \alpha(\mathcal{M}))$ since $\alpha(\mathcal{M}) \subseteq \beta(\mathcal{M})$ so we may also view u as an $\mathcal{S}$-homomorphism $u : \mathcal{M} \to \beta(\mathcal{M})$ i.e. $u \in Hom_{\mathcal{S}}(\mathcal{M}, \beta(\mathcal{M}))$. So we have $Hom_{\mathcal{S}}(\mathcal{M}, \alpha(\mathcal{M})) \subseteq Hom_{\mathcal{S}}(\mathcal{M}, \beta(\mathcal{M}))$ and therefore by hypothesis $\alpha\mathbb{E} \subseteq \beta\mathbb{E}$.
3)⇒1) Consider $\Phi : \mathcal{M} \to \alpha(\mathcal{M})$ an $\mathcal{S}$-homomorphism, so $\Phi(\mathcal{M}) \subseteq \alpha(\mathcal{M})$ by hypothesis $\Phi\mathbb{E} \subseteq \alpha\mathbb{E}$. Since $\Phi \in \Phi\mathbb{E} \subseteq \alpha\mathbb{E}$ so $\Phi = \alpha u$, for some $u \in \mathbb{E}$. Hence $\mathcal{M}$ is quasi-principally projective.

Theorem 16. *Let $\mathcal{M}$ be a semi-projective right $\mathcal{S}$-act and $\alpha, \beta \in \mathbb{E}$. Then:*
1) If $\alpha(\mathcal{M})$ embeds into $\beta(\mathcal{M})$ then $\alpha\mathbb{E}$ can be embedded into $\beta\mathbb{E}$.
2) If $\beta(\mathcal{M})$ is a homomorphic image of $\alpha(\mathcal{M})$ then $\beta\mathbb{E}$ is a homomorphic image of $\alpha\mathbb{E}$.
3) If $\alpha(\mathcal{M}) \cong \beta(\mathcal{M})$ then $\alpha\mathbb{E} \cong \beta\mathbb{E}$.

Proof. 1) Let $f : \alpha(\mathcal{M}) \to \beta(\mathcal{M})$ be an $\mathcal{S}$-homomorphism. Since $\mathcal{M}$ is semi projective so $\beta\mathbb{E} = Hom_{\mathcal{S}}(\mathcal{M}, \beta(\mathcal{M}))$. As $f\alpha : \mathcal{M} \to \beta(\mathcal{M})$, so $f\alpha \in Hom_{\mathcal{S}}(\mathcal{M}, \beta(\mathcal{M})) = \beta\mathbb{E}$ and so $f\alpha = \beta g$, for some $g \in \mathbb{E}$. Define $\phi : \alpha\mathbb{E} \to \beta\mathbb{E}$ by $\phi(\alpha u) = \beta gu$ for $u \in \mathbb{E}$. We can see that ϕ is well-defined. Indeed, for any $u, v \in \mathbb{E}$ such that $\alpha u = \alpha v$ we have $f\alpha u = f\alpha v$ implies $\beta gu = \beta gv$ and so $\phi(\alpha u) = \phi(\alpha v)$. Clearly $\mathbb{E}$-homomorphism. To show injectivity we take $\phi(\alpha u) = \phi(\alpha v)$ then $\beta gu = \beta gv$. As $f\alpha = \beta g$ therefore $f\alpha u = f\alpha v$. Since f is 1-1 $\alpha u = \alpha v$. Hence ϕ is embbedding.

2) Let f, g and ϕ be as in part 1). For $\beta u \in \beta\mathbb{E}$ there exists $\psi \in Hom_{\mathcal{S}}(\mathcal{M}, \beta(\mathcal{M}))$ such that $\beta u = \psi$. For $m \in \mathcal{M}$ we have $\beta u(m) = \psi(m) \in \beta(\mathcal{M})$ so there exist m' such that $\beta u(m) = \psi(m) = \beta(m')$. As $f\alpha(\mathcal{M}) = \beta(\mathcal{M})$ so there exists m'' such that

$$\beta u(m) = \psi(m) = \beta(m') = f\alpha(m'') = \beta g(m'').$$

Hence we can define $\gamma : \mathcal{M} \to \mathcal{M}$ by $\gamma(m) = m''$,for $m, m'' \in \mathcal{M}$, whenever $\beta u(m) = \beta g(m'')$. Let us see that γ is well-defined.
Let $m_1 = m_2$, where $\gamma(m_1) = m_1''$ and $\gamma(m_2) = m_2''$. This holds only when $\beta u(m_1) = \beta g(m_1'')$ and $\beta u(m_2) = \beta g(m_2'')$, $\beta u(m_2) = \beta u(m_1) = \beta g(m_1'')$, which implies $\gamma(m_2) = m_1'' = \gamma(m_1)$, hence γ is well defined. Moreover $\beta u(m) = \beta g(m'') = \beta g\gamma(m)$ for all $m \in \mathcal{M}$. Thus $\beta u = \beta g\gamma = \phi(\alpha\gamma)$ so ϕ is epimorphism.
3) Follows directly form 1) and 2).

Theorem 17. *Let $\mathcal{M}$ be a semi-projective(quasi-principally projective) right $\mathcal{S}$-act then there is a one-one correspondence between $\mathcal{M}$-cyclic sub-acts of $\mathcal{M}$ and principal right ideals of E.*

Proof. Let $\widetilde{A}$ be a collection of all $\mathcal{M}$-cyclic sub-acts of $\mathcal{M}$ and $\widetilde{B}$ be a collection all principal right ideals of E. Define $\Phi : \widetilde{A} \to \widetilde{B}$ by $\Phi(\alpha(\mathcal{M})) = \alpha\mathbb{E}$. Now Φ is well defined because if $\alpha(\mathcal{M}) = \beta(\mathcal{M})$ for $\alpha, \beta \in \mathbb{E}$, then $Hom_{\mathcal{S}}(\mathcal{M}, \alpha(\mathcal{M})) = Hom_{\mathcal{S}}(\mathcal{M}, \beta(\mathcal{M}))$, since $\mathcal{M}$ is semi-projective so $\alpha\mathbb{E} = \beta\mathbb{E}$ and $\Phi(\alpha(\mathcal{M})) = \Phi(\beta(\mathcal{M}))$. To show injectivity we let $\Phi(\alpha(\mathcal{M})) = \Phi(\beta(\mathcal{M}))$ so $\alpha\mathbb{E} = \beta\mathbb{E}$ and $Hom_{\mathcal{S}}(\mathcal{M}, \alpha(\mathcal{M})) = Hom(\mathcal{M}, \beta(\mathcal{M}))$ which clearly implies $\alpha(\mathcal{M}) = \beta(\mathcal{M})$. Surjection is trivial.

Theorem 18. *There is a one-one correspondence between $f \in Hom_{\mathcal{S}}(\alpha(\mathcal{M}), \mathcal{M})$ and $F \in Hom_{\mathcal{S}}(\alpha\mathbb{E}, \mathbb{E})$ with $\ker F(\alpha) \supset \ker\alpha$ in such away that $F(\alpha u) = f\alpha u$, for all $u \in \mathbb{E}$ and $f(\alpha(m)) = F(\alpha)(m)$, for all $\alpha \in \mathbb{E}$ and $m \in \mathcal{M}$.*

Proof. Let us fix $\alpha \in \mathbb{E}$. We claim that for every $f \in Hom_{\mathcal{S}}(\alpha(\mathcal{M}), \mathcal{M})$. we can define a unique $F : \alpha\mathbb{E} \to \mathbb{E}$ by $F(\alpha u) = f\alpha u$, for all $u \in \mathbb{E}$. Also we can see that $\ker\ F(\alpha) \supset \ker\alpha$. Indeed, for $(x, y) \in \ker\alpha$ we have $\alpha(x) = \alpha(y)$, which implies $f\alpha(x) = f\alpha(y)$ and therefore $F(\alpha)(x) = F(\alpha)(y)$ i.e.$(x, y) \in \ker F(\alpha)$.

Hence we can define a map $\Phi : Hom_{\mathcal{S}}(\alpha(\mathcal{M}), \mathcal{M}) \to Hom_{\mathcal{S}}(\alpha\mathbb{E}, \mathbb{E})$ as $\Phi(f) = F$. We will show that Φ is the required one-one correspondence. Let us begin by proving that Φ is well defined. Let $f, f' \in Hom_{\mathcal{S}}(\alpha(\mathcal{M}), \mathcal{M})$ such that $f = f'$. Then $f\alpha u = f'\alpha u$ and so $F(\alpha u) = F'(\alpha u)$,for all $\alpha u \in \alpha\mathbb{E}$. Hence $F = F'$ i.e. $\Phi(f) = \Phi(f')$.
To show that Φ is 1-1. Let $f, f' \in Hom_{\mathcal{S}}(\alpha(\mathcal{M}), \mathcal{M})$ such that $\Phi(f) = \Phi(f')$ i.e. $F = F'$. This implies that $F(\alpha u) = F'(\alpha u)$, for all $\alpha u \in \alpha\mathbb{E}$. Therefore $f\alpha u = f'\alpha u$, for all $u \in \mathbb{E}$. In particular for $u = i_{\mathcal{M}}$, it follows that $f\alpha = f'\alpha$ and hence $f(\alpha(m)) = f'(\alpha(m))$ for all $\alpha(m) \in \alpha(\mathcal{M})$, and $f = f'$. Hence Φ is 1-1. To see surjectivity, let $F \in Hom_{\mathcal{S}}(\alpha\mathbb{E}, \mathbb{E})$. Define $f(\alpha\mathcal{M}(m)) = F(\alpha)(m)$ for all $m \in \mathcal{M}$. Clearly f is $\mathcal{S}$-homomorphism and $\Phi(f) = F$. Thus we established the 1-1 correspondence.

Theorem 19. *For a semi-projective $\mathcal{S}$-act and $\alpha \in \mathbb{E}$, $\alpha(\mathcal{M})$ is simple. Converse is true for those $\mathcal{S}$-acts for which $m\mathcal{S}$ is $\mathcal{M}$-cyclic for all $m \in \mathcal{M}$.*

Proof. Let $\alpha(\mathcal{M})$ be simple. Suppose contrary that $\alpha\mathbb{E}$ is not simple, so there exists $\gamma \in \mathbb{E}$ such that $\theta \neq \alpha\gamma\mathbb{E} \subsetneq \alpha\mathbb{E}$, and therefore $\alpha\gamma(\mathcal{M}) \subsetneq \alpha(\mathcal{M})$ contradiction. Hence $\alpha\mathbb{E}$ is simple. conversely assume that $\alpha\mathbb{E}$ is simple and $\mathcal{M}$ is, as mentioned in the theorem. Let $m\mathcal{S} = \gamma(\mathcal{M})$, $\gamma \in \mathbb{E}$. Now $\theta \neq \alpha\gamma(\mathcal{M}) \subsetneq \alpha(\mathcal{M}) \to \theta \neq \alpha\gamma\mathbb{E} \subsetneq \alpha\mathbb{E}$ contradiction. Hence the result.

Theorem 20. *If $\mathcal{N} = \bigoplus\limits_{i\in I}\mathcal{N}_i$ is quasi-principally projective (respectively semi-projective) then each $\mathcal{N}_i$ is quasi-principally projective (respectively semi-projective), for all $i \in I$.*

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