EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 3, 2018, 793-802
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

On the Continuity of Orthogonal Sets in the Sense of Operator Orthogonality

M. Iranmanesh[1,*], M. Saeedi Khojasteh[1], M. K. Anwary[2]

[1] *Department of Pure Mathematics, Shahrood University of Technology, Iran*
[2] *Department of Pure Mathematics, Ferdowsi University of Mashhad, Iran*

Abstract. In this paper, we introduce the operator approach for orthogonality in linear spaces. In particular, we represent the concept of orthogonal vectors using an operator associated with them, in normed spaces. Moreover, we investigate some of continuity properties of this kind of orthogonality. More precisely, we show that the set valued function

$$F(x;y) = \{\mu : \mu \in \mathbb{C}, p(x-\mu y, y) = 1\}$$

is upper and lower semi continuous, where

$$p(x,y) = \sup\{p_{z_1,\ldots,z_{n-2}}(x,y) : z_1,\ldots,z_{n-2} \in X\}$$

and

$$p_{z_1,\ldots,z_{n-2}}(x,y) = \|P_{x,z_1,\ldots,z_{n-2},y}\|^{-1}$$

where $P_{x,z_1,\ldots,z_{n-2},y}$ denotes the projection parallel to y from X to the subspace generated by $\{x, z_1, \ldots, z_{n-2}\}$. This can be considered as an alternative definition for numerical range in linear spaces.

Key Words and Phrases: Birkhoff orthogonality, Minkowski plane, set valued function, upper semi continuous, lower semi continuous

1. Introduction

Orthogonality, is one of the important concepts in mathematical and numerical analysis. Perhaps, it is the main property in linear spaces, normed spaces and inner product spaces. There are some various kinds of orthogonality. In fact, it has been defined different kinds in mathematical spaces.

In inner product spaces, it is easily said that two vectors x, y are orthogonal if

$$\langle x, y\rangle = 0.$$

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v11i3.3237

Email addresses: m.iranmanesh2012@gmail.com (M. Iranmanesh),
m.saeedi64@gmail.com (M. Saeedi Khojasteh), abdh1248@gmail.com (M. K. Anwary)

But, in normed spaces, there is no simple tool for define orthogonal vectors. However, there are some good suggestions. One of them, is the Birkhoff James orthogonality [2].

Let X be a real normed space, and x, y be in X. We say that x is Birkhoff orthogonal to y if for every constant a,

$$\|x\| \leqslant \|x + ay\|. \tag{1}$$

It is not difficult to show that this definition is the same in inner product spaces [6].

In 1993, Milicic [7] introduced g-orthogonality in normed spaces via Gateaux derivatives. In fact, one has the notion of g-angle related to g-orthogonality.

In this paper, the authors define a new type of orthogonality in a linear space by using projection operators.

Let X be a Minkowski plane. Denote by $\|.\|$ the norm of X. Fix a basis $\{e_1, e_2\}$ of X. Then we can write each $x \in X$ as $x = (x_1, x_2)$ under this basis, where $x_1, x_2 \in \mathbb{R}$. Moreover, $\{\delta_{e_1}, \delta_{e_2}\}$ is a basis of the dual space X^*, where δ_{e_i} for $i = 1, 2$ is a bounded linear function on X with

$$\delta_{e_i}(e_j) = \begin{cases} 0 & i \neq j; \\ 1 & i = j. \end{cases}$$

Denote by $L(X)$ the set of all bounded linear operators from X to X. For $T \in L(X)$, the operator $T^* \in L(X^*)$ is said to be the Banach conjugate operator of T if for any $z \in X$ and any $z^* \in X^*$, there must be $(T^*z^*)(z) = z^*(Tz)$. Note that if we use the following notation

$$f(x) = \langle x, f \rangle$$

then the property of conjugate can be rewritten as the following way

$$\langle x, T^* f \rangle = \langle Tx, f \rangle$$

as usual in inner product spaces.

Recall that an operator P is an orthogonal projection if it is idempotent and self-adjoint, i.e. $P^2 = P$ and $P^* = P$. In an inner product space it is equivalent to

$$\langle Px, x \rangle = \langle Px, Px \rangle = \langle x, Px \rangle.$$

Suppose that $x = (x_1, x_2)^T$ and $y = (y_1, y_2)^T$ are two linearly independent vectors in X under the basis $\{e_1, e_2\}$. Put

$$D_{xy} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

notice

$$|D_{xy}| = x_1 y_2 - x_2 y_1 \neq 0$$

since x and y are linearly independent. Define by P_{xy} the projection parallel to y from X to the subspace $\{\lambda x; \lambda \in \mathbb{R}\}$. Then P_{xy} depends only on the vectors x and y, and has the following presentation under the basis $\{e_1, e_2\}$:

$$P_{xy} = D_{xy} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot {D_{xy}}^{-1} = \frac{1}{|D_{xy}|} \begin{bmatrix} x_1 y_2 & -x_1 y_1 \\ x_2 y_2 & -x_2 y_1 \end{bmatrix}.$$

It is clear for any two linearly independent vectors x and y in X,

$$1 \leq \|P_{xy}\| < +\infty.$$

Note that if x, y are orthogonal, in the sense of inner product space, then P_{xy} is an orthogonal projection.

Furthermore, denote

$$p(x,y) = \begin{cases} 0 & x \text{ and } y \text{ are linearly dependent;} \\ \|P_{xy}\|^{-1} & x \text{ and } y \text{ are linearly independent.} \end{cases}$$

For any $x, y \in X$, the p-angle between x and y is defined by

$$A_p(x,y) = arcsin(p(x,y)).$$

In an inner product space $(X, \langle ., . \rangle)$, obviously

$$p(x,y) = \frac{\langle x, y \rangle}{\|x\|\|y\|},$$

and consequently, the p-angle is identical with the usual angle.

Let $(X, \|.\|)$ Be a complex Banach space, $(X^*, \|.\|)$ be its dual space, and $B(X)$ be the algebra of all bounded linear operators acting on X. Define the set of normalized states $\Omega = \{\omega \in B(X)^* : \omega(I) = \|\omega\| = 1\}$, where I denotes the identity operator. For any operator $A \in B(X)$, the *(algebraic) numerical range* (also known as *field of values*) of A is defined by

$$F(A) = \{\omega(A) : \omega \in \Omega\}.$$

In the finite-dimensional case $(X, \|.\|) = (\mathbb{C}^n, \|.\|_2)$, where $\|.\|_2$ is the spectral norm, the numerical range of a square matrix $A \in \mathbb{C}^{n\times n}$ is also written

$$F(A) = \{x^* Ax \in \mathbb{C} : x \in \mathbb{C}^n, x^* x = 1\}.$$

The suggested references on numerical ranges of operators and matrices are [3] and [5].

We recall that for two compact subsets Ω_1 and Ω_2 of a metric space (X, ρ), the Hausdorff distance between Ω_1 and Ω_2 is defined by

$$d_H(\Omega_1, \Omega_2) = \max\{\max_{x_1\in\Omega_1} \min_{x_2\in\Omega_2} \rho(x_1, x_2), \max_{x_2\in\Omega_2} \min_{x_1\in\Omega_1} \rho(x_1, x_2)\}$$

For any $x_0 \in X$ and $\delta > 0$, we define the closed ball $B(x_0, \delta) = \{x \in X : \rho(x_0, x) \leq \delta\}$.

Definition 1. *[1] Suppose (X, ρ_X) is a metric space and (Y, ρ_Y) is a complete metric space. Consider a multi-valued mapping $F : X \to Y$, and let $x_0 \in X$.*

(i) F is called upper semi-continuous at x_0 if for every neighborhood $N(F(x_0)) \subset Y$ of the set $F(x_0)$, there is a neighborhood $N(x_0) \subset X$ of x_0 such that

$$F(x) \subset N(F(x_0)), \qquad \forall x \in N(x_0).$$

(ii) F is called lower semi-continuous at x_0 if for every $y_0 \in F(x_0)$ and every neighborhood $N(y_0) \subset Y$ of y_0, there exist a neighborhood $N(x_0) \subset X$ of x_0 such that

$$F(x) \cap N(y_0) \neq \emptyset, \qquad \forall x \in N(x_0).$$

(iii) F is said to be semi continuous at x_0 if it is upper and lower semi-continuous.

The following example is showing some of the common behavior of upper semi continuous functions.

Example 1. *The following functions is upper semi continuous*

$$f(\alpha) = \{x : x \in X, \|x\| \leq |\alpha|\}.$$

Note that this function is increasing in the mean that if $|\alpha| \leq |\beta|$ then $f(\alpha)$ is contained in $f(\beta)$.

We will investigate the upper semi continuity at $\alpha_0 = 1$. Investigating other points are similar.

Assume that $N(f(1))$ is an open set containing $f(1)$. Since $f(1)$ is closed, there is a scalar β such that

$$f(1) \subseteq f(\beta) = \{x : x \in X, \|x\| \leq |\beta|\} \subseteq N(f(1)).$$

It is clear that $1 < |\beta|$. Now consider the following open set

$$N(1) = \{\alpha : \alpha \in \mathbb{C}, |\alpha| < |\beta|\},$$

It is clear that for any α in $N(1)$, $f(\alpha)$ is contained in $f(\beta)$, since f is increasing. So $f(\alpha) \subseteq N(f(1))$.

2. Main Results

Definition 2. *Let X be a linear space with dimension n. Suppose that*

$$x_k = (x_{k_1}, \ldots, x_{k_n})^T \quad , \quad k = 1, \ldots, n$$

are n linearly independent vectors in X. Put

$$D_{x_1,\ldots,x_2} = \begin{bmatrix} x_{11} & \ldots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \ldots & x_{nn} \end{bmatrix}$$

since $x_1, \ldots, x_2$ are linearly independent, we have

$$|D_{x_1,\ldots,x_2}| \neq 0.$$

Definition 3. *Let X be a linear space with dimension n. Suppose that*

$$x = (x_1, \dots, x_n)^T \quad , \quad y = (y_1, \dots, y_n)^T$$

are two linearly independent vectors in X. Extend x, y to a basis for X by adding $n-2$ vector as

$$z_k = (z_{k_1}, \dots, z_{k_n})^T \quad , \quad k = 1, \dots, n-2.$$

Denote by $P_{x,z_1,\dots,z_{n-2},y}$ the projection parallel to y from X to the subspace generated by $x, z_1, \dots, z_{n-2}$. Since the vectors $x, z_1, \dots, z_{n-2}, y$ are the eigenvectors of $P_{x,z_1,\dots,z_{n-2},y}$, it turn implies that $P_{x,z_1,\dots,z_{n-2},y}$ is similar to the following

$$\begin{bmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

In fact, $P_{x,z_1,\dots,z_{n-2},y}$ has a representation as follows

$$P_{x,z_1,\dots,z_{n-2},y} = D_{x,z_1,\dots,z_{n-2},y} \cdot \begin{bmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \cdot D^{-1}_{x,z_1,\dots,z_{n-2},y}.$$

Proposition 1. *For any two linearly independent vectors x and y in X,*

$$1 \leq \|P_{x,z_1,\dots,z_{n-2},y}\| < +\infty$$

in other words, $P_{x,z_1,\dots,z_{n-2},y}$ is a bounded operator.

Furthermore, denote

$$p_{z_1,\dots,z_{n-2}}(x, y) = \|P_{x,z_1,\dots,z_{n-2},y}\|^{-1}$$

and let

$$p(x, y) = \sup\{p_{z_1,\dots,z_{n-2}}(x, y) \,:\, z_1, \dots, z_{n-2} \in X\}.$$

It is obvious that

$$p(x, y) = \max\{\|P_{x,z_1,\dots,z_{n-2},y}\|^{-1} \,:\, z_1, \dots, z_{n-2} \in X \,,\, \|z_1\| = 1, \dots, \|z_{n-2}\| = 1\}.$$

Definition 4. *For any linearly independent x, y in X , the p-angle between x, y is defined by*

$$A_p(x, y) = \arcsin(p(x, y)).$$

Note that p-angle is not depending on selected vectors $z_1, \dots, z_{n-2}$.

Definition 5. *For linearly independent vectors x, y in X, we say that x is p-orthogonal to y if*

$$A_p(x,y) = \frac{\pi}{2}.$$

It is clear that x, y are p-orthogonal if there exist suitable vectors $z_1, \ldots, z_{n-2}$ such that

$$\|P_{x,z_1,\ldots,z_{n-2},y}\| = 1.$$

Theorem 1. *The concept of p-orthogonality is compatible with the usual orthogonality in the inner product spaces.*

Proof. Let X be an inner product space. First, assume that x, y are orthogonal. We shall show that

$$\|P_{x,z_1,\ldots,z_{n-2},y}\| = 1$$

for suitable choice of $z_1, \ldots, z_{n-2}$. To this end, extending x, y to a basis as $\{x, z_1, \ldots, z_{n-2}, y\}$ to an orthogonal basis for X, we show that

$$\|P\| = 1$$

where $P = P_{x,z_1,\ldots,z_{n-2},y}$ is the orthogonal projection associated with the subspace generated by $\{x, z_1, \ldots, z_{n-2}, y\}$.

Since

$$y \in [span\{x, z_1, \ldots, z_{n-2}\}]^{\perp}$$

and for any z in X, we have

$$Pz \in span\{x, z_1, \ldots, z_{n-2}\}$$

we conclude that

$$y \perp Pz$$

and we have

$$\|z\|^2 = \|z - Pz + Pz\|^2 = \|z - Pz\|^2 + \|Pz\|^2 \geq \|Pz\|^2$$

therefore

$$\|p\| \leq 1,$$

now, taking $z = x$, we have

$$Px = x$$

so

$$\|Px\| = \|x\|$$

hence

$$\|P\| = 1.$$

Next, assume that x, y are not orthogonal. We shall show that

$$\|P_{x,z_1,\dots,z_{n-2},y}\| > 1$$

for all choices of $z_1, \dots, z_{n-2}$. Since

$$\{y\}^{\perp} \neq span\{x, z_1, \dots, z_{n-2}\}$$

there exists a nonzero vector z in $\{y\}^{\perp}$ that does not belong to $span\{x, z_1, \dots, z_{n-2}\}$. For this z we have

$$Pz - z \perp z.$$

We conclude that

$$\|Pz\|^2 = \|Pz - z + z\|^2 = \|Pz - z\|^2 + \|z\|^2 > \|z\|^2$$

therefore

$$\|P\| > 1$$

as claimed.

For a complex linear space, we have already defined the operator orthogonality. We denote this kind of orthogonality by notation $\perp_p$.

Let x, y be two vectors in X. Similar to [**?**], we consider the following set in $\mathbb{C}$ as the orthogonality set of x with respect to y:

$$F(x; y) = \{\mu : \mu \in \mathbb{C}, (x - \mu y) \perp_p y\}$$

or equivalently

$$F(x; y) = \{\mu : \mu \in \mathbb{C}, \|P_{x-\mu y,z_1,\dots,z_{n-2},y}\| = 1\}.$$

Moreover, we can involve an other parameter α for more benefits:

$$F(x; y; \alpha) = \{\mu : \mu \in \mathbb{C}, (\alpha x - \mu y) \perp_p y\}$$

or equivalently

$$F(x; y; \alpha) = \{\mu : \mu \in \mathbb{C}, \|P_{\alpha x-\mu y,z_1,\dots,z_{n-2},y}\| = 1\}.$$

Lemma 1. *For any non zero α, we have the following*

$$F(x; y; \alpha) = \alpha F(x; y; 1).$$

Proof. By definition, we have

$$\begin{aligned} F(x; y; \alpha) &= \{\mu : \mu \in \mathbb{C}, \|P_{\alpha x-\mu y,z_1,\dots,z_{n-2},y}\| = 1\} \\ &= \{\mu : \mu \in \mathbb{C}, \|P_{\alpha(x-\frac{\mu}{\alpha}y),z_1,\dots,z_{n-2},y}\| = 1\} \\ &= \{\mu : \mu \in \mathbb{C}, \|P_{(x-\frac{\mu}{\alpha}y),z_1,\dots,z_{n-2},y}\| = 1\} \end{aligned}$$

since P is homogenized. It turn implies that if $\mu \in F(x;y;\alpha)$, then $\frac{\mu}{\alpha} \in F(x;y;1)$; or $\mu \in \alpha F(x;y;1)$; it completes the proof.

In the following theorems we will see upper semi continuity and lower semi continuity of $F(x;y;\alpha)$ in α, using lemma 1.

Theorem 2. *The set valued function which maps α to $F(x;y;\alpha)$, is upper semi continuous.*

Proof. With out loss of generality, we will show upper continuity at $\alpha_0 = 1$. Continuity at other points are similar.

For more simplicity, fix x, y and let $f(\alpha) = F(x;y;\alpha)$.

Assume that $N(f(1))$ is an open set containing $f(1)$. The following scalars are well defined

$$\beta_1 = \inf\{\beta : f(\beta) \text{ is contained in } N(f(1))\}$$
$$\beta_2 = \sup\{\beta : f(\beta) \text{ is contained in } N(f(1))\}.$$

On the other hand, since $f(1)$ is closed, we have $\beta_1 < 1 < \beta_2$. Now consider the open set around 1,

$$N(1) = \{\alpha : \alpha \in \mathbb{C}, \beta_1 < |\alpha| < \beta_2\}.$$

It is clear that for any α in $N(1)$, $f(\alpha)$ is contained in $N(f(1))$.

Theorem 3. *The set valued function which maps α to $F(x;y;\alpha)$, is lower semi continuous*

Proof. With out loss of generality, we will show lower continuity at $\alpha_0 = 1$. Continuity at other points are similar.

For more simplicity, fix x, y and let $f(\alpha) = F(x;y;\alpha)$.

Assume that $\beta_0 \in f(1)$ is an arbitrary point. Moreover, assume that $N(\beta_0)$ is an open neighborhood of β_0. The following scalars are well defined

$$\beta_1 = \inf\{\beta : f(\beta) \text{ intersects } N(\beta_0)\}$$
$$\beta_2 = \sup\{\beta : f(\beta) \text{ intersects } N(\beta_0)\}.$$

On the other hand, since $f(1)$ is closed, we have $\beta_1 < 1 < \beta_2$. Now consider the open set around 1,

$$N(1) = \{\alpha : \alpha \in \mathbb{C}, \beta_1 < |\alpha| < \beta_2\}.$$

It is clear that for any α in $N(1)$, $f(\alpha)$ intersects $N(\beta_0)$.

Corollary 1. *The set valued function which maps α to $F(x;y;\alpha)$, is semi continuous*

As we proved, this functions is both upper and lower semi continuous, the corollary is hold.

As we saw in theorem 1 , the concept of operator orthogonal vectors in the inner product spaces, is the same of usual orthogonal vectors, i.e. for vector x, y in an inner product space X, we have

$$p(x, y) = 1$$

if and only if

$$\langle x, y\rangle = 0$$

This leads us to a simple computation for the set $F(x; y; \alpha)$ in an inner product space.

Example 2. *In an inner product space we have*

$$F(x; y; \alpha) = \{\mu : \mu \in \mathbb{C}, \langle \alpha x - \mu y, y\rangle = 0\}.$$

It turn implies that

$$F(x; y; \alpha) = \{\mu : \mu \in \mathbb{C}, \alpha\langle x, y\rangle = \mu\langle y, y\rangle\}.$$

or equivalently

$$F(x; y; \alpha) = \{\frac{\alpha\langle x, y\rangle}{\langle y, y\rangle}\}.$$

This means that in inner product spaces, $F(x; y; \alpha)$ is a singleton set. Therefore the concept of semi continuity of this set valued function is the same of its usual continuity.

Example 3. *Let $1 \leq r \leq \infty$. For any two vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in ${l_2}^r$, it has been shown in [8] that the operator angle between x, y is the following*

$$\arcsin\left(\frac{|x_1y_2 - x_2y_1|}{\|x\|_r\|y\|_{\frac{r-1}{r}}}\right).$$

It implies that x, y are operator orthogonal if

$$\|x\|_r\|y\|_{\frac{r-1}{r}} = |x_1y_2 - x_2y_1|.$$

Therefore, in this case we have

$$\begin{aligned} F(x; y; \alpha) &= \{\mu : \mu \in \mathbb{C}, (\alpha x - \mu y) \perp_p y\} \\ &= \{\mu : \mu \in \mathbb{C}, \|\alpha x - \mu y\|_r\|y\|_{\frac{r-1}{r}} = |(\alpha x_1 - \mu y_1)y_2 - (\alpha x_2 - \mu y_2)y_1|\}. \end{aligned}$$

So by definition of $\|x\|_r$, we have

$$F(x; y; \alpha) = \{\mu : \mu \in \mathbb{C}, (|\alpha x_1 - \mu y_1|^r + |\alpha x_2 - \mu y_2|^r)^{\frac{1}{r}} (|y_1|^{\frac{r-1}{r}} + |y_2|^{\frac{r-1}{r}})^{\frac{r}{r-1}} = |(\alpha x_1 - \mu y_1)y_2 - (\alpha x_2 - \mu y_2)y_1|\}.$$

Specially, for $r = 2$,

$$F(x; y; \alpha) = \{\mu : \mu \in \mathbb{C}, (|\alpha x_1 - \mu y_1|^2 + |\alpha x_2 - \mu y_2|^2)^{\frac{1}{2}} (|y_1|^{\frac{1}{2}} + |y_2|^{\frac{1}{2}})^2 = |(\alpha x_1 - \mu y_1)y_2 - (\alpha x_2 - \mu y_2)y_1|\}.$$

Giving $x_1, x_2, y_1, y_2, \alpha$, this is an equation on μ; In fact we have

$$A((\alpha x_1 - \mu y_1)^2 + (\alpha x_2 - \mu y_2)^2) = ((\alpha x_1 - \mu y_1)y_2 - (\alpha x_2 - \mu y_2)y_1)^2$$

where

$$A = (|y_1|^{\frac{1}{2}} + |y_2|^{\frac{1}{2}})^4.$$

It leads to the following

$$\begin{aligned} &A(({y_1}^2 + {y_2}^2)\mu^2 - 2\alpha(x_1y_1 + x_2y_2)\mu + \alpha^2({x_1}^2 + {x_2}^2)) \\ &\quad = \alpha^2({x_1}^2{y_2}^2 - 2x_1x_2y_1y_2 + {x_2}^2{y_1}^2), \end{aligned}$$

and μ is obtained from this equation.

References

[1] M.V. Balashov, Geometric difference of multivalued maps, *Mathematical Notes* **70** (2001), 147–153.

[2] G. Birkhoff, Orthogonality in normed linear spaces, *Duke Math. J.*, **1** (1935), 169–172.

[3] Bonsall F.F. and Duncan J., Numerical Ranges II,*London Mathematical Society Lecture Note Series*, Cambridge University Press, New York (1973).

[4] Christos Chorianopoulos and Panayiotis J. Psarrakos, On the continuity of Birkhoff-James ϵ -orthogonality sets, *Linear Multilinear Algebra*, **61** (2013).

[5] Gustafson K.E. and Rao D.K.M., Numerical Range. The Field of Values of Linear Operators and Matrices, *Springer-Verlag* , New York, (1997).

[6] D. H. Ji, and S. L. Wu, Quantitative characterization of the difference between Birkhoff orthogonality and Isosceles orthogonality, *J. Math. Ana. Appl.*, **323** (2006), 17.

[7] P. M. Milicic, Sur le g-angle dans un espace norme, *Mat. Vesnik*, **45** (1993), 4348.

[8] Chen Zhi-zhi, Lin Wei and Luo Lu-lin, Projections, Birkhoff Orthogonality and Angles in normed spaces, *Communications in Mathematical Research* **27(4)** (2011), 378–384.