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Computing μ-values for representations of symmetric groups in engineering systems

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Abstract. In this article we consider the matrix representations of finite symmetric groups S_n over the filed of complex numbers. These groups and their representations also appear as symmetries of certain linear control systems [5]. We compute the structure singular values (SSV) of the matrices arising from these representations. The obtained results of SSV are compared with well-known MATLAB routine mussv.

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1. Introduction

In [5], Danielson used symmetric groups to design model predictive controllers with reduced complexity for constrained linear control systems. In model predictive control, the control input is obtained by solving a constrained finite time optimal control problem. For a piecewise affine control law symmetries are state-space and input-space transformations that relate controller pieces. Using symmetry he could discard some of the pieces of a given controller. These discarded pieces can also be reconstructed using symmetry. Using symmetries of the control system he was able to reduce the complexity of the controller and save memory without sacrificing performance. It was also noted that the amount of reduction in complexity depends on the number of symmetries possessed by the system. For systems with large symmetry groups the techniques presented in [5] can significantly reduce the complexity of the piecewise affine control-law produced using explicit model predictive control.

In this paper we consider the characters of the groups of finite number of symmetries and construct their representations. These representations in particular give us matrix generators for these groups. We then compute the μ-values (structured singular values)

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for these matrices. The structured singular value [11] is very important tool being used in linear control theory which allows to discuss mathematical problems related to both stability and instability analysis of feedback systems subject to a certain class of perturbations. A well defined complex/real Linear Fractional Transformation LFT's can be used to cover parametric perturbations addressed by SSV incorporated into feedback systems. We suggest to read [1, 3, 8–11, 13] on more about SSV and its application in system theory.

The exact computation of the SSV, especially in higher dimensions is notoriously hard, in fact Non-deterministic Polynomial time that is NP hard [2] to investigate. The numerical methods available in literature provides the approximation of both upper and lower bounds of structured singiular values. The message from upper bound is to provide the conditions which guarantee stability of linear systems, while on the other hand the message from a lower bound is to provide sufficient conditions for instability analysis of the feedback systems in the linear control theory.

The well-known MATLAB function mussv available in the Matlab Control Toolbox can be used to approximate an upper bounds of SSV by help of both diagonal balancing and Linear Matrix Inequlaity techniques [6]. While the computation of lower bound is quite possible by help of generalized version of power method, see in [12] for more detail.

In this paper the main contribution is towards the comparison of both lower and upper bounds of SSV subject to class of mixed real and complex uncertainties. We also consider the case when pure real and pure complex uncertainties are under consideration.

1.1. Group Representations

Let G be a group, k be a field and $GL(n, k)$ be the group of invertible $n \times n$ matrices. A representation ρ of G is a homomorphism from G to $GL(n, k)$. Let $V = k^n$ ($n-$dimensional vector space over k) then we can make V into a $kG-$module by defining $g \cdot v = \rho(g)v$. Equivalently every $kG-$module V gives rise to a representation σ of G. In this paper we will write $G-$module instead of $kG-$module. We will say that a $G-$module is irreducible if it has no nontrivial $G-$submodules. A $G-$module is said to be completely reducible if it can be written as a direct sum of irreducible $G-$submodules. Moreover a representation is irreducible (completely reducible) if the corresponding $G-$module is irreducible (completely reducible). It is an important problem in representation theory of groups to classify all possible irreducible representations of a given group G. For a detailed account on representation theory see [15].

For the rest of this paper let $k = \mathbb{C}$, the field of complex numbers and G be a finite group. The Mashke's theorem states that every nonzero $G-$module is completely reducible. This theorem guarantees that finding the irreducible representations of G gives us all possible representations of G. It is worth mentioning here that in general Mashke's theorem does not hold for infinite group or for fields other then the field of complex numbers. Also note that writing down all possible irreducible representations of G is not always easy.

Let $\rho : \ G \to GL(n, k)$ be a representation of G. A function $\chi : \ G \to k$ defined by

g_i	1	(12)	(123)
$\lvert C_G(g_i)\rvert$	6	2	3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

g_i	1	(12)	(123)	(12)(34)	(1234)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

$\chi(g) = \text{trace}(\rho(g))$ is called the character of the representation ρ. The characters of a group G are the characters of its representations. A character χ is said to be irreducible if it corresponds to an irreducible representation. It is true in general that the number of irreducible representations of G is equal to number of conjugacy classes of G (finite). For characters χ and ψ of G we can define an inner product of character of G by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \Sigma_{g \in G} \chi(g) \overline{\psi(g)}.$$

Suppose χ is a character of a $G-$module V then V is irreducible if and only if $\langle \chi, \chi \rangle = 1$. We can classify all possible irreducible characters of G and this in turn gives us a classification of irreducible representations of G. A character table of G is a table which lists character values for all irreducible characters of G.

We now turn our attention to the special case when $G = S_n$, the symmetric group on n letters. The symmetric group S_3 has three conjugacy classes and hence three irreducible characters. The character table for S_3 is given below

In this table the character χ_3 can be obtained from the permutation representation and the two linear characters correspond to the Abelian group $G/G^{'}$, where $G^{'}$ is the derived subgroup of G.

The symmetric group S_4 has five conjugacy classes and the character table of S_4 is given below

In the above table linear characters correspond to the Abelian group $G/G^{'}$ and the character χ_4 can be obtained from the permutation character of G. More over χ_5 is the product of irreducible characters χ_2 and χ_4. Moreover the character χ_3 of G can be obtained by lifting the character of the subgroup of S_4 generated by the permutation (12)(34).

1.2. From Characters To Representations

Finding an irreducible representation corresponding to a given character is a historical problem which has been around since late eighteenth century. For a detailed history of the problem see [4]. It is not always straight forward to come up with the right representation although it always exists. Several methods has been proposed to construct these representations with various limitations. In this paper we use an algorithm given by [4] to compute the representations of S_n for $n = 3, 4$. His algorithm has been implemented in the GAP [7] package "RepSn".

For the group S_3 the only representation we have to compute is for the character χ_3. The matrix representation is given by the matrices A and B where

$$A = \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{bmatrix}; \quad B = \begin{bmatrix} 0 & \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \end{bmatrix}.$$

It is worth mentioning here that A and B are matrices of order 3 and 2 respectively and therefore are matrix generators for S_3.

For the group S_4 we need to construct representations for χ_3, χ_4 and χ_5. The matrix representation for χ_3 is given by the matrices A_1 and B_1 where

$$A_1 = \begin{bmatrix} 0 & \frac{-1}{2} + \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

These are matrix generators of S_4. For χ_4 we get

$$A_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad B_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Similarly for χ_5 the matrix representation of S_5 is given by the matrices A_3 and B_3 where

$$A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}; \quad B_3 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

In this paper we will compute the $\mu-$values for the matrices obtained from the representations of symmetric group S_n for $n = 3, 4$. In the following subsection we give definition of SSV. In the subsequent sections we present the algorithm for SSV and discuss our main results.

Consider n-dimensional real (or complex) matrix M and a set of block diagonal matrices $\Delta_{\mathbb{B}}$, where

$$\Delta_{\mathbb{B}} = \{diag(\Gamma_i, \alpha_j I_j) : \Gamma_i \in \mathbb{C}^{m_i,m_i}(\mathbb{R}^{m_i,m_i}), \alpha_j \in \mathbb{C}(\mathbb{R})\},$$

where I_j denotes the identity matrix with the dimension j. We give following definition of SSV.

Definition 1.2 [11]. For a $n \times n$ dimensional real or complex matrix M and consider a family of block diagonal matrices $\Delta_{\mathbb{B}}$. Then the structured singular value denoted by $\mu_{\Delta_{\mathbb{B}}}(M)$ is given as

$$\mu_{\Delta_{\mathbb{B}}}(M) := \frac{1}{\min\{\|\Delta\|_2 : \Delta \in \Delta_{\mathbb{B}}, |(I - M\Delta)| = 0\}}. \tag{1}$$

In Definition 1.2, $|\cdot|$ denotes the determinant of a matrix while matrix 2-norm is give by $\|\cdot\|$. From above definition of SSV, its clear that $\mu_{\Delta_{\mathbb{B}}}(M) = 0$ if $|(I - M\Delta)| \neq 0$ for all $\Delta \in \Delta_{\mathbb{B}}$. The most important case under consideration is when $\Delta_{\mathbb{B}}$ allows only pure complex uncertainties. In this case we write $\Delta_{\mathbb{B}}^*$ instead of $\Delta_{\mathbb{B}}$. The pure complex uncertainties $\Delta \in \Delta_{\mathbb{B}}^*$ gives $exp(i\varphi)\Delta \in \Delta_{\mathbb{B}}^*$ for any $\varphi \in \mathbb{R}$. As a result, this gives us a suitable choice of $\Delta \in \Delta_{\mathbb{B}}^*$ such that spectral radius attains the maximum value 1 that is $\rho(M\Delta) = 1$ iff there is $\Delta' \in \Delta_{\mathbb{B}}^*$, which possesses the same matrix 2-norm so that $M\Delta'$ posses an eigenvalue 1, this in turn implies that the matrix $(I - M\Delta')$ is singular. From above discussion on pure complex uncertainties we have following alternative expression for SSV:

$$\Delta_{\mathbb{B}}^* = \frac{1}{\min\left\{\|\Delta\|_2 : \Delta \in \Delta_{\mathbb{B}}^*, \rho(M\Delta) = 1\right\}}, \tag{2}$$

where $\rho(\cdot)$ is the spectral radius of a matrix $M\Delta$, that is, maximum absolute value of eigenvalue of $M\Delta$.

2. SSV based on structured spectral sets

Structured spectral value set of a given n dimensional complex M with respect to ϵ_0, the perturbation level is given as

$$\Lambda_{\epsilon_0}^{\Delta_{\mathbb{B}}}(M) = \{\lambda \in \Lambda(\epsilon_0 M\Delta) : \Delta \in \Delta_{\mathbb{B}}\}, \tag{3}$$

where $\Lambda(\cdot)$ contains all eigenvalues of a matrix while Δ has a unit 2-norm. The above set $\Lambda_{\epsilon_0}^{\Delta_{\mathbb{B}}}(M)$ is simply a disk centered at the origin say O when $\Delta_{\mathbb{B}}^*$ contains only pure complex uncertainties. The structured spectral values set

$$\Sigma_{\epsilon}^{\Delta_{\mathbb{B}}}(M) = \{\xi = 1 - \lambda : \lambda \in \Lambda_{\epsilon_0}^{\Delta_{\mathbb{B}}}(M)\}, \tag{4}$$

allows us to write down the structured singular values defined in Equ. (2) as below when both real and complex uncertainties are under consideration

$$\mu_{\Delta_{\mathbb{B}}}(M) = \frac{1}{\arg\min\{0 \in \Sigma_{\epsilon_0}^{\Delta_{\mathbb{B}}}(M)\}}, \tag{5}$$

For a purely complex uncertainties $\Delta_{\mathbb{B}}^*$, one can rewrite Equ. (3)to express SSV as

$$\mu_{\Delta_{\mathbb{B}}^*}(M) = \frac{1}{\arg\min\{\max|\lambda| = 1\}}, \tag{6}$$

where $\lambda \in \Lambda_{\epsilon_0}^{\Delta_{\mathbb{B}}^*}(M)$.

2.1. Mathematical Problem

Consider the following optimization problem

$$\xi(\epsilon_0) = \arg\min |\xi|. \tag{7}$$

In Equ. (7), $\xi \in \Sigma_{\epsilon_0}^{\Delta_{\mathbb{B}}}(M)$. From above discussion we obtain a fact that SSV $\mu_{\Delta_{\mathbb{B}}}(M)$ is the reciprocal of the minimum value of perturbation level for which $\xi(\epsilon_0) = 0$. In order to overcome this difficulty we give a two-level algorithm, that is inner and outer algorithm: By the help of inner algorithm, we obtain a solution corresponding to the optimization problem as addressed in Equ. (7) while outer algorithm helps to vary the perturbation level by using fast Newton iteration. We first construct a gradient system of Ordinary Differential Equations (ODE's) and then solve it. This gradient system of ODE's in turn solve the optimization problem addressed in Equ. (7). While the case of a purely complex uncertainties $\Delta_{\mathbb{B}}^*$ can be addressed by taking an inner algorithm to compute a local optima for the following maximization problem

$$\lambda(\epsilon_0) = \arg\max |\lambda|, \tag{8}$$

where $\lambda \in \Lambda_{\epsilon_0}^{\Delta_{\mathbb{B}}^*}(M)$ This produces a lower bound for $\mu_{\Delta_{\mathbb{B}}^*}(M)$.

3. Purely Complex uncertainties [14]

This section is devoted to the case of pure complex perturbations. We estimate SSV $\mu_{\Delta_{\mathbb{B}}^*}(M)$ for the given n-dimensional complex matrix M while taking inner problem discussed in Equ. (8) into account. The set of purely complex perturbations is defined as

$$\Delta_{\mathbb{B}}^* = \{diag(\alpha_1 I_1, ..., \alpha_n I_n; \Delta_1, ..., \Delta_F) : \alpha_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j, m_j}\}. \tag{9}$$

We make use of the following eigenvalue perturbation result in order to compute the derivative of an eigenvalue $\lambda(t)$.

Lemma 3.1. Let $\tau : \mathbb{R} \to \mathbb{C}^{n,n}$ and consider that $\lambda(t)$ is an eigenvalue of $\tau(t)$ which converges towards a simple eigenvalue λ_0 of $\tau_0 = \tau(0)$ as $t \to 0$. Then the simple eigenvalue $\lambda(t)$ is analytic near $t = 0$ with

$$\dot{\lambda}(t)|_{t=0} = \frac{w_0^* \tau_1 v_0}{w_0^* v_0},$$

where $\tau_1 = \dot{\tau}(0)$ and v_0, w_0 are right and left eigenvectors of τ_0 associated to λ_0, that is, $(\tau_0 - \lambda_0 I)v_0 = 0$ and $w_0^*(\tau_0 - \lambda_0 I) = 0$. Since our main objective is to solve the optimization problem as discussed in Equ. (8). For this we need to compute a perturbation Δ_{local} so that $\rho(\epsilon A \Delta_{local})$ has the maximum growth among all $\Delta \in \Delta_{\mathbb{B}}^*$ while $\|\Delta\|$ posses a unit a unit 2-norm. In the following we give definition of local extremizer of a structured spectral value set.

Definition 3.2 [14]. A matrix $\Delta \in \Delta_{\mathbb{B}}^{*}$ so that $\|\Delta\|$ possesses a unit 2-norm while $(\epsilon_0 M\Delta)$ has a maximum eigenvalue which maximizes (locally) $\Lambda_{\epsilon_0}^{\Delta_{\mathbb{B}}^{*}}(M)$ is known a local extremizer of structured spectral value set. We give following theorem in order to compute the local extremizer of structured spectral value set.

Theorem 3.3 [14]. Let's suppose that

$$\Delta_{local} = diag(\alpha_1 I_1, ..., \alpha_n I_n; \Delta_1, ..., \Delta_F), \ \ \|\Delta_{local}\|_2 = 1,$$

is an extremizer of structured epsilon spectral value set $\Lambda_{\epsilon}^{\Delta_{\mathbb{B}}^{*}}(A)$. Further consider that $\epsilon_0 M\Delta_{local}$ possesses a simple greatest eigenvalue $\lambda = |\lambda|e^{i\theta}$ having v and w as right and left eigenvectors which are scaled as $s = e^{i\theta}w^{*}v > 0$. Upon the partitioning, we have

$$v = (v_1^{\mathrm{T}}, \ \ldots, \ v_n^{\mathrm{T}}, \ v_{n+1}^{\mathrm{T}}, \ \ldots \ , v_{n+F}^{\mathrm{T}})^{\mathrm{T}}; \quad u = A^{*}w = (u_1^{\mathrm{T}}, \ \ldots \ , u_n^{\mathrm{T}}, \ u_{n+1}^{\mathrm{T}}, \ \ldots \ , u_{n+F}^{\mathrm{T}})^{\mathrm{T}}, \tag{10}$$

Also consider that,

$$u_k^{*}v_k \neq 0 \quad \forall \ k = 1, \ldots, n \tag{11}$$

$$\|u_{n+l}\|_2 \cdot \|v_{n+l}\|_2 \neq 0 \quad \forall \ l = 1, \ldots, F. \tag{12}$$

Then

$$|s_k| = 1 \quad \forall \ k = 1, \ldots, n \quad \text{and} \quad \|\Delta_l\|_2 = 1 \quad \forall l = 1, \ldots, F,$$

this means that all blocks of Δ_{local} possesses unit 2-norm. In following theorem we replace full blocks in local extremizer with rank-1 matrices, in turn, this allow us to work to Forbenius norm instead with matrix 2-norm.

Theorem 3.4 [14]. Let's suppose that

$$\Delta_{local} = diag(\alpha_1 I_1, ..., \alpha_n I_n, \Delta_1, ..., \Delta_F)$$

is a local maximizer and also consider that λ, v, u as defined and partitioned in the previous given theorem. Further assume that Equ. (12) holds true and every single block Δ_h possesses a singular value 1 which having the singular vectors $q_l = \gamma_l w_{n+l}/\|w_{n+l}\|_2$ and $r_l = \gamma_l u_{n+l}/\|u_{n+l}\|_2$ for $|\gamma_l| = 1$. Furthermore, the matrix

$$\widehat{\Delta} = diag(\alpha_1 I_1, ..., \alpha_n I_n, a_1 b_1^{*}, ..., a_F b_F^{*})$$

is also a maximizer, that is $\rho(\epsilon M\Delta_{local}) = \rho(\epsilon M \widehat{\Delta})$.

3.1. A system of ODEs to approximate the extremal points of structured spectral values set.

First of all we compute a matrix valued function denoted as $\Delta(t)$. This matrix valued function will help to have a maximum growth for the largest eigenvalue $|\lambda(t)|$, with $\lambda(t) \in \Lambda_{\epsilon_0}^{\Delta_{\mathbb{B}}*}(M)$ and then finally we construct and solve a system of ordinary differential equations. For this system of ODE's the matrix valued function $\Delta(t)$ acts as the initial approximation.

3.2. The local optimization problem.

Consider that $\lambda = |\lambda|e^{i\theta}$ is eigenvalue with algebraic multiplicity 1 and having the eigenvectors v, w which are normalized such that

$$\|w\| = \|v\| = 1, \qquad w^*v = |w^*v|e^{-i\theta}. \tag{13}$$

As a result of the previous Lemma 3.1, we have

$$\frac{d}{dt}|\lambda|^2 = 2|\lambda| Re\Big(\frac{u^*\dot{\Delta}v}{e^{i\theta}w^*v}\Big) = \frac{2|\lambda|}{|w^*v|} Re(u^*\dot{\Delta}v), \tag{14}$$

where $u = M^*w$. In fact the dependence on t is intentionally omitted.

Now we consider $\Delta \in \Delta_{\mathbb{B}}$ and we aim to compute $\dot{\Delta} = U$. This direction will locally maximizes the growth of trajectory of the modulus of $\lambda(t)$. As a result, finally we get

$$U = diag(\omega_1 I_{r_1}, \ldots, \omega_s I_{r_N}, \Omega_1, \ldots, \Omega_F). \tag{15}$$

The result as given in Equ. (15) is the solution of the following maximization problem

$$U_* = \arg\max\{Re(u^*Ux)\}$$

$$\text{subject to } \ Re(\overline{\delta}_i \omega_i) = 0, \ \ i = 1 : N,$$

$$\text{and } \quad Re\langle \Delta_j, \Omega_j \rangle = 0, \ \ j = 1 : F. \tag{16}$$

We give following Lemma 3.5 in order to solve the maximization problem as discussed in above Equ. (8).

Lemma 3.5. We make use of the notation as already introduced in above theorem and partitioning of the v, u as earlier, a solution of the maximization problem discussed in Equ. (15) is given by

$$U_* = diag(\omega_1 I_{r_1}, \ldots, \omega_N I_{r_N}, \Omega_1, \ldots, \Omega_F), \tag{17}$$

with

$$\omega_i = \nu_i \left(v_i^* u_i - Re\left(v_i^* u_i \bar{s}_i\right) s_i\right), \quad i = 1, \ldots, N \tag{18}$$

$$\Omega_j = \zeta_j \left(u_{N+j} v_{N+j}^* - Re\langle \Delta_j, u_{N+j} v_{N+j}^* \rangle \Delta_j\right), \quad j = 1, \ldots, F. \tag{19}$$

The coefficient ν_i is strictly positive. The ν_i is the reciprocal of the absolute value of the right-hand side in Equ. (18), if this is different from zero, and $\nu_i = 1$ otherwise. In a similar fashion, ζ_j is also obtained as strictly positive and is the reciprocal of the Frobenius norm of the matrix on the right hand side in Equ. (19), if this is different from zero, and $\zeta_j = 1$ otherwise. The result of previous Lemma 3.5 can be expressed as

$$U_* = D_1 P_{\Delta_{\mathbb{B}}^*}(uv^*) - D_2 \Delta. \tag{20}$$

In Equ. (20), $P_{\Delta_{\mathbb{B}}^*}(\cdot)$, denotes the orthogonal projection while the matrices $D_1, D_2 \in \Delta_{\mathbb{B}}^*$ are diagonal matrices with D_1 having structure such that all of its eigenvalues are positive.

4. System of Ordinary Differential Equation's

The Lemma 3.5, as discussed previously suggest us to consider the differential equation $\dot{\Delta}(t)$ on the manifold $\Delta_{\mathbb{B}}^*$:

$$\dot{\Delta}(t) = D_1 P_{\Delta_{\mathbb{B}}^*}(uv^*) - D_2 \Delta(t). \tag{21}$$

In Equ. (21), $v(t)$ is an eigenvector possesses the unit 2-norm and is associated to the simple eigenvalue $\lambda(t)$ of $\epsilon_0 M \Delta(t)$. Here we also note that fact that $u(t), D_1(t), D_2(t)$ depend on $\Delta(t)$. The differential equation as obtained in Equ. (21) generates a system of ODE's. This system of ODE's is a gradient system of ODE's because, by definition, the right-hand side is nothing but the projected gradient of $U \mapsto Re(u^* U v)$.

4.1. Computation of initial value matrix $\Delta(t)$ and ϵ_1 [14].

In order to obtain a suitable choice for the initial valued matrix function $\Delta_0(t)$ which acts as the initial value to solve the gradient system of ODE's is given as below.

$$\Delta_0 = D\, P_{\Delta_{\mathbb{B}}}(wv^*), \tag{22}$$

where D is positive diagonal matrix. The matrix D is chosen so that $\Delta_0 \in \Delta_{\mathbb{B}}$. On the other hand, a straight forward but a very natural choice for ϵ_0 is given by just computing the reciprocal of the upper bound $\widehat{\mu}_{\Delta_{\mathbb{B}}}(M)$ of SSV by using mussv function. That is,

$$\epsilon_0 = \frac{1}{\widehat{\mu}_{\Delta_{\mathbb{B}}}(M)} \tag{23}$$

4.2. Outer algorithm to compute SSV

In this section of the article, we give a brief discussion on the outer algorithm. As the principles are the same, one can deal with the purely complex perturbations in a great. The same discussion is true for the when we have mixed real and complex uncertainties.

5. Numerical Experimentation

In this section, we give the main contribution towards our article. We establish the numerical results for the approximation of both lower and upper bounds of SSV for a set of matrices obtianed by the representation of symmetric groups S_n for $n = 3.4$. Finally, we compare our obtained results with the one ontained by Matlab function mussv.

Example 1. Consider the following two dimensional complex matrix A.

$$A = \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{bmatrix}.$$

We take the perturbation set as,

$$\Delta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{2,2}\}.$$

First, by using mussv function, we obtain the perturbation structure $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} 0 & 0 \\ 0 & -0.5000 - 0.8660i \end{bmatrix}.$$

Here, $\|\widehat{\Delta}\|_2 = 1$. The value of upper bound is obtained as, that is, $\mu_{PD}^{upper} = 1.0000$ while the same lower bound is obtained, that is, $\mu_{PD}^{lower} = 1.0000$.
Now, by using algorithm [14], we obtain the perturbation structure $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} 0 & 0 \\ 0 & -0.5000 - 0.8660i \end{bmatrix}.$$

Here, $\epsilon^* = 1.0000$ and $\|\Delta^*\|_2 = 1$, we got the same lower bound, that is, $\mu_{New}^{lower} = 1.0000$. The bounds of structured singular value for above matrix A when the perturbation set is considered as,

$$\Delta_{\mathbb{B}} = \{diag(\delta_1 I_2) : \delta_1 \in \mathbb{C}\},$$

are as follows. First, we apply the mussv function and we obtain the perturbation $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} -0.5000 + 0.8660i & 0 \\ 0 & -0.5000 + 0.8660i \end{bmatrix}.$$

Here, $\|\widehat{\Delta}\|_2 = 1$. For this particular example, we obtain the upper bound $\mu_{PD}^{upper} = 1.0000$. The value of lower bound also remain same, that is,$\mu_{PD}^{lower} = 1.0000$.
Now, by using algorithm [14], we obtain the perturbation structure $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -0.5000 - 0.8660i & 0 \\ 0 & -0.5000 - 0.8660i \end{bmatrix}.$$

Here, $\epsilon^* = 1.0000$ and $\|\Delta^*\|_2 = 1$. The same lower bound is obtained for this particular example, that is, $\mu_{New}^{lower} = 1.0000$, the one approximated by MATLAB function mussv.

Example 2. In Figure 1, we show the comparison of lower bounds computed by algorithm [14] with the bounds (Lower and Upper) computed by mussv function for matrix valued function $B(w)$ for $w = 1 : 4$, where $w \in \Omega$ and Ω denotes the frequency range of interest which is usually $\mathbb{R}^+$. The frequency response w is the quantitative measure of output of $(M - \Delta)$ system. We use mussv function to compute μ as a function of frequency response.

Example 3. Consider the following two dimensional complex matrix A_1.

$$A_1 = \begin{bmatrix} 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \end{bmatrix}.$$

We take the perturbation set as,

$$\Delta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{2,2}\}.$$

First, by using mussv function, we obtain the perturbation structure $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} 0 & 0 \\ -0.5000 - 0.8660i & 0 \end{bmatrix}.$$

Here, $\|\widehat{\Delta}\|_2 = 1$. The value of upper bound is obtained as $\mu_{PD}^{upper} = 1.0000$ while the same lower bound as obtained, that is, $\mu_{PD}^{lower} = 1.0000$.
Now, by using algorithm [14], we obtain the perturbation structure $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -0.5000 - 0.0000i & -0.2500 + 0.4330i \\ -0.2500 - 0.4330i & -0.5000 - 0.0000i \end{bmatrix}.$$

Here, $\epsilon^* = 1.0000$ and $\|\Delta^*\|_2 = 1$. For this particular example, we got the same lower bound, that is, $\mu_{New}^{lower} = 1.0000$.
The bounds of structured singular value for above matrix A_1 when the perturbation set is considered as,

$$\Delta_{\mathbb{B}} = \{diag(\delta_1 I_2) : \delta_1 \in \mathbb{C}\},$$

are as follows. First, we apply the mussv function and we obtain the perturbation $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} -0.2000 + 0.8660i & 0 \\ 0 & -0.2000 + 0.8660i \end{bmatrix}.$$

Here $\|\widehat{\Delta}\|_2 = 1$. For this particular example, we obtain the upper bound $\mu_{PD}^{upper} = 1.0000$. The value of lower bound remains same, that is, $\mu_{PD}^{lower} = 1.0000$.
Now, by using algorithm [14], we obtain the perturbation structure $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -0.4000 - 0.8660i & 0 \\ 0 & -0.4000 - 0.8660i \end{bmatrix}.$$

Here, $\epsilon^* = 1.0000$ and $\|\Delta^*\|_2 = 1$. The same lower bound is obtained for this particular example, that is, $\mu_{New}^{lower} = 1.0000$, the one approximated by MATLAB function mussv.

Example 4. In Figure 2, we show the comparison of lower bounds computed by algorithm [14] with the bounds (Lower and Upper) computed by mussv function for matrix valued function $B_1(w)$ for $w = 1 : 7$, where $w \in \Omega$ and Ω denotes the frequency range of interest which is usually $\mathbb{R}^+$. The frequency response w is the quantitative measure of output of $(M - \Delta)$ system. We use mussv function to compute μ as a function of frequency response.

Example 5. Consider the following three dimensional real valued matrix A_2.

$$A_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We take the perturbation set as,

$$\Delta_{\mathbb{B}} = \{diag(\delta_1 I_1, \delta_2 I_1, \delta_3 I_1) : \delta_1, \delta_2, \delta_3 \in \mathbb{R}\}.$$

First, by using mussv function, we obtain the perturbation structure $\widehat{\Delta}$ with

$$\widehat{\Delta} = 1.0e + 050 \begin{bmatrix} 5.0000 & 0.0000 & 0.0000 \\ 0.00000 & 5.0000 & 0.00000 \\ 0.0000 & 0.0000 & 5.00000 \end{bmatrix}.$$

Here, $\|\widehat{\Delta}\|_2 = 5.0000e+050$. For this particular example, we obtain the upper bound, that is, $\mu_{PD}^{upper} = 1.0000$. In this case the obtained lower bound is $\mu_{PD}^{lower} = 0.0000$.
Now, by using algorithm [14], we obtain the perturbation structure $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -1.0000 & 0.0000 \\ 0.0000 & 0.0000 & -1.0000 \end{bmatrix}.$$

Here $\epsilon^* = 1.0000$ and $\|\Delta^*\|_2 = 1.0000$. In this case, same lower bound is obtained, that is, $\mu_{New}^{lower} = 1.0000$, as the one obtained by mussv function.

Example 6. Consider the following three dimensional real valued matrix B_2.

$$B_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We take the perturbation set as,

$$\Delta_{\mathbb{B}} = \{diag(\delta_1 I_1, \Delta_1) : \delta_1 \in \mathbb{R}, \Delta_2 \in \mathbb{C}^{2,2}\}.$$

First, by using mussv function, we obtain the perturbation structure $\widehat{\Delta}$ with

$$\widehat{\Delta} = 1.0e + 050 \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.00000 & 0.0000 & 0.00000 \\ 0.0000 & 0.0000 & -1.00000 \end{bmatrix}.$$

Here, $\|\widehat{\Delta}\|_2 = 1.0000$. For this particular example, we obtain the upper bound $\mu_{PD}^{upper} = 1.0000$ while a same lower bound is obtained, that is, $\mu_{PD}^{lower} = 1.0000$.
Now, by algorithm [14], we have obtained the perturbation structure $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

Here, $\epsilon^* = 1.0000$ and $\|\Delta^*\|_2 = 1.0000$. In this case, the same lower bound is obtained $\mu_{New}^{lower} = 1.0000$ as the one obtained by mussv function. The obtained bounds for SSV for above matrix M when the perturbation structure takes the form,

$$\Delta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{3,3}\},$$

are as follows. Applying the mussv function, we obtain the perturbation structure $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -1.0000 \end{bmatrix}.$$

Here, $\|\widehat{\Delta}\|_2 = 1$. For this particular example, we have obtained an upper bound $\mu_{PD}^{upper} = 1.0000$. The same lower bound is obtained, that is $\mu_{PD}^{lower} = 1.0000$, while applying mussv function.
Now, by applying algorithm [14], we obtain the perturbation structure $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -0.5000 & -0.5000 & 0.0000 \\ -0.5000 & -0.5000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

Here, $\epsilon^* = 1.0000$ and $\|\Delta^*\|_2 = 1$, while same lower bound is obtained $\mu_{New}^{lower} = 1.0000$ as the one obtained by mussv function.

Example 7. Consider the following three dimensional real valued matrix B_3.

$$B_3 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We take the perturbation set as,

$$\Delta_{\mathbb{B}} = \{diag(\delta_1 I_1, \Delta_1) : \delta_1 \in \mathbb{R}, \Delta_2 \in \mathbb{C}^{2,2}\}.$$

First, by making use of the well-known Matlab routine mussv, we have obtained the perturbation structure $\widehat{\Delta}$ with

$$\widehat{\Delta} = 1.0e + 050 \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.00000 & 0.0000 & 0.00000 \\ 0.0000 & 0.0000 & -1.00000 \end{bmatrix}.$$

Here, $\|\widehat{\Delta}\|_2 = 1.0000$. For this particular example, we have obtained an upper bound $\mu_{PD}^{upper} = 1.0000$, while a same lower bound is obtained, that is, $\mu_{PD}^{lower} = 1.0000$.
Now, by making use of algorithm [14], we obtain perturbation structure $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -1.0000 & 0.0000 & 0.0000 \\ 0.0000 & -1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

Here, $\epsilon^* = 1.0000$ and $\|\Delta^*\|_2 = 1.0000$, while a same lower bound is achieved, that is, $\mu_{New}^{lower} = 1.0000$ as the one obtained by mussv function. The obtained bounds of SSV for above given matrix M when the perturbation set takes the form,

$$\Delta_{\mathbb{B}} = \{diag(\Delta_1) : \Delta_1 \in \mathbb{C}^{3,3}\},$$

are as follows. First, by using mussv function, we obtain the perturbation structure $\widehat{\Delta}$ with

$$\widehat{\Delta} = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -1.0000 \end{bmatrix}.$$

Here, $\|\widehat{\Delta}\|_2 = 1$. For this particular example, we have obtained an upper bound $\mu_{PD}^{upper} = 1.0000$ while lower bound is approximated, that is, $\mu_{PD}^{lower} = 1.0000$. Now, by making use of algorithm [14], we have obtained obtain the perturbation structure $\epsilon^*\Delta^*$ with

$$\Delta^* = \begin{bmatrix} -0.5000 & -0.5000 & 0.0000 \\ -0.5000 & -0.5000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

Here, $\epsilon^* = 1.0000$ and $\|\Delta^*\|_2 = 1$, while a same lower bound is approximated, that is, $\mu_{New}^{lower} = 1.0000$ as the one obtained by mussv function.

Example 8. In Figure 3, we show the comparison of lower bounds computed by algorithm algorithm [14] with the bounds (Lower and Upper) computed by mussv function for matrix valued function $A_3(w)$ for $w = 1:5$, where $w \in \Omega$ and Ω denotes the frequency range of interest which is usually $\mathbb{R}^+$. The frequency response w is the quantitative measure of output of $(M - \Delta)$ system. We use mussv function to compute μ as a function of frequency response.

6. Conclusion

We have considered the approximation of SSV for the matrix representations of finite symmetric groups S_3 and S_4 over the filed of complex numbers. For the comparison of bounds of SSV, we have done experiments on family of matrices. The experimental results shows the comparison of both lower and upper bounds with once computed by MATLAB funtion mussv and our numerical alogorithm [14].

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Appendix

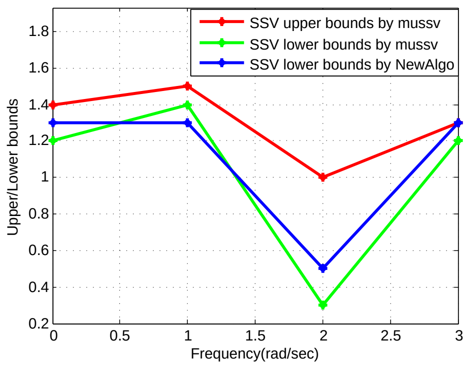

Figure 1: Comparison of bounds of SSV

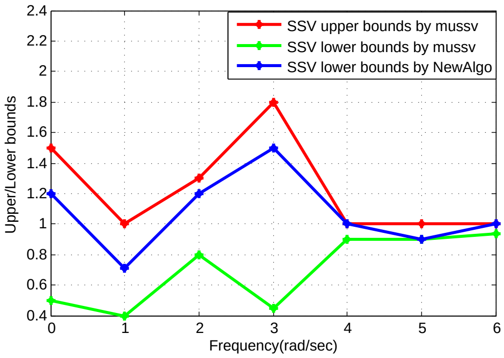

Figure 2: Comparison of bounds of SSV

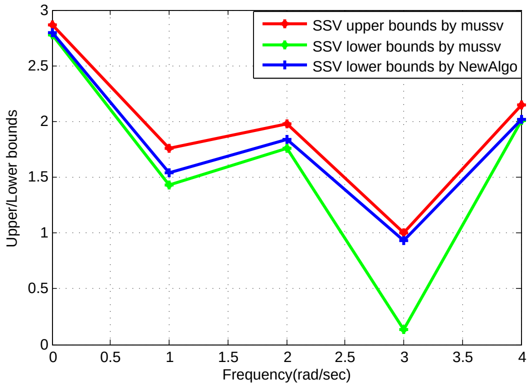

Figure 3: Comparison of bounds of SSV