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On Concomitants of Dual Generalized Order Statistics for a Bivariate Inverse Exponential Distribution

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Abstract. The concomitants of Dual Generalized Order Statistics for Inverse Exponential distribution have been studied. Specifically the distributional properties of r–th concomitant and joint distribution of r–th and s–th concomitant of dual generalized order statistics have been studied when sample is available from a bivariate inverse exponential distribution.

2010 Mathematics Subject Classifications: 62F15, 62C12

Key Words and Phrases: Concomitants, dual generalized order statistics, bivariate inverse exponential distribution

1. Introduction

The Dual Generalized Order Statistics (DGOS) has been discussed by Burkschat et al. (2003) as a unified model for descendingly ordered variables. The joint distribution of n DGOS given by Burkschat et al. (2003) is

$$\begin{aligned} f_{1,2,\ldots,n:n,m,k}\left(x_1,x_2,\ldots,x_n\right) &= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left\{F\left(x_n\right)\right\}^{k-1}f\left(x_n\right) \\ &\quad\times\left[\prod_{j=1}^{n-1}\left\{F\left(x_i\right)\right\}^{m}f\left(x_i\right)\right], \end{aligned} \tag{1}$$

where $\gamma_j = k + (n-j)(m+1)$. Using (1) the marginal distribution of rth DGOS is given by Burkschat et al. (2003) as

$$f_{r:n,m,k}\left(x\right) = \frac{C_{r-1}}{(r-1)!}f\left(x\right)\left\{F\left(x\right)\right\}^{\gamma_r-1}g_{m(d)}^{r-1}\left[F\left(x\right)\right]. \tag{2}$$

The joint distribution of rth and sth DGOS is given by Burkschat et al. (2003) as

$$f_{r,s:n,m,k}\left(x_1,x_2\right)= \frac{C_{s-1}}{(r-1)!\left(s-r-1\right)!}f\left(x_1\right)f\left(x_2\right)\left\{F\left(x_1\right)\right\}^{m}g_m^{r-1}\left\{F\left(x_1\right)\right\}$$

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$$\times \left\{F\left(x_2\right)\right\}^{\gamma_s-1}\left[h_{m(d)}\left\{F\left(x_1\right)\right\}-h_{m(d)}\left\{F\left(x_2\right)\right\}\right]^{s-r-1}, \tag{3}$$

where $C_{r-1}=\prod_{j=1}^{r}\gamma_r$ and

$$h_m(x)=\begin{cases}\frac{x^{m+1}}{m+1};\ m\neq-1\\ \ln x;\quad m=-1\end{cases}\ ;\ x\in[0,1)$$

$$g_m(x)=\begin{cases}\frac{1}{m+1}\left[1-x^{m+1}\right];\ m\neq-1\\ -\ln x;\qquad m=-1\end{cases}\ ;\ x\in[0,1).$$

The DGOS reduces to reversed order statistics for $m=0$ and $k=1$. The lower record values given by Chandler (1952) emerges as special case of DGOS for $m=-1$.

Since its emergence, several authors have studied the distribution of DGOS for various distributions. Pawlas and Szynal (2001) have obtained the recurrence relations for moments of DGOS for inverse Weibull distribution. Khan et al. (2008) have obtained recurrence relations for single and product moments for exponentiated Weibull distribution. Athar and Faizan (2011) have obtained recurrence relations for moments of DGOS for power function distribution. Kotb et al (2013) have obtained the recurrence relations for single and product moments of DGOS for a general class of inverted distributions which provide relations for inverse Rayleigh and Inverse Weibull distribution as a special case.

The concomitants of DGOS emerge from DGOS when sample from a bivariate distribution is available. The distribution of rth concomitant of DGOS is given in Shahbaz et al. (2016) as

$$f_{[r:n,m,k]}(y)=\int_{-\infty}^{\infty}f(y|x)\,f_{r:n,m,k}(x)\,dx, \tag{4}$$

where $f_{r:n,m,k}(x)$ is given in (2). The joint distribution of two concomitants is given as

$$f_{[r,s:n,m,k]}(y_1,y_2)=\int_{-\infty}^{\infty}\int_{x_1}^{\infty}f(y_1|x_1)\,f(y_2|x_2)\,f_{r,s:n,m,k}(x_1,x_2)\,dx_2dx_1, \tag{5}$$

where $f_{r,s:n,m,k}(x_1,x_2)$ is given in (3). The concomitants of DGOS provide concomitants of lower record values; given by Ahsanullah and Nevzorov (2001); for $m=-1$. The concomitants of reversed order statistics; given by Arnold et al. (2008); appear as special case of (4) and (5) for $m=0$ and $k=1$.

The concomitants of DGOS have been studied by several authors. Ahsanullah and Beg (2006) have studied concomitants of GOS for Gumbel's bivariate exponential distribution. Beg and Ahsanullah (2008) have studied distributional properties for concomitants of Farlie-Gumbel-Morgensterm distributions which have been further studied by BuHamra and Ahsanullah (2013). The concomitants of Farlie-Gumbel-Morgensterm Inverse Rayleigh distribution have been studied by Athar and Nayabuddin (2015) and the concomitants of Farlie-Gumbel-Morgensterm Rayleigh distribution have been studied by Tahmasebi et al. (2016).

Filus and Filus (2001, 2006) have introduced a new class of multivariate distributions as linear function of independent random variables and named them as pseudo distributions.

These distributions have found useful applications where conventional multivariate distributions are not useful. The distribution of concomitants for pseudo distributions have been studied by some authors. Mohsin et al. (2009) have studied the concomitants of lower record values for bivariate pseudo Inverse Rayleigh distribution. Hanif and Shahbaz (2016) have studied the concomitants of GOS for pseudo exponential distribution.

In this paper we have studied the concomitants of DGOS for a bivariate Inverse Exponential distribution. The distribution is defined in the following.

2. A Bivariate Inverse Exponential Distribution

Fillus and Fillus (2001 and 2006) have proposed a method to generate bivariate and multivariate distributions as linear combination of independent variates. They have introduced Pseudo–Normal and Pseudo–Gamma distributions as linear combinations of the normal and gamma random variables. We have defined a Bivariate Inverse Exponential distribution as a compound distribution of two random variables. The distribution is defined below.

Let a random variable X has an inverse Exponential distribution with parameter β. The density function of X is

$$f(x;\beta) = \frac{1}{\beta x^2}\exp\left(-\frac{1}{\beta x}\right) \ ; \ x > 0 \ ; \ \beta > 0. \tag{6}$$

Further, let random variable Y has inverse Exponential distribution with parameter $\phi(x)$, where $\phi(x)$ is some function of random variable X. The density function of Y is

$$f\{y;\phi(x)\} = \frac{1}{\phi(x)\,y^2}\exp\left[-\frac{1}{\phi(x)\,y}\right] \ ;\phi(x) > 0, y > 0. \tag{7}$$

The compound distribution of (6) and (7) will be referred to as a bivariate inverse exponential distribution and is given as

$$f(x,y) = \frac{1}{\beta\phi(x)\,x^2y^2}\exp\left[-\left\{\frac{1}{\beta x} + \frac{1}{\phi(x)\,y}\right\}\right] ;x,y,\beta,\phi(x) > 0. \tag{8}$$

Various choices of $\phi(x)$ provide various bivariate distributions and hence increase the domain of applications. Using $\phi(x) = x$ in (8), we obtain the following Bivariate inverse Exponential distribution

$$f(x,y) = \frac{1}{\beta x^3y^2}\exp\left[-\left\{\frac{1}{\beta x} + \frac{1}{xy}\right\}\right] ;x,y,\beta,\phi(x) > 0. \tag{9}$$

The simple moments for the distribution (9) do not exist. The expression for inverse moments is given in the following.

$$\mu_{-p,-q} = E\left(X^{-p}Y^{-q}\right) = \int_0^\infty\int_0^\infty x^{-p}y^{-q}f(x,y)\,dxdy$$

$$= \int_0^\infty \int_0^\infty x^{-p} y^{-q} \frac{1}{\beta x^3 y^2} \exp\left[-\left\{\frac{1}{\beta x} + \frac{1}{xy}\right\}\right] dxdy,$$

which after simplification becomes

$$\mu_{-p,-q} = \beta^{p-q} \Gamma(p-q+1) \Gamma(q+1). \tag{10}$$

The expression (10) is same as the product moments of pseudo exponential distribution as given by Shahbaz and Shahbaz (2010). The ratio moments for the distribution are given as

$$\begin{aligned} \mu_{q|p} &= E\left(\frac{Y^q}{X^p}\right) = \int_0^\infty \int_0^\infty x^{-p} y^{-q} f(x,y)\, dxdy \\ &= \int_0^\infty \int_0^\infty x^{-p} y^{q} \frac{1}{\beta x^3 y^2} \exp\left[-\left\{\frac{1}{\beta x} + \frac{1}{xy}\right\}\right] dxdy, \end{aligned}$$

which after simplification becomes

$$\mu_{q|p} = \beta^{p+q} \Gamma(p+q+1) \Gamma(1-q). \tag{11}$$

From (10) and (11) we can see that the simple and inverse moments for Y do not exist. The expression or inverse moments for random variable X is readily obtained from (10) or (11) by using $q = 0$ and is given as

$$\mu_{-p} = E\left(X^{-p}\right) = \beta^p \Gamma(p+1).$$

In the following section the distribution of concomitant of DGOS has been derived for (9).

3. Distribution of r–th Concomitant and its Properties

A bivariate inverse exponential distribution has been given in (8) and (9). In this section the distribution of r–th concomitants of dual generalized order statistics (*dgos*) for the distribution given in (9) has been obtained. The distribution of the rth concomitants of *dgos* can be obtained by using (4). In order to obtain the required distribution we first obtain the distribution of rth *dgos* for the distribution given in (6). The distribution of rth *dgos* is obtained by using (2). For this we first note that the cdf, $F(x)$, for the distribution (6) is

$$F(x) = \int_0^x f(t)\, dt = \int_0^x \frac{1}{\beta t^2} \exp\left(-\frac{1}{\beta t}\right) dt = \exp\left(-\frac{1}{\beta x}\right).$$

We also have

$$g_m\{F(x)\} = \frac{1}{m+1}\left[1 - \exp\left(-\frac{m+1}{\beta x}\right)\right].$$

The distribution of rth *dgos* for distribution (6) is therefore

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(m+1)^{r-1}(r-1)!} \frac{1}{\beta x^2} \exp\left(-\frac{\gamma_r}{\beta x}\right)$$

$$\times\left[1-\exp\left(-\frac{m+1}{\beta x}\right)\right]^{r-1}.$$

Expanding last term binomially, we have

$$\begin{aligned} f_{r:n,m,k}(x) &= \frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j} \\ &\quad\times\frac{1}{\beta x^2}\exp\left(-\frac{w_1}{\beta x}\right), \end{aligned} \tag{12}$$

where $w_1=(m+1)j+\gamma_j$.

Also from (6) and (9), the conditional distribution of Y given X is

$$f(y|x)=\frac{1}{xy^2}\exp\left(-\frac{1}{xy}\right)\ ;\ x,y>0. \tag{13}$$

Now using (12) and (12) in (4) we have

$$\begin{aligned} f_{[r:n,m,k]}(y) &= \frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\sum_{j=0}^{r-1}\binom{r-1}{j} \\ &\quad\times\int_0^\infty\frac{1}{\beta x^3y^2}\exp\left[-\frac{1}{x}\left\{\frac{w_1}{\beta}+\frac{1}{y}\right\}\right]dx, \end{aligned}$$

which after simplification becomes

$$f_{[r:n,m,k]}(y)=\frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\sum_{j=0}^{r-1}\binom{r-1}{j}\frac{\beta}{(\beta+w_1y)^2}\ ;\ y>0. \tag{14}$$

It is interesting to note that the distribution of rth concomitant of *dgos* for bivariate inverse exponential distribution provide certain distributions as special case. The distribution of rth concomitant for reversed order statistics can be obtained from (14) by using $m=0$ and $k=1$ in (14). The distribution of rth concomitant of lower record values for bivariate inverse exponential distribution can be obtained from (14) by setting $m=-1$. The pth moment of rth concomitant of *dgos* is immediately written as

$$\mu^p_{[r:n,m,k]}=\frac{\beta^p pC_{r-1}\Gamma(p)\Gamma(1-p)}{(r-1)!(m+1)^{r-1}}\sum_{j=0}^{r-1}\binom{r-1}{j}w_1^{-(p+1)}, \tag{15}$$

which exist for $|p|<1$. Expression of moments for special cases can be obtained from (15).

4. Joint Distribution of the Concomitants and Moments

In this section we have derived the joint distribution of the concomitants of the *dgos* for bivariate inverse exponential distribution given in (9). The distribution of bivariate concomitant can be obtained by using (5). In order to obtain the distribution (5), we first obtain the joint distribution of two *dgos* by using (3). The joint distribution of two *dgos* for inverse exponential distribution given in (6) is

$$\begin{aligned} f_{r,s:n,m,k}(x_1,x_2) &= \frac{C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!}\frac{1}{\beta x_1^2}\exp\left(-\frac{m+1}{\beta x_1}\right) \\ &\times\frac{1}{\beta x_2^2}\exp\left(-\frac{\gamma_s+1}{\beta x_2}\right)\left[1-\exp\left(-\frac{m+1}{\beta x_1}\right)\right]^{r-1} \\ &\times\left[\exp\left(-\frac{m+1}{\beta x_1}\right)-\exp\left(-\frac{m+1}{\beta x_2}\right)\right]^{s-r-1}. \end{aligned}$$

On simplification we have

$$\begin{aligned} f_{r,s:n,m,k}(x_1,x_2) &= \frac{C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!}\frac{1}{\beta^2x_1^2x_2^2} \\ &\times\sum_{j=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \\ &\times\exp\left\{-\frac{(m+1)(s-r-j+i)}{\beta x_1}\right\}\exp\left\{-\frac{(m+1)j+\gamma_s}{\beta x_2}\right\} \end{aligned}$$

or

$$\begin{aligned} f_{r,s:n,m,k}(x_1,x_2) &= \frac{C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!}\frac{1}{\beta^2x_1^2x_2^2} \\ &\times\sum_{j=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \\ &\times\exp\left\{-\frac{w_2}{\beta x_1}\right\}\exp\left\{-\frac{w_3}{\beta x_2}\right\},\ 0<x_1<x_2<\infty \end{aligned} \tag{16}$$

where $w_2=(m+1)(s-r-j+i)$ and $w_3=(m+1)j+\gamma_s$. Now using (13) and (16) in (4), the joint distribution of two concomitants of *dgos* for inverse exponential distribution is given as

$$\begin{aligned} f_{[r,s:n,m,k]}(y_1,y_2) &= \frac{C_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!}\frac{1}{\beta^2x_1^2x_2^2} \\ &\times\sum_{j=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j} \\ &\times\int_0^\infty\int_{x_1}^\infty\frac{1}{\beta^2x_1^3y_1^2}\frac{1}{x_2^3y_2^2}\exp\left\{-\frac{1}{x_1}\left(\frac{w_2}{\beta}+\frac{1}{y_1}\right)\right\} \\ &\times\exp\left\{-\frac{1}{x_1}\left(\frac{w_3}{\beta}+\frac{1}{y_2}\right)\right\}dx_2dx_1, \end{aligned}$$

On simplification we have

$$f_{[r,s:n,m,k]}(y_1,y_2) = \frac{C_{s-1}\beta^2}{(m+1)^{s-2}(r-1)!(s-r-1)!}$$

$$\times \sum_{j=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j}$$
$$\times \frac{y_1^2 \{4\beta + (3w_2 + w_3)\, y_1\}}{y_2^2 (\beta + w_2 y_1)^2 \{2\beta + (w_2 + w_3)\, y_1\}^3}. \quad (17)$$

The product moments can be obtained by using (17).

5. Conclusions and Recommendations

In this paper we have obtained the distribution of concomitants and joint distribution of two concomitants of dual generalized order statistics when a sample is available from bivariate inverse exponential distributions. We have seen that the distribution of concomitant of dual generalized order statistics is a linear combination of Burr type distributions. The positive integer moments for distribution of concomitant does not exist but the negative moments can be computed upto any fractional number.

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