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On the symmetric block design with parameters (306,61,12) admitting a group of order 61

Menderes Gashi

Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Prishtina, Avenue Mother Teresa 5, 10000 Prishtina, Kosova

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Abstract. In this paper we have proved that up to isomorphism there are exactly two orbit structures for a putative symmetric block design $\mathcal{D}$ with parameters (306,61,12), constructed by group G of order 61. Also the full automorphism groups for these orbit structures are given.

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1. Introduction and Preliminaries

A $2-(v,k,\lambda)$ design $(\mathcal{P},\mathcal{B},I)$ is said to be *symmetric* if the relation $|\mathcal{P}| = |\mathcal{B}| = v$ holds and in that case we often speak of a symmetric design with parameters (v,k,λ). The collection of the parameter sets (v,k,λ) for which a symmetric $2-(v,k,\lambda)$ design exists is often called the "spectrum". The determination of the spectrum for symmetric designs is a widely open problem. For example, a finite projective plane of order n is a symmetric design with parameters $(n^2+n+1, n+1, 1)$ and it is still unknown whether finite projective planes of non–prime–power order may exist at all.

The existence/non-existence of a symmetric design has often required "ad hoc" treatments even for a single parameter set (v,k,λ). The most famous instance of this circumstance is perhaps the non-existence of the projective plane of order 10, see [10].

It is of interest to study symmetric designs with additional properties, which often involve the assumption that a non–trivial automorphism group acts on the design under consideration, see for instance [4].

Among symmetric block designs of square order, a study of symmetric block designs of order 49 is of a particular interest. There are 15 possible parameters (v,k,λ) for symmetric designs of order 49, but until now only a few results are known (see [3], [5]).

Due to the fact that symmetric designs of order 49 have a big number of points (blocks), the study of sporadic cases is very difficult, except, possibly, when the existence of a collineation group is assumed.

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Email address: menderes_gashi@yahoo.com (M. Gashi)

A few methods for the construction of symmetric designs are known and all of them have shown to be effective in certain situations. Here, we shall use the method of tactical decompositions, assuming that a certain automorphism group acts on the design we want to construct, used by Z.Janko in [7] ; see also [6, 8].

The present paper is concerned with a symmetric design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ with parameters $(306, 61, 12)$: the existence/non–existence of such a design is still in doubt as far as we know. We shall further assume that the given design admits a certain automorphism group of order 61. We assume the reader is familiar with the basic facts of design theory, see for instance [9], [2] and [11]. If g is an automorphism of a symmetric design $\mathcal{D}$ with parameters (v, k, λ), then g fixes an equal number of points and blocks, see [11, Theorem 3.1, p.78]. We denote the sets of these fixed elements by $F_{\mathcal{P}}(g)$ and $F_{\mathcal{B}}(g)$ respectively, and their cardinality simply by $|F(g)|$. We shall make use of the following upper bound for the number of fixed points, see [11, Corollary 3.7, p. 82]:

$$|F(g)| \leq k + \sqrt{k - \lambda}. \tag{1}$$

It is also known that an automorphism group G of a symmetric design has the same number of orbits on the set of points $\mathcal{P}$ as on the set of blocks $\mathcal{B}$: [11, Theorem 3.3, p.79]. Denote that number by t.

We adopt the notation and terminology of Section 1 in [4]: we repeat some fundamental relations here for the reader's sake. Let $\mathcal{D}$ be a symmetric design with parameters (v, k, λ) and let G be a subgroup of the automorphism group $Aut\mathcal{D}$ of $\mathcal{D}$. Denote the point orbits of G on $\mathcal{P}$ by $\mathcal{P}_1, \mathcal{P}_2, \ldots \mathcal{P}_t$ and the line orbits of G on $\mathcal{B}$ by $\mathcal{B}_1, \mathcal{B}_2, \ldots \mathcal{B}_t$. Put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. Obviously,

$$\sum_{r=1}^{t} \omega_r = \sum_{i=1}^{t} \Omega_i = v. \tag{2}$$

Let γ_{ir} be the number of points from $\mathcal{P}_r$, which lie on a line from $\mathcal{B}_i$; clearly this number does not depend on the chosen line. Similarly, let Γ_{js} be the number of lines from $\mathcal{B}_j$ which pass through a point from $\mathcal{P}_s$. Then, obviously,

$$\sum_{r=1}^{t} \gamma_{ir} = k \text{ and } \sum_{j=1}^{t} \Gamma_{js} = k. \tag{3}$$

By [2, Lemma 5.3.1. p.221], the partition of the point set $\mathcal{P}$ and of the block set $\mathcal{B}$ forms a tactical decomposition of the design $\mathcal{D}$ in the sense of [2, p.210]. Thus, the following equations hold:

$$\Omega_i \cdot \gamma_{ir} = \omega_r \cdot \Gamma_{ir}, \tag{4}$$

$$\sum_{r=1}^{t} \gamma_{ir} \Gamma_{jr} = \lambda \Omega_j + \delta_{ij}(k - \lambda), \tag{5}$$

$$\sum_{i=1}^{t} \Gamma_{ir} \gamma_{is} = \lambda \omega_s + \delta_{rs}(k - \lambda), \tag{6}$$

where δ_{ij}, δ_{rs} are the Kronecker symbols.

For a proof of these equations, the reader is referred to [2] and [4]. Equation (5), together with (4) yields

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij}(k-\lambda). \tag{7}$$

Definition 1. *The $(t \times t)$-matrix (γ_{ir}) is called the orbit structure of the design $\mathcal{D}$.*

An automorphism of a orbit structure is a permutation of rows followed by a permutation of columns leaving that matrix unchanged. It is clear that the set of all such automorphisms is a group, which we call the automorphism group of that orbit structure.

The first step in the construction of a design is to find all possible orbit structures. The second step of the construction is usually called indexing. In fact for each coefficient γ_{ir} of the orbit matrix one has to specify which γ_{ir} points of the point orbit $\mathcal{P}_r$ lie on the lines of the block orbit $\mathcal{B}_i$. Of course, it is enough to do this for a representative of each block orbit, as the other lines of that orbit can be obtained by producing all G-images of the given representative.

2. Main results

Denote $\mathcal{D}$ the symmetric block design with parameters (306,61,12). Since $v = 1+5\cdot 61$, in order to construct the symmetric block design $\mathcal{D}$ we use the the cyclic group $G = \langle\rho|\rho^{61} = 1\rangle$ of order 61 as a collineation group.

Lemma 1. *Let ρ be an element of G with $o(\rho) = 61$. Then $\langle\rho\rangle$ fixes precisely one point and one block.*

Proof. By [11, Theorem 3.1] the group $\langle\rho\rangle$ fixes the same number of points and blocks. Denote that number by f. Obviously$f \equiv 306(\text{mod } 61)$, i.e.$f \equiv 1(\text{mod } 61)$. The upper bound (1) for the number of fixed points yeilds $f \in \{1, 62\}$. As $o(\rho) > \lambda$, an application of a result of M. Aschbacher [1, Lemma 2.6, p.274] forces the fixed structure to be a subdesign of $\mathcal{D}$. But there is no symmetric design with $v = 62$ and $\lambda = 12$ (there is no $k \in N$ which satisfies $12 \cdot (v-1) = k \cdot (k-1)$). Hence, f is equal to 1.

We put $\mathcal{P}_I = \{I_0, I_1, \cdots, I_{60}\}$, $I = 1, 2, 3, 4, 5$, for the non–trivial orbits of the group G. Thus, G acts on these point orbits as a permutation group in a unique way. Hence, for the generator of G we may put

$$\rho = (\infty)(I_0, I_1, \cdots, I_{60}), I = 1, 2, 3, 4, 5,$$

where ∞ is the fixed point of collineation, whereas non–trivial $\langle\rho\rangle$-orbits are numbers 1, 2, 3, 4, 5 and $\infty, 1_0, 1_1, \cdots, 5_{60}$ are all points of the symmetric block design $\mathcal{D}$.

In what follows, we are going to construct a representative block for each block orbit.

The $\langle\rho\rangle-$fixed block can be writen in the form:

$$L_1 = (1_0 1_1 \cdots 1_{60})$$

or

$$L_1 = 1_{61}.$$

Let L_2, L_3, L_4, L_5, L_6 be the representative blocks for the five non–trivial block orbits.

The second orbit block L_2 of design D, constructed by collineation can be written as

$$L_2 = \infty 1_{a_1} 2_{a_2} 3_{a_3} 4_{a_4} 5_{a_5},$$

where $a_i, i = 1,2,3,4,5$ denote the multiplicities of the appearance of orbit numbers $1,2,3,4$ and 5 in the orbit block L_2.

The multiplicities of the appearance of orbit numbers satisfy the following conditions:

$$a_1 + a_2 + a_3 + a_4 + a_5 = 60,$$

Because $|L_1 \cap L_2| = 12$, we have $a_1 = 12$. From (7) we have

$[L_2, L_2] = 61/1 \cdot 1 \cdot 1 + 61/61 \cdot a_1^2 + 61/61 \cdot a_2^2 + 61/61 \cdot a_3^2 + 61/61 \cdot a_4^2 + 61/61 \cdot a_5^2$ $= 12 \cdot 61 + 61 - 12 = 781$, i.e.

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 = 781$$

or

$$a_2^2 + a_3^2 + a_4^2 + a_5^2 = 576.$$

From the last relation, for the multiplicities of appearance in the block L_2, we obtain the reductions $0 \le a_i \le 24, i = 2,3,4,5$.

In order to reduce isomorphic cases that may appear in the orbit structures at the last stage, without loss of generality, for block L_2, we may assume that the inequalty $a_2 \ge a_3 \ge a_4 \ge a_5$ hold.

Using the computer we have proved that there exists exactely one orbit type for the block L_2 that satisfies the above mentioned conditions:

	a_1	a_2	a_3	a_4	a_5
1.	12	12	12	12	12

The third orbit block L_3, constructed with the collineation ρ, has the form:

$$L_3 = 1_{b_1} 2_{b_2} 3_{b_3} 4_{b_4} 5_{b_5},$$

where $b_i, i = 1,2,\cdots,5$ are multiplicities of the appearance of orbit numbers 1,2,3, 4 and 5 in orbit block L_3.

The multiplicities of orbit numbers satisfy the following conditions: $b_1 + b_2 + b_3 + b_4 + b_5 = 61$.

$[L_1 \cap L_3] = 12$ implies $b_1 = 12$. From (7) we hawe

$[L_3, L_3] = b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 = 12 \cdot 61 + 61 - 12 = 781$

or

$b_2^2 + b_3^2 + b_4^2 + b_5^2 = 637$.

From the last relation we obtain the reductions $0 \le b_i \le 25, i = 2,3,4,5$.

$[L_2, L_3] = a_1b_1 + a_1b_1 + a_1b_1 + a_1b_1 + a_1b_1 = 12 \cdot 61 = 732.$

Using the computer we have proved that there are exactly twenty–eight orbit types for the block L_3 satisfying the above mentioned conditions:

	b_1	b_2	b_3	b_4	b_5
1.	12	16	14	11	8
2.	12	16	14	8	11
3.	12	16	11	14	8
4.	12	16	11	8	14
5.	12	16	8	14	11
6.	12	16	8	11	14
7.	12	14	16	11	8
8.	12	14	16	8	11
9.	12	14	14	14	7
10.	12	14	14	7	14
11.	12	14	11	16	8
12.	12	14	11	8	16
13.	12	14	8	16	11
14.	12	14	8	11	16
15.	1	214	7	14	14
16.	12	11	16	14	8
17.	12	11	16	8	14
18.	12	11	14	16	8
19.	12	11	14	8	16
20.	12	11	8	16	14
21.	12	11	8	14	16
22.	12	8	16	14	11
23.	12	8	16	11	14
24.	12	8	14	16	11
25.	12	8	14	11	16
26.	12	8	11	16	14
27.	12	8	11	14	16
28.	12	7	14	14	14

It is clear that among the candidates for the block L_3 are also blocks L_4, L_5, L_6. Therefore, we investigate quadruples of blocks $\{L_3, l_4, L_5, L_6\}$ which are pairwise compatible. In this way, we have found that, up to isomorphism, there are exactely two orbit structures for the symmetric block design with parameters $(306, 61, 12)$ acting with the collineation ρ of order 61:

First orbit structure:

SO1	1	61	61	61	61	61
	0	61	0	0	0	0
	1	12	12	12	12	12
	0	12	16	14	11	8
	0	12	14	7	14	14
	0	12	11	14	8	16
	0	12	8	14	16	11

Directly from orbit structure we find these automorphisms:

1. $(1)(L_1)$
2. $(3\ 5\ 6)(L_3\ L_6\ L_5)$
3. $(3\ 6\ 5)(L_3\ L_5\ L_6)$

and the full automporphism group of the orbit stucture SO1 is:

$$Aut(SO1) = \{1, (3\ 5\ 6)(\bar{3}\ \bar{6}\ \bar{5}), (3\ 6\ 5)(\bar{3}\ \bar{5}\ \bar{6})\}.$$

Second orbit structure:

SO2	1	61	61	61	61	61
	0	61	0	0	0	0
	1	12	12	12	12	12
	0	12	14	14	14	7
	0	12	14	14	7	14
	0	12	14	7	14	14
	0	12	7	14	14	14

Full automporphism group of the orbit stucture SO2 is:
$Aut(SO2) \cong \Sigma_{\{3,4,5,6\}}$ of order $|Aut(SO2)| = 24$.
Thus we have

Theorem 1. *Up to isomorphism, there are exactly two orbit structures for the symmetric block design $\mathcal{D}$ with parameters $(306, 61, 12)$ admitting the group G of order 61.*

Remark 1. *The actual indexing of these two orbit structures in order to produce an example is still an open problem.*

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