EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 3, No. 2, 2010, 282-294
ISSN 1307-5543 – www.ejpam.com

Estimates for Resonant Frequencies Under Boundary Deformation in Multi-dimensional Space

Abdessatar Khelifi* and M. Nour. Shamma

Department of mathematics & Faculty of Sciences & Arts, Al Gassim University, Al-Rass Province, Kingdom of Saudi Arabia

Abstract. In multi-dimensional space, we address the integral equation method to investigate the interplay between the geometry, boundary conditions and the properties of the resonant frequencies and their associated eigenfunctions under boundary variations of domain. We provide a rigorous derivation of asymptotic expansions for eigenfunctions and we establish error estimations for both resonant frequencies and eigenfunctions of the Helmholtz eigenvalue problem.

2000 Mathematics Subject Classifications: 35C15, 35P05, 45C05

Key Words and Phrases: Resonant frequencies, Boundary integral equation, Error estimates

1. Introduction

The resonant frequencies may evolve under shape deformation, as separated, distinct eigenvalues of the Helmholtz eigenvalue problem. But, the main difficulty in solving eigenvalue problems relates to the continuation of multiple eigenvalues of the unperturbed configuration. The properties of eigenvalue problems under shape deformation have been the subject of comprehensive studies [9, 22] and the area continues to carry great importance to this day [4, 7, 6, 8, 12, 13, 14, 15]. A substantial portion of these investigations relate to properties of smoothness and analyticity of eigenvalues and eigenfunctions with respect to perturbations. Bruno and Reitich have presented in [4, Theorem 2, p.172 and Section 3, pp.180-183] some explicit constructions of high-order boundary perturbation expansions for eigenelements in two dimensions. Their algorithm is based on certain properties of joint analytic dependence on the boundary perturbations and spatial variables of the eigenfunctions. In a series of papers [19]-[21], Ozawa derived the leading-order term in the asymptotic expansions of simple eigenvalues in domain with a specific geometry. Nevertheless, in our paper we remove the condition that eigenvalue is simple and provide more accurate asymptotic expansions for eigenfunctions in domain with more general shape. Recently, Lanza de Cristoforis

*Corresponding author.

Email addresses: abdessatar.khelifi@fsb.rnu.tn (A. Khelifi), shamman01@yahoo.com. (M. Shamma)

and Lamberti have developed in [13] some preliminary abstract results for the dependence of the eigenvalues upon perturbation. Their applications to the Dirichlet eigenvalue problem for the Laplace operator appear clearly in Section 3 of their paper and in Theorem 3.21 when they justify the analyticity result for some symmetric functions of eigenvalues. Our analysis and uniform asymptotic formulas of the eigenfunctions, which are represented by the single-layer potential involving the Green function, are considerably different from those in [12, 13, 10]. Next, Our method differ, essentially, from the classical methods used to study the analytic dependence of the eigenfunctions of a real or complex parameter and used to give the asymptotic formulae for the eigenvalues.
The main goal of this paper is to justify and to give formulae for the convergence estimates for both resonant frequencies and eigenfunctions associated to Helmholtz eigenvalue oroblem. Compared to papers in this fields [15, 17, 18], one can notice that our results in section 4, are important and give an idea to evaluate the speed of convergence.
The paper is organized as follows. In Section 2 we describe the central problem in this work, and we remember some well-known results concerning the analyticity of the eigenvalues with respect to ϵ. In Section 3 we develop a boundary integral formulation for solving the eigenvalue problem (2). From results found in [11] in two dimensional space, we end this Section by presenting the main theorem which gives the analyticity and the uniform asymptotic expansion for the eigenfunctions. Section 4 contains the main results of our paper which are deeply based on the Osborn's theorem. We then prove some error estimates for the convergence of resonant frequencies.

2. Problem Description

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a connected Lipschitz boundary $\partial\Omega$ and ν denotes the unit outward normal to $\partial\Omega$. Let $T_i > 0$, for all $i \in \{1, \cdots, d-1\}$ and let $\gamma(t), \quad \beta(t) : t = (t_1, \cdots, t_{d-1}) \in [0, T_1] \times \cdots \times [0, T_{d-1}] \to \mathbb{R}^d$, be two analytic, T_i-periodic (in each composite t_i) functions and satisfying the following assumption:

$$\langle \gamma'(t), \beta(t) \rangle = 0, \quad \text{for all} t = (t_1, \cdots, t_{d-1}) \in [0, T_1] \times \cdots \times [0, T_{d-1}]. \tag{1}$$

where $\langle ., . \rangle$ denotes the usual product scalar in $\mathbb{R}^d$. We introduce,

$$\gamma_\epsilon(t) = \gamma(t) + \epsilon\beta(t), \quad \epsilon \in \mathbb{R}.$$

With this definition, $(t, \epsilon) \mapsto \gamma_\epsilon(t)$ is an analytic function on $[0, T_1] \times \cdots \times [0, T_{d-1}] \times \mathbb{R}$, T_i-periodic in each composite t_i.
We consider the bounded domain Ω_ϵ in $\mathbb{R}^d$ with smooth boundary $\partial\Omega_\epsilon$ parameterized by the function $\gamma_\epsilon(t)$:

$$\partial\Omega_\epsilon = \{\gamma_\epsilon(t), \quad t = (t_1, \cdots, t_{d-1}) \in [0, T_1] \times \cdots \times [0, T_{d-1}]\}.$$

Let μ_0 and ε_0 denote the permeability and the permittivity of the background medium $\Omega \equiv \Omega_{\epsilon=0}$, and assume that $\mu_0 > 0$ and $\varepsilon_0 > 0$ are positive constants. Let $\mu_1 > 0$ and $\varepsilon_1 > 0$

denote the permeability and the permittivity of $\Omega_\epsilon \backslash \Omega_0$. Introduce the piecewise-constant electric permittivity

$$\varepsilon_\epsilon(x) = \begin{cases} \varepsilon_0, & x \in \Omega_0, \\ \varepsilon_1, & x \in \Omega_\epsilon \backslash \Omega_0. \end{cases}$$

If we allow the degenerate case $\epsilon = 0$, then the function $\varepsilon_0(x)$ equals the constant ε_0. The piecewise constant magnetic permeability, $\mu_\epsilon(x)$ is defined analogously.

In this paper, we deal with the asymptotic behavior associated with the following Helmholtz eigenvalue problem:

$$div(\frac{1}{\mu_\epsilon}\mathbf{grad}\ u(\epsilon)) + \omega^2(\epsilon)\varepsilon_\epsilon u(\epsilon) = 0 \quad \text{in } \Omega_\epsilon, \quad \text{and } u(\epsilon) = 0 \quad \text{on } \partial\Omega_\epsilon, \tag{2}$$

where the function u represents some electric field or magnetic field (or rather, the transversal strength) and $\omega^2(\epsilon)$ is the *perturbed resonant frequency (eigenfrequency)* associated to the above problem (2). The eigenfunction u_0, in the absence of any deformation, satisfies the following equations:

$$-\Delta u_0(x) = \omega_0^2(\varepsilon_0\mu_0)u_0(x), \quad x \in \Omega, \quad \text{and } u_0(x) = 0, \quad x \in \partial\Omega, \tag{3}$$

It is well known that the operator $-\Delta$ on $L^2(\Omega_\epsilon)$ with domain $H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ is self-adjoint with compact resolvent. Consequently, its spectrum consists entirely of isolated, real and positive eigenvalues with finite multiplicity, and there are corresponding eigenfunctions which make up an orthonormal basis of $L^2(\Omega_\epsilon)$. By extending our approach for treating the eigenvalue problem (3) we will investigate the splitting of the eigenvalues and derive their asymptotic expansions under boundary perturbations.
Let $\omega_0^2 > 0$ denote an eigenfrequency of the eigenvalue problem (3) for $\epsilon = 0$ with geometric multiplicity m in the domain $\Omega \equiv \Omega_0$. There exists a small constant $r_0 > 0$ such that ω_0^2 is the unique eigenfrequency of (3) for $\epsilon = 0$ in the set $\{\omega^2, \omega \in D_{r_0}(\omega_0)\}$, where $D_{r_0}(\omega_0)$ is a disk of center ω_0 and radius r_0. Let us call the ω_0-group the *totality of the perturbed eigenfrequencies of (3) for* $\epsilon > 0$ generated by splitting from ω_0^2 and chosen to be an increasing family. The following analyticity result is well-known [9]-[22].

Theorem 1. *(Kato [12, §VII.6], Rellich [25, §§II.2 and II.6]) There exits $\epsilon_0 > 0$ such that for $|\epsilon| < \epsilon_0$, the ω_0-group consists of $m-$ eigenfrequencies, $\omega_j^2(\epsilon), j = 1, \ldots, m$ (repeated according to their multiplicity). Moreover, they are analytic functions with respect to ϵ satisfying $\omega_j^2(0) = \omega_0^2, j = 1, \ldots, m$. The normalized eigenfunctions associated to the ω_0-group of eigenfrequencies are analytic and their values at 0 ($\{u_0^j\}_{1\le j\le m}$) are m linearly independent solutions of the unperturbed eigenvalue problem.*

Classical regularity results and the previous theorem imply that the eigenfunctions associated to the ω_0-group of eigenvalues are separately analytic in the small parameter ϵ and the spatial variable x. By an integral equation technique we also established [13, Thm. 5.4, p.1219] the joint analytic dependence of these functions with respect to (x, ϵ).

We will develop a boundary integral formulation for solving the eigenvalue problem (2). The integral equations applying to this problem will be obtained from a study of the layer potentials for the Helmholtz equation. 2 For $\lambda > 0$, a fundamental solution $\Gamma_\lambda(x)$ to the Helmholtz operator $\Delta + \lambda^2$ in $\mathbb{R}^d$, $d = 2, 3$, is given by

$$\Gamma_\lambda(x) = \begin{cases} -\frac{i}{4} H_0^{(1)}(\lambda \|x\|), & d = 2, \\ -\dfrac{e^{i\lambda\|x\|}}{4\pi\|x\|}, & d = 3, \end{cases}$$

for $x \neq 0$, where $H_0^{(1)}$ is the Hankel function of the first kind of order 0.
Suppose that $G(x, y) = \Gamma_{\omega(\epsilon)\sqrt{\mu_\epsilon \varepsilon_\epsilon}}(x - y)$. The singularity of this function has the form:

$$G(x, y) \sim \begin{cases} \frac{1}{2\pi} \log\|x - y\| + \cdots \quad \text{as}\ , x \to y & d = 2, \\ \frac{1}{4\pi\|x-y\|} + \cdots \quad \text{as}\ x \to y & d = 3. \end{cases}$$

The following operator is well defined [2, 16]

$$S(\omega) : H^{-1/2}(\partial\Omega_\epsilon) \to H^{1/2}(\partial\Omega_\epsilon)$$

where

$$S(\omega) : g \to \int_{\partial\Omega_\epsilon} G(\cdot, y) g(y) d\sigma(y).$$

For such g and every $x \in \partial\Omega_\epsilon$, we denote by $g_+(x)$ and $g_-(x)$ the limits of $g(y)$ as $y \to x$, from $y \in \Omega_\epsilon$ and $y \in \mathbb{R}^d \setminus \overline{\Omega}_\epsilon$, respectively, when these limits exist. It is a well-known classical result that, for $x \in \partial\Omega_\epsilon$,

$$S(\omega)g(x) = (Sl(\omega)g)_+(x) = (Sl(\omega)g)_-(x)$$

where the operator $Sl(\omega)$ called single-layer potential (see [5], [16]) and $S(\omega)$ is pseudo-differential operator of order -1.
Throughout this paper, we use for simplicity the notation
$H^\varsigma_\sharp(]0, T_1[\times \cdots \times]0, T_{d-1}[) = H^\varsigma(\mathbb{R}^{d-1}/]0, T_1[\times \cdots \times]0, T_{d-1}[)$, for $\varsigma \in \mathbb{R}$, where
$H^\varsigma(\mathbb{R}^{d-1}/]0, T_1[\times \cdots \times]0, T_{d-1}[)$ denotes the classical Sobolev H^ς-space on the quotient
$\mathbb{R}^{d-1}/]0, T_1[\times \cdots \times]0, T_{d-1}[$ (Adams [1]).
Using change of variables and integral equations, the following result immediately holds (see [23]).

Proposition 1. *Let* $A_\epsilon(\omega) : H_\sharp^{-1/2}(]0, T_1[\times \cdots \times]0, T_{d-1}[) \to H_\sharp^{1/2}(]0, T_1[\times \cdots \times]0, T_{d-1}[)$ *be defined as follows:*

$$\begin{aligned} A_\epsilon(\omega) f(t) &= \Big(S(\omega) f(\gamma_\epsilon^{-1})\Big)(\gamma_\epsilon(t)) \\ &= \int_{]0,T_1[\times\cdots\times]0,T_{d-1}[} G(\gamma_\epsilon(t), \gamma_\epsilon(s)) |\nabla\gamma_\epsilon(s)| f(s) ds \quad \textit{for } f \in H_\sharp^{-1/2}(]0, T_1[\times \cdots \times]0, T_{d-1}[). \end{aligned}$$

Then the operator-valued function $A_\epsilon(\omega)$ is Fredholm analytic with index 0 in $\mathbb{C} \setminus i\mathbb{R}^-$. Moreover, $A_\epsilon^{-1}(\omega)$ is a meromorphic function and its poles are in $\{\Im(z) \leq 0\}$, where $\Im(z)$ means the imaginary part of z and $\Re(z)$ is the real part.

Using the proprieties of the operator-valued function A_ϵ given by Proposition 1 and using the Lemma 5.3 found in [11], we can easily prove the following results.

Theorem 2. *Let $\mathscr{K}_0$ be a bounded neighborhood of $\overline{\Omega}_0$ in $\mathbb{R}^d$. Then there exists a constant $\epsilon_1 > 0$ smaller than ϵ_0 such that an orthonormal basis of eigenfunctions $(u_j(\epsilon))_j$ corresponding to the $\omega_0 - group$, $(\omega_j^2(\epsilon))_j$, in $H_0^1(\Omega_\epsilon)$ can be chosen to depend holomorphically in $(x, \epsilon) \in \mathscr{K}_0 \times] - \epsilon_1, \epsilon_1[$. Moreover these eigenfunctions satisfy the following uniform expansion: for $x \in \mathscr{K}_0$,*

$$u_j(\epsilon) = u_0^j + \sum_{n\geq 1} u_n^{(j)} \epsilon^n,$$

where the family u_0^j builds a basis of eigenfunctions of (3) associated to ω_0^2 and normalized in $L^2(\Omega_0)$. The terms $u_n^{(j)}$ are computed from the Taylor coefficients of the normal derivatives.

3. Convergence Estimate

In this Section we are in a position to use the Theorems 1 and 2 in order to establish certain estimates for the convergence of the eigenfunctions $u_j(\epsilon)$ and the corresponding eigenfrequencies $\omega_j^2(\epsilon)$, for all $j = 1, \cdots, m$. Let $\alpha_1 > 0$ be the smaller positive constant such that $\Omega_0 \subset \Omega_\epsilon$ and $\partial\Omega_\epsilon \cap \partial\Omega_0 = \emptyset$, for $0 < \epsilon < \alpha_1$ and define the open, bounded domain $\tilde{\Omega}_\epsilon \equiv \Omega_\epsilon \backslash \overline{\Omega}_0$.

Lemma 1. *Let the functions $u_j(\epsilon)$ and u_0^j, for $j = 1, \cdots, m$, be given by Theorem 2. Then, there exist some positive constants $\epsilon_2 < \epsilon_1$ and C_j, such that*

$$\|\nabla(u_j(\epsilon) - u_0^j)\|_{L^2(\tilde{\Omega}_\epsilon)} \leq C_j |\Omega_\epsilon \backslash \Omega_0|^{1/2},$$

for $0 < \epsilon < \epsilon_2$. The constant C_j depends on ω_0 and u_0^j, but is otherwise independent of ϵ.

Proof. Define the function $U(\epsilon) = u_j(\epsilon) - u_0^j$, for $0 < \epsilon < \inf(\epsilon_1, \alpha_1)$ where ϵ_1 is given by Theorem 2 and combine the equations (2) and (3) and let for simplicity $\omega = \omega_j(\epsilon)\sqrt{\mu_\epsilon \varepsilon_\epsilon}$, we compute that $U(\epsilon)$ solves:

$$-\Delta U = \omega^2 U + (\omega^2 - \omega_0^2 \varepsilon_0 \mu_0) u_0^j \quad \text{in } \Omega_\epsilon. \tag{4}$$

For $z \in \mathbb{R}$, we define the function ϑ by

$$\vartheta(z) = \omega^2 z + (\omega^2 - \omega_0^2 \varepsilon_0 \mu_0) \|u_0^j\|_{L^\infty(\Omega_0)}.$$

Then, we trivially remark,

$$|\vartheta(z)| \leq |\vartheta(0)| + \omega^2 |z|, \quad \forall z \in \mathbb{R},$$

and consequently,

$$|\vartheta(U(\epsilon))| \leq |\vartheta(0)| + \omega^2 |U(\epsilon)|. \quad (5)$$

The fact that $u_j(\epsilon) \to u_0^j$ implies that there exists $0 < \alpha_2 < \inf(\epsilon_1, \alpha_1)$ such that for $0 < \epsilon < \alpha_2$,

$$|U(\epsilon)(x)| \leq 2\|u_0^j\|_{L^\infty(\Omega_0)}, \text{ for } x \in \Omega_\epsilon. \quad (6)$$

Moreover, we remember that $\omega^2(\epsilon) \to \omega_0^2 \varepsilon_0 \mu_0$, then there exists $\alpha_3 \geq 0$ such that: $\omega^2 \leq \omega_0^2 \varepsilon_0 \mu_0 + \frac{1}{3}$, for $0 \leq \epsilon \leq \alpha_3$.
Now, it is useful to introduce the following function:

$$\tilde{\vartheta}(U) = \omega^2 U + (\omega^2 - \omega_0^2 \varepsilon_0 \mu_0) u_0^j,$$

where U is the solution of (4). If we examine each term on the right hand side of (5) separately, we find out that the first term is bounded by

$$|\vartheta(0)| \leq \frac{1}{3}\|u_0^j\|_{L^\infty(\Omega_0)}, \text{ for } 0 < \epsilon < \alpha_3.$$

The second term is bounded by

$$\omega^2 |U(\epsilon)| \leq 2(\omega_0^2 \varepsilon_0 \mu_0 + \frac{1}{3})\|u_0^j\|_{L^\infty(\Omega_0)}, \text{ for } 0 < \epsilon < \epsilon_2 = \inf(\alpha_2, \alpha_3).$$

These estimates give

$$\|\tilde{\vartheta}(U(\epsilon))\|_{L^\infty(\tilde{\Omega}_\epsilon)} \leq (1 + 2\omega_0^2 \varepsilon_0 \mu_0)\|u_0^j\|_{L^\infty(\Omega_0)}, \text{ for } 0 < \epsilon < \epsilon_2. \quad (7)$$

Next, the relation (4) implies

$$-\Delta U(\epsilon) = \tilde{\vartheta}(U(\epsilon)), \quad \text{in} \Omega_\epsilon.$$

By integrating by parts in $\tilde{\Omega}_\epsilon$, we find that the function $U(\epsilon)$ is solution to the following problem:

$$\forall v \in H^1(\tilde{\Omega}_\epsilon), \quad \int_{\tilde{\Omega}_\epsilon} \nabla U(\epsilon) \bar{\nabla} v \, dx = \int_{\tilde{\Omega}_\epsilon} \tilde{\vartheta}(U(\epsilon)) \bar{v} \, dx + \int_{\partial \tilde{\Omega}_\epsilon} \frac{\partial U}{\partial \nu} \bar{v} \, ds(x). \quad (8)$$

On the other hand, it is not hard to see that (Trace Theorem),

$$\Big| \int_{\partial \tilde{\Omega}_\epsilon} \frac{\partial U}{\partial \nu} \bar{v} \, ds(x) \Big| \leq |\tilde{\Omega}_\epsilon|^{1/2} \sup_{z \in \partial \tilde{\Omega}_\epsilon} |\frac{\partial U(z)}{\partial \nu}|.\|v\|_{L^2(\tilde{\Omega}_\epsilon)}.$$

But, the relation (6) implies that $\sup_{z \in \partial \tilde{\Omega}_\epsilon} |\frac{\partial U(z)}{\partial \nu}|$ is a positive constant c_* independent of ϵ. If we choose $v = U(\epsilon)$ and if we consider relations (7) and (8) we deduce that,

$$\|\nabla U\|^2_{L^2(\tilde{\Omega}_\epsilon)} \leq [c_* + (1 + 2\omega_0^2 \varepsilon_0 \mu_0)\|u_0^j\|_{L^\infty(\Omega_0)}]|\tilde{\Omega}_\epsilon|^{1/2}\|U\|_{L^2(\tilde{\Omega}_\epsilon)}. \quad (9)$$

By Poincare's inequality, there exists some positive constant $C(\tilde{\Omega}_\epsilon)$ such that

$$\|U\|_{L^2(\tilde{\Omega}_\epsilon)} \leq C(\tilde{\Omega}_\epsilon)\|\nabla U\|_{L^2(\tilde{\Omega}_\epsilon)}.$$

The fact U and ∇U are uniformly bounded on Ω_ϵ implies there exists some constant C_0 independent of ϵ (e.g.[13, p.33]) such that

$$C(\tilde{\Omega}_\epsilon) \leq C_0, \tag{10}$$

and therefore the relation (9) becomes

$$\|\nabla U\|_{L^2(\tilde{\Omega}_\epsilon)} \leq C_0[c_* + (1+2\omega_0^2\varepsilon_0\mu_0)\|u_0^j\|_{L^\infty(\Omega_0)}]|\tilde{\Omega}_\epsilon|^{1/2}.$$

We take $C_j = C_0[c_* + (1+2\omega_0^2\varepsilon_0\mu_0)\|u_0^j\|_{L^\infty(\Omega_0)}]$ which concludes the proof.

The following main result holds.

Theorem 3. *Let γ, β and Ω_ϵ be defined as in Section 2 and let the functions $u_j(\epsilon)$ and u_0^j, for $j = 1, \cdots, m$, be given by Theorem 2. Then, there exist some constant $0 < \epsilon_3 \leq 1/M$, $M = \max_{t\in[0,T_1]\times\cdots\times[0,T_{d-1}]}|\beta(t)|$ and some positive constant κ_j dependent on ω_0, u_0^j, $|\gamma|$ and M but otherwise independent of ϵ such that,*

$$\|u_j(\epsilon) - u_0^j\|_{L^2(\tilde{\Omega}_\epsilon)} \leq \kappa_j\epsilon^{1/2},$$

for $0 < \epsilon < \epsilon_3$.

Proof. For simplicity we can suppose that Ω_0 is a disk with radius $\varrho_0 > 0$ in $\mathbb{R}^2$. It then follows that $|\gamma(t)| = \varrho_0$. The proof is simple if we calculate the area $|\tilde{\Omega}_\epsilon|$ of the domain $\tilde{\Omega}_\epsilon$. But it is not hard to see that, in polar coordinates (ϱ, θ), there exists a regular function $\Upsilon : [0, 2\pi] \to \mathbb{R}_+$; $\Upsilon(\theta) = |\beta(\theta)|$ such that the boundary $\partial\Omega_\epsilon$ can be re-parameterized by

$$\varrho = \varrho(\epsilon, \theta) = \varrho_0 + \epsilon\Upsilon(\theta); \quad \theta \in [0, 2\pi].$$

Therefore

$$|\tilde{\Omega}_\epsilon| = \int_0^{2\pi}[\int_{\varrho_0}^{\varrho_0+\epsilon\Upsilon(\theta)}\varrho d\varrho]d\theta = \frac{1}{2}\int_0^{2\pi}[\epsilon^2\Upsilon^2(\theta) + 2\epsilon\varrho_0\Upsilon(\theta)]d\theta. \tag{11}$$

For $\epsilon < \epsilon_3 = \inf(\epsilon_2, 1/M)$, we can write: $\epsilon^2 \leq \epsilon/M$; which implies

$$\epsilon^2 M^2 \leq \epsilon M.$$

Consequently the equality (11) gives

$$|\tilde{\Omega}_\epsilon| \leq \pi M(1 + 2\varrho_0)\epsilon. \tag{12}$$

By Poincare's inequality and Lemma 1 we write,

$$\|u_j(\epsilon) - u_0^j\|_{L^2(\tilde{\Omega}_\epsilon)} \leq C(\tilde{\Omega}_\epsilon)C_j|\tilde{\Omega}_\epsilon|^{1/2}.$$

Finally, we obtain the desired result if we consider the relations (10) and (12) and if we choose the constant $\kappa_j = C_j C_0 \sqrt{\pi M(1+2\varrho_0)}$.

To derive the corresponding formulae for the eigenvalues we will use an idea close to the theorem of Osborn [18] which gives estimates for the convergence of the eigenvalues of a sequence of compact operators. For our case we consider the Hilbert space $L^2(\Omega_\epsilon)$ with the standard inner product $\langle .,.\rangle$.
For any $\varphi \in L^2(\Omega_\epsilon)$, define the operator $T_\epsilon\varphi = v_\epsilon$, where v_ϵ is the solution to the problem

$$\begin{cases} -\Delta v_\epsilon = \varphi \quad \text{in } \Omega_\epsilon, \\ v_\epsilon = 0 \quad \text{on } \partial\Omega_\epsilon. \end{cases} \tag{13}$$

and we define the operator $T_0\varphi = v_0$, where v_0 is the solution to the problem

$$\begin{cases} -\Delta v_0 = \varphi \quad \text{in } \Omega_0, \\ v_0 = 0 \quad \text{on } \partial\Omega_0. \end{cases} \tag{14}$$

The function $\varphi \mapsto (-\Delta)^{-1}\varphi$ is continuous from $L^2(\Omega_\epsilon)$ to $H_0^1(\Omega_\epsilon)$. Clearly T_ϵ and T_0 are compact operators from $L^2(\Omega_0)$ to $L^2(\Omega_0)$. From the standard H^1 estimates for v_ϵ which are independent of ϵ, we see that the set $\{T_\epsilon\}$ is collectively compact. Hence all hypotheses hold for the theorem of Osborn. Now if we set,

$$\lambda_0 = \frac{1}{\omega_0^2} \quad \text{and } \lambda_j(\epsilon) = \frac{1}{\omega_j^2(\epsilon)},$$

then according to the problem (13)(resp. (14)) we can see that $(\lambda_j(\epsilon), u_j(\epsilon))$ (resp.(λ_0, u_0^j)) is eigenpairs of T_ϵ (resp. of T_0) with $\varphi = \frac{1}{\omega_j^2(\epsilon)} u_j(\epsilon)$. We remember that ω_0^2 is an eigenfrequency of multiplicity m with a corresponding set of orthonormal eigenfunctions $\{u_0^j\}$ and then $R(P(0))$ is just the $m-$dimensional subspace generated by $\{u_0^j\}$(where $P(0)$ means the spectral projection associated with T_0 and means the projection onto the space associated to $\{u_0^j\}$).
Although each of the eigenvalues $\lambda_1(\epsilon), \cdots, \lambda_m(\epsilon)$ are close to λ_0, their arithmetic mean is generally a closer approximation [3]. Thus we define

$$\hat{\lambda}(\epsilon) = \frac{1}{m}\sum_{j=1}^{m} \frac{1}{\omega_j^2(\epsilon)}. \tag{15}$$

In the terminology of [9] this is the weighted mean of the λ_0-group. The next Lemma gives an estimate for $\lambda_0 - \hat{\lambda}(\epsilon)$ which will be useful to prove our main result.

Lemma 2. *Let ϵ_3 be the positive constant given by Theorem 3. Then there exists a positive constant K_1 such that for $\epsilon < \epsilon_3$,*

$$|\lambda_0 - \hat{\lambda}(\epsilon)| \leq K_1 \epsilon^{1/2}.$$

Proof. We write,

$$\|T_\epsilon u_0^j - T_0 u_0^j\|_{L^2(\tilde{\Omega}_\epsilon)} = \|T_\epsilon u_j(\epsilon) + T_\epsilon u_0^j - T_\epsilon u_j(\epsilon) - T_0 u_0^j\|_{L^2(\tilde{\Omega}_\epsilon)}$$

$$\leq \|\frac{1}{\omega_j^2(\epsilon)} u_j(\epsilon) - \frac{1}{\omega_0^2} u_0\|_{L^2(\tilde{\Omega}_\epsilon)} + \|T_\epsilon u_0^j - T_\epsilon u_j(\epsilon)\|_{L^2(\tilde{\Omega}_\epsilon)}. \quad (16)$$

If we set $z_j(\epsilon) = \frac{1}{\omega_j^2(\epsilon)} u_j(\epsilon)$ and $z_0^j = \frac{1}{\omega_0^2} u_0^j$, we see that $z_j(\epsilon)$ and z_0^j are solutions to the problems (3) and (2) respectively and $z_j(\epsilon) \to z_0^j$ as ϵ tends to 0. Therefore, Theorem 3 gives for $\epsilon < \epsilon_3$,

$$\|z_j(\epsilon) - z_0^j\|_{L^2(\tilde{\Omega}_\epsilon)} \leq \kappa_j \epsilon^{1/2}.$$

In other words, for reasons of compactness of T_ϵ and according to Theorem 3 we have

$$\begin{aligned} \|T_\epsilon u_0^j - T_\epsilon u_j(\epsilon)\|_{L^2(\tilde{\Omega}_\epsilon)} &= \|T_\epsilon (u_0^j - u_j(\epsilon))\|_{L^2(\tilde{\Omega}_\epsilon)} \\ &\leq K\|u_0^j - u_j(\epsilon)\|_{L^2(\tilde{\Omega}_\epsilon)} \leq K.\kappa_j \epsilon^{1/2}. \end{aligned}$$

Then, the relation (16) becomes

$$\|T_\epsilon u_0^j - T_0 u_0^j\|_{L^2(\tilde{\Omega}_\epsilon)} \leq \kappa_j (1+K)\epsilon^{1/2}. \quad (17)$$

Inserting all this information into the theorem of Osborn [21, Thm.3], we obtain

$$\frac{1}{\omega_0^2} - \frac{1}{m}\sum_{j=1}^m \frac{1}{\omega_j^2(\epsilon)} = \frac{1}{m}\sum_{j=1}^m \langle (T_0 - T_\epsilon)u_0^j, u_0^j\rangle + \epsilon^{1/2} O(1).$$

The proof follows by reconsidering again relation (17).

Next, to estimating $|\omega_0^2 - \omega_j^2(\epsilon)|$ we may, firstly, estimate $|\lambda_0 - \lambda_j(\epsilon)|$ for each j.

Lemma 3. *There exist some constants* $0 < \epsilon_4 \leq \epsilon_3$ *and* $k_j^{(1)}$ *such that for all* $j = 1, \cdots, m$,

$$|\lambda_0 - \lambda_j(\epsilon)|^m \leq k_j^{(1)} \epsilon^{5/2}, \quad \textit{for } 0 \leq \epsilon \leq \epsilon_4.$$

Proof. Let $T_\epsilon(y_\epsilon) = \lambda_j(\epsilon) y_\epsilon$ such that $\|y_\epsilon\| = 1$. We can then choose $\varpi^* \in Ker((\lambda_0 - T_0^*)^m)$ in such a way that $\langle y_\epsilon, \varpi^* \rangle = 1$. Then,

$$\langle (\lambda_0 - T_0^*)^m \varpi^*, y_\epsilon \rangle = 0$$

and therefore,

$$|\langle (\lambda_0 - \lambda_j(\epsilon))^m \varpi^*, y_\epsilon \rangle| = |\langle (\lambda_0 - \lambda_j(\epsilon))^m \varpi^*, y_\epsilon \rangle - \langle (\lambda_0 - T_0^*)^m \varpi^*, y_\epsilon \rangle|$$

$$= |-\sum_{l=0}^{m-1} (\lambda_0 - \lambda_j(\epsilon))^l \langle (\lambda_0 - T_0^*)^{m-1-l} (\lambda_j(\epsilon) - T_0^*) \varpi^*, y_\epsilon \rangle|. \quad (18)$$

Now we prove that $|\lambda_0 - \lambda_p(\epsilon)| \le |\lambda_0 - \hat{\lambda}(\epsilon)|$, for each positive integer p. But, the relation,

$$\lambda_0 - \hat{\lambda}(\epsilon) - (\lambda_0 - \lambda_p(\epsilon)) = \lambda_p(\epsilon) - \frac{1}{m}\sum_{j=1}^{m} \lambda_j(\epsilon)$$

implies that, for p and q the integers such that $\lambda_p = \sup_{1\le j\le m} |\lambda_j(\epsilon)|$ and $\lambda_q = \inf_{1\le j\le m} |\lambda_j(\epsilon)|$, the following relation holds(the family $(\omega_j^2)_j$ is increasing):

$$|\lambda_0 - \hat{\lambda}(\epsilon)| \ge \sup(|\lambda_0 - \lambda_p(\epsilon)|, |\lambda_0 - \lambda_q(\epsilon)|)$$

which gives that $|\lambda_0 - \lambda_j(\epsilon)| \le |\lambda_0 - \hat{\lambda}(\epsilon)|$ for $1 \le j \le m$. Therefore, the fact that $\lambda_0 - \hat{\lambda}(\epsilon) \to 0$ implies that there exists $\delta_1 > 0$ such that for all $0 \le l \le m-1$,

$$|\lambda_0 - \lambda_j(\epsilon)|^l \le |\lambda_0 - \hat{\lambda}(\epsilon)|, \quad \text{for } 0 < \epsilon < \delta_1. \tag{19}$$

Next, if we insert the relation (19) into (18) we obtain the following inequality

$$|\lambda_0 - \lambda_j(\epsilon)|^m \le |\lambda_0 - \hat{\lambda}(\epsilon)| \sum_{l=0}^{m-1} \|\lambda_0 - T_0^*\|^{m-1-l} \max_{\|\psi^*\|=1} |\langle(\lambda_j(\epsilon) - T_0)y_\epsilon, \psi^*\rangle|, \quad \text{for } 0 < \epsilon < \delta_1. \tag{20}$$

For any $\psi^* \in R(P(0)^*)$, with $\|\psi^*\| = 1$, and the fact that $P_j(\epsilon)^{-1}P_j(\epsilon)$ is the identity on $R(P(0))$ (where $P_j(\epsilon)$ means the spectral projection associated with T_ϵ and is a projection onto the direct sum of the spaces of the eigenvectors corresponding to T_ϵ) we write,

$$|\langle(\lambda_j(\epsilon) - T_0)y_\epsilon, \psi^*\rangle| = |\langle P_j(\epsilon)^{-1}P_j(\epsilon)(T_\epsilon - T_0)y_\epsilon, \psi^*\rangle| = |\langle(T_\epsilon - T_0)y_\epsilon, (P_j(\epsilon)^{-1}P_j(\epsilon))^*\psi^*\rangle|.$$

Due to the estimate (4.10) found in [21, p.722], we obtain

$$|\langle(T_\epsilon - T_0)y_\epsilon, (P_j(\epsilon)^{-1}P_j(\epsilon))^*\psi^*\rangle| \le c.\epsilon^2\|y_\epsilon\|_{L^2}.\|(P_j(\epsilon)^{-1}P_j(\epsilon))^*\psi^*\|_{L^2}, \quad \text{for } 0 < \epsilon < \delta_1,$$

where c is a positive constant. Then,

$$|\langle(T_\epsilon - T_0)y_\epsilon, (P_j(\epsilon)^{-1}P_j(\epsilon))^*\psi^*\rangle| \le c.\epsilon^2\|y_\epsilon\|_{L^2}.\|P_j(\epsilon)^{-1}\|.\|P_j(\epsilon)\|.\|\psi^*\|_{L^2}.$$

The norm $\|P_j(\epsilon)\|$ is bounded in ϵ since $P_j(\epsilon) \to P(0)$ pointwise. In other words, for ϵ small enough and for $f \in R(P(0))$ with $\|f\| = 1$, we have

$$1 - \|P_j(\epsilon)f\| = \|P(0)f\| - \|P_j(\epsilon)f\| \le \|(P(0) - P_j(\epsilon))f\| \le \frac{1}{2}$$

and hence $\|P(\epsilon)f\| \ge \frac{1}{2}$ which implies $\|P(\epsilon)^{-1}\| \le 2$ for ϵ small enough, say for $0 \le \epsilon \le \delta_2$ where δ_2 is a positive constant. Then the relation (20) becomes,

$$|\lambda_0 - \lambda_j(\epsilon)|^m \le c'.\epsilon^2|\lambda_0 - \hat{\lambda}(\epsilon)|. \tag{21}$$

The proof is achieved if we use Lemma 2 and we take $\epsilon_4 = \inf(\epsilon_3, \delta_1, \delta_2)$ and $k_j^{(1)} = c'K_1$.

The main estimate will be given in the following theorem which its proof follows easily by Lemma 3.

Theorem 4. *Let $\omega_0^2 > 0$ be the eigenfrequency of the problem (2) with geometric multiplicity m, and $\omega_j^2(\epsilon)$, for $j = 1, \cdots, m$ given by Theorem 1. Then, there exist some positive constants $0 < \epsilon_5 \leq \epsilon_4$ and $k_j^{(2)}$ such that*

$$|\omega_0^2 - \omega_j^2(\epsilon)| \leq k_j^{(2)} \epsilon^{\frac{5}{2m}},$$

for $0 < \epsilon \leq \epsilon_5$.

Proof. It is not hard to see that for all $j = 1, \cdots, m$,

$$|\lambda_0 - \lambda_j(\epsilon)| = \Big| \frac{\omega_j^2(\epsilon) - \omega_0^2}{\big(\omega_0 \omega_j(\epsilon)\big)^2} \Big| \tag{22}$$

But the fact $\omega_j^2(\epsilon) \to \omega_0^2$ as ϵ tends to 0 implies that there exists some constant $\delta_3 > 0$ such that $|\big(\omega_j(\epsilon)\omega_0\big)^2| < \dfrac{3}{2}\omega_0^4$, for $0 < \epsilon < \delta_3$.
The relation (22) implies,

$$|\omega_j^2(\epsilon) - \omega_0^2| \leq \frac{3}{2}\omega_0^4 |\lambda_0 - \lambda_j(\epsilon)|, \quad \text{for } 0 < \epsilon < \delta_3.$$

The theorem follows immediately by considering Lemma 3, we take $\epsilon_5 = \inf(\delta_3, \epsilon_4)$ and we choose $k_j^{(2)} = \dfrac{3}{2}\omega_0^4 (k_j^{(1)})^{1/m}$.

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975 .

[2] H. Ammari, and H. Kang, *Reconstruction of small inhomogeneities from boundary measurements*, Lecture Notes in Mathematics, Vol.1846, Springer-Verlag, Berlin (2004).

[3] J.H. Bramble and J.E. Osborn, *Rate of convergence estimates for nonselfadjoint eigenvalue approximations* , Math. Comp. Vol. 27 (1973), 525-549.

[4] O. P. Bruno and F. Reitich, *Boundary-variation solution of eigenvalue problems for elliptic operators*, J. Fourier Anal. Appl. 7 (2001), 169-187.

[5] D. Colton and R. Kress, *Integral equation methods in Scattering theory*, New York: John Wiley (1983).

[6] C. Conca, M. Duran and J. Rappaz, *Rate of convergence estimates for the spectral approximation of a generalized eigenvalue problem* , Numer. Math. 79, no3, (1998), 349-369.

[7] S. J. Cox,*The Generalized gradient at a multiple eigenvalue*, J. Func. Anal, 133, (1995), 30-40.

[8] R. R. Gadyl'shin and A. M. Il'in, *Asymptotic behavior of the eigenvalues of the Dirichlet problem in a domain with a narrow Crack,* Sbornik Math. 189 (1998), 503-526.

[9] T. Kato, *Perturbation theory for linear operators*, 2end edition, Springer-Verlag, Berlin (1980).

[10] A. Khelifi, *Diffraction d'ondes électromagnétiques par des inhomogénéités diélectriques*, PhD. Thesis, Applied Mathematics, Ecole Polytechnique - France, February (2002).

[11] A. Khelifi, *The Integral Equation Methods for the Perturbed Helmholtz Eigenvalue Problems,* Int. J. Math. Math. Sc. 80,(2005), 1201-1220.

[12] P.D. Lamberti, and M. Lanza de Cristoforis, *A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Dirichlet problem for the Laplace operator* . J. Nonlin. Con. Anal. Vol 5, no 1,(2004), 19-42.

[13] P.D. Lamberti, and M. Lanza de Cristoforis, *An analyticity result for the dependence of multiple eigenvalues and eigenspaces of the Laplace operator upon perturbation of the domain* . Glasgow Math J. no 1,(2002), 29-43.

[14] D. Lupo, and A. M. Micheletti, *On multiple eigenvalues of selfadjoint compact operators,* J. Math. Anal. Appl. 172, (1993), 106-116.

[15] S. Moskow and M. Vogelius, *First Order correction to the homogenized eigenvalues of a periodic composite medium. The case of Neumann boundary conditions*, to appear in J. Indiana. Univ., (2006).

[16] J. C. Nédélec, *Acoustic and Electromagnetic equations. Integral representation for Harmonic problem,* Springer-Verlag, New York, 2001.

[17] K. Neymeyr, *A posteriori error estimation for elliptic eigenproblems,* Num. Lin. Alg. Appl. 9 (2002), 263-279.

[18] J.E. Osborn, *Spectral Approximations for Compact Operators,* Math. Comp., Vol 29 (1975), 712-725.

[19] S. Ozawa, *Eigenvalues of the Laplacian under singular variation of domains-the Robin problem with obstacle of general shape,* Proc.Japan Acad. Ser.A Math.Sci.72 (1996), 124-134.

[20] S. Ozawa, *Asymptotic property of eigenfunction of the Laplacian under singular variation of domains-the Neumann condition,* Osaka J.Math.22(1985), 639-655.

[21] S. Ozawa, *An asymptotic formula for the eigenvalues of the Laplacian in a three-dimensional domain with a small hole,* J. Fac. Sci. Univ. Tokyo Sect. IA 30 (1983), 243-257.

[22] F. Rellich, *Perturbation theory of eigenvalue problems*, Gordon and Breach Science Publishers, New York (1969).

[23] M. E. Taylor, *Partial differential equations II, Qualitative studies of Linear equations,* Applied Mathematical Sciences 116, Springer- Verlag, (1996).