



Some Results on Projective Curvature Tensor of Nearly Cosymplectic Manifold

Nawaf Jaber Mohammed¹, Habeeb Mtashar Abood^{2,*}

^{1,2} *Department of Mathematics, Faculty of Education for Pure Sciences, University of Basra, Basra, Iraq*

Abstract. In the nearly cosymplectic manifold, defined a tensor of type $(4,0)$, it's called a projective curvature tensor. In this article we discuss an interesting question; what the geometric meaning of this tensor when it's act on nearly cosymplectic manifold? The answer of this question leads to get an application on Einstein space. In particular, the necessary and sufficient conditions that a projective tensor is vanishes are found.

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1. Introduction

Almost contact manifold contains many varieties, one of the most important of them is called a nearly cosymplectic manifold (NC -manifold). There have many studies about this manifold. In 1974, Blair and Showers [4] have got some of the characteristics of NC -manifold. Later, appeared many studies on NC -manifold, for more details we refer to [2], [7] and [8]. In 2011, Kirichenko and Kusova [14] studied NC -manifold in G -adjoined structure space. This method allowed the researchers to study different geometric properties.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. In 1953, Yano and Bochner [19], proved that a manifold is projectively flat if and only if, it is of constant curvature. Thus the projective tensor measures a Riemannian manifold to be of constant curvature. In 2009, Abood and Mohammed [1], studied the projective tensor on almost Hermitian manifold and they are found some properties of this tensor. In 2010, Ghosh [9] found some properties of the projective curvature tensor on (k, μ) -contact manifolds. In 2012, De and De A. [6] studied the projective curvature tensor on K - contact manifolds.

*Corresponding author.

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Email addresses: nawafjaber80@yahoo.com (N. J. Mohammed),
iraqsafwan2006@gmail.com (H. M. Abood)

2. Preliminaries

This section shows a simple overview of the basic concepts that pertain to the subject of our study.

Definition 2.1. [3] Suppose that M is $2n + 1$ -dimensional smooth manifold. The set of smooth manifold and tensors (M, η, ξ, Φ, g) is called an almost contact metric manifold (AC-manifold) if such that: $\eta(\xi) = 1, \Phi(\xi) = 0, \eta \circ \Phi = 0$ and $\Phi^2 = -id + \eta \otimes \xi$, where η is differential 1-form called a contact form, ξ be a vector field called a characteristic, Φ endomorphism of $X(M)$ called a structure endomorphisim and there is a Riemannian structure $g = \langle \cdot, \cdot \rangle$ on M such that: $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), X, Y \in X(M)$.

Definition 2.2. [5] Almost contact manifold is called a nearly cosymplectic manifold (NC-manifold) if the equality

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0, X, Y \in X(M),$$

holds.

Definition 2.3. [12] Let (M, η, Φ, g) be an almost contact metric manifold (AC-manifold). In the module $X^c(M)$ (complexification of the module $X(M)$) define two endomorphisms σ and $\bar{\sigma}$ as follows: $\sigma = \frac{1}{2}(id - \sqrt{-1}\Phi)$ and $\bar{\sigma} = -\frac{1}{2}(id + \sqrt{-1}\Phi)$. Further, depending on σ , we can define two projections as follows:

$$\Pi = \sigma \circ \ell = -\frac{1}{2}(\Phi^2 - \sqrt{-1}\Phi) \text{ and } \bar{\Pi} = \bar{\sigma} \circ \ell = \frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi),$$

where $\sigma \circ \Phi = \Phi \circ \sigma = i\sigma$ and $\bar{\sigma} \circ \Phi = \Phi \circ \bar{\sigma} = -i\bar{\sigma}$. Therefore, If we denote $Im\Pi = D_\Phi^{\sqrt{-1}}$ and $Im\bar{\Pi} = D_\Phi^{-\sqrt{-1}}$, then

$$X^c(M) = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}} \oplus D_\Phi^0,$$

where $D_\Phi^{\sqrt{-1}}, D_\Phi^{-\sqrt{-1}}$ and D_Φ^0 are proper submodules of endomorphism Φ with proper values $\sqrt{-1}, -\sqrt{-1}$ and 0 respectively.

Definition 2.4. [15] At each point $p \in M$, we can construct a frame in $T_p^c(M)$ by the form $(p, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}})$, where $\varepsilon_a = \sqrt{2}\sigma_p(e_p), \varepsilon_{\hat{a}} = \sqrt{2}\bar{\sigma}(e_p)$ and $\varepsilon_0 = \xi_p$. The frame $(p, \varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \varepsilon_{\hat{1}}, \dots, \varepsilon_{\hat{n}})$ is called an A-frame.

The principle fiber bundle of all A-frames with structure group $\{1\} \times U(n)$ is called an G -adjoined structure space.

Lemma 2.1. [13] Given an AC-manifold. Then the matrices of the tensors Φ and Riemannian metric g in A-frame are given by the following forms:

$$(\Phi_j^i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n .

The following theorem describes the structure equations of NC -manifold in the G -adjoined structure space.

Theorem 2.1. [14] *In G -adjoined structure space, the structure equations of NC -manifold are given by the following forms:*

- (i) $d\omega^a = \omega_b^a \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c + \frac{3}{2}C^{ab}\omega_b \wedge \omega;$
- (ii) $d\omega_a = -\omega_a^b \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c + \frac{3}{2}C_{ab}\omega^b \wedge \omega;$
- (iii) $d\omega = C^{bc}\omega_b \wedge \omega_c + C_{bc}\omega^b \wedge \omega^c;$
- (iv) $d\omega_b^a = \omega_c^a \wedge \omega_b^c + [A_{bc}^{ad} - 2B^{adh}B_{hbc} + \frac{3}{2}C^{ad}C_{bc}]\omega^c \wedge \omega_d ,$

where $B^{abc} = \frac{\sqrt{-1}}{2}\Phi_{b,\hat{c}}^a, C^{ab} = \sqrt{-1}\Phi_{0,\hat{b}}^a, C_{ab} = -\sqrt{-1}\Phi_{b,0}^{\hat{a}}$ and $B_{abc} = -\frac{\sqrt{-1}}{2}\Phi_{b,c}^{\hat{a}}$.

The tensors B, C and A are called the first, second and third structure tensors respectively.

Definition 2.5. [16] *A Riemann-Christoffel tensor of a smooth manifold M is a tensor of type $(4, 0)$ which is defined by*

$$R(X, Y, Z, W) = g(R(Z, W)Y, X),$$

where $R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]})Z$, and satisfies the following properties:

- (i) $R(X, Y, Z, W) = -R(Y, X, Z, W)$
- (ii) $R(X, Y, Z, W) = -R(X, Y, W, Z)$
- (iii) $R(X, Y, Z, W) = R(Z, W, X, Y)$
- (iv) $R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0.$

The components of Riemann-Christoffel tensor of NC -manifold are given in theorem below.

Theorem 2.2. [14] *In the G -adjoined structure space, the components of Riemann-Christoffel tensor of NC -manifold have the following forms:*

- (i) $R_{\hat{a}bcd} = 0;$
- (ii) $R_{abcd} = -2B_{ab[cd]};$
- (iii) $R_{\hat{a}\hat{b}cd} = -2B^{abh}B_{hcd};$
- (iv) $R_{\hat{a}0b0} = C^{ac}C_{bc};$
- (v) $R_{\hat{a}bcd\hat{a}} = A_{bc}^{ad} - B^{adh}B_{hbc} - \frac{5}{3}C^{ad}C_{bc}.$

The other components of Riemann-Christoffel tensor R can be obtained by the property of symmetry for R or equal to zero.

Definition 2.6. [18] A tensor of type $(2, 0)$ which is a contracting of Riemann-Christoffel tensor and defined as

$$r_{ij} = R_{ijk}^k = g^{kl} R_{kijl}$$

is called a Ricci tensor.

Lemma 2.2. [14] In the G -adjoined structure space, The components of Ricci tensor of NC -manifold are given by the following forms:

- (i) $r_{ab} = 0$;
- (ii) $r_{a\hat{b}} = -A_{ac}^{cb} + 3B^{cbh}B_{hac} + \frac{2}{3}C^{bc}C_{ac}$;
- (iii) $r_{a0} = 0$;
- (iv) $r_{oo} = -2C^{cd}C_{cd}$.

and the others are conjugate to the above components or equal to zero.

The previous definitions of Riemann-Christoffel and Ricci tensors completed the requirements of the projective tensor which is embodied in the next definition.

Definition 2.7. [10] Let M be an AC -manifold. A tensor of type $(4, 0)$ which is defined as

$$P_{ijkl} = R_{ijkl} - \frac{1}{2n}[r_{ik}g_{jl} - r_{jk}g_{il}]$$

is called a projective curvature tensor, where $P_{ijkl} = -P_{jikl} = -P_{ijlk} = P_{klij}$.

We will demonstrate the projective tensor on one of the AC -manifolds which is NC -manifold.

Definition 2.8. [12] An AC -manifold M is called vanishing projective tensor, if the projective tensor is vanishes.

Definition 2.9. An NC -manifold has Φ -invariant Ricci tensor, if $\Phi \circ r = r \circ \Phi$.

Lemma 2.3. An NC -manifold has Φ -invariant Ricci tensor if and only if, in the G -adjoined structure space the following condition

$$r_{\hat{b}}^{\hat{a}} = r_{ab} = 0$$

holds.

Definition 2.10. [11] Let M be a Riemannian manifold, t be a non-zero tensor field of the type (r, s) on M . A tensor t is said to be a recurrent if there is 1-form ρ on M such that $\nabla t = \rho \otimes t$, where ∇ is the Riemannian connection on M . The 1-form ρ is called a recurrence covector. An NC -manifold which allows a field of the recurrent tensor t is called t -recurrent.

Lemma 2.4. [11] *If $\rho = 0$, then the manifold is called t -symmetrical, and if $\rho \neq 0$ then it is called nontrivially t -symmetrical.*

Now, we are in position to introduce the next definition.

Definition 2.11. *Let M be NC-manifold, M is called P_r -recurrent if M is P -recurrent and r -recurrent with the same recurrence convector.*

Definition 2.12. [17] *A Riemannian manifold is called an Einstein manifold, if the Ricci tensor satisfies the equation $r_{ij} = eg_{ij}$, where, e is an Einstein constant.*

Lemma 2.5. [12] *In the G -adjoined structure space, an NC-manifold is a manifold of class*

- (i) CR_1 if and only if, $R_{abcd} = R_{\hat{a}bcd} = R_{\hat{a}\hat{b}cd} = 0$;
- (ii) CR_2 if and only if, $R_{abcd} = R_{\hat{a}bcd} = 0$;
- (iii) CR_3 if and only if, $R_{\hat{a}bcd} = 0$.

It easy to see that $CR_1 \subset CR_2 \subset CR_3$

Concerning the projective tensor, we defined three special classes of NC-manifold which are given in the definition below.

Definition 2.13. *In the G -adjoined structure space, an NC-manifold is a manifold of class*

- (i) PR_1 if and only if, $P_{abcd} = P_{\hat{a}bcd} = P_{\hat{a}\hat{b}cd} = 0$;
- (ii) PR_2 if and only if, $P_{abcd} = P_{\hat{a}bcd} = 0$;
- (iii) PR_3 if and only if, $P_{\hat{a}bcd} = 0$.

3. The main results

In the present section, we concentrate our attention on projective tensor of NC-manifold, and study the notion of projective-recurrent NC-manifold.

Lemma 3.1. *In the G -adjoined structure space, the components of projective curvature tensor of NC-manifold are given by the following forms:*

- (i) $P_{abcd} = -2B_{ab[cd]}$;
- (ii) $P_{\hat{a}bcd} = -2B^{abh}B_{hcd} - \frac{1}{2n}[r_c^a\delta_d^b - r_c^b\delta_d^a]$;
- (iii) $P_{\hat{a}bcd} = A_{bc}^{ad} - B^{adh}B_{hac} - \frac{5}{3}C_{ac}^{ad} - \frac{1}{2n}r_c^a\delta_a^d$;
- (iv) $P_{\hat{a}0b0} = C^{ac}C_{bc} - \frac{1}{2n}r_b^a$.

and the others are conjugate to the above components or equal to zero.

Proof: By using the Theorem 2.2 , Lemma 2.2 and Definition 2.7 , directly we obtain the above components.

Theorem 3.1. *Let M be NC-manifold with vanishing projective tensor. If M is a manifold of vanishing Ricci tensor, then the first structure tensor is vanishing in the first canonical connection.*

Proof: Suppose that M is projectively vanishing NC-manifold. Making use of Definition 2.8 and Lemma 3.1, we have

$$-2B^{abh}B_{hcd} - \frac{1}{2n}[r_c^a\delta_d^b - r_c^b\delta_d^a] = 0 \tag{3.1}$$

Since M has vanishing Ricci tensor, so (3.1) becomes

$$-2B^{adh}B_{hcd} = 0 \tag{3.2}$$

Contracting the equation (3.2) by the induces (a, c) and (b, d) , it follows that

$$-2B^{abh}B_{hab} = 0$$

Since B^{abh} and B_{hab} are antisymmetric tensors, then we get

$$\sum_{a,b,h} |B^{abh}|^2 = 0$$

Consequently, we deduce that $B^{abh} = 0$.

Theorem 3.2. *If M is a projectively vanishing NC-manifold and Φ -invariant Ricci tensor, then the necessary and sufficient condition that M is an Einstein manifold is $A_{ac}^{ad} = \frac{5}{3}C_{ac}^{ad} + C_0\delta_c^d$, where $C_0 = \frac{e}{2n}$.*

Proof: Let M be projectively vanishing NC-manifold. According to the Definition 2.8 and Theorem 3.1, we have

$$A_{bc}^{ad} - B^{adh}B_{hbc} - \frac{5}{3}C_{bc}^{ad} - \frac{1}{2n}r_c^a\delta_b^d = 0 \tag{3.3}$$

Symmetrizing and antisymmetrizing the equation (3.3) by the induces (h, d) , we deduce

$$A_{bc}^{ad} - \frac{5}{3}C_{bc}^{ad} - \frac{1}{2n}r_c^a\delta_b^d = 0 \tag{3.4}$$

Suppose that M is Einstein manifold.

Using the Definition 3.2, so the equation (3.4) becomes

$$A_{bc}^{ad} - \frac{5}{3}C_{bc}^{ad} - \frac{e}{2n}\delta_c^a\delta_b^d = 0 \tag{3.5}$$

Contracting (3.5) by induces (a, b) , it follows that

$$A_{ac}^{ad} = \frac{5}{3}C_{ac}^{ad} + C_0\delta_c^d \tag{3.6}$$

Conversely, let the equation (3.6) holds.

Contracting the equation (3.4) by indices (a,b) , we deduce

$$A_{ac}^{ad} - \frac{5}{3}C_{ac}^{ad} - \frac{1}{2n}r_c^d = 0 \tag{3.7}$$

Making use of the equations (3.6) and (3.7), it follows that

$$r_c^a = e\delta_c^a$$

According to the Φ -invariant Ricci tensor, we get that M is Einstein manifold.

Theorem 3.3. *Suppose that M is a projectively vanishing NC-manifold and Φ -invariant Ricci tensor. If M is an Einstein manifold then the first structure tensor is vanishing in the first canonical connection.*

Proof: Let M be NC-manifold with vanishing projective tensor.

Making use of the Definition 2.8 and Lemma 3.1, then we have

$$A_{bc}^{ad} - B^{adh}B_{hbc} - \frac{5}{3}C_{bc}^{ad} - \frac{1}{2n}r_c^a\delta_b^d = 0 \tag{3.8}$$

Contracting (3.8) with respect to the induces (a, b) , it follows that

$$A_{ac}^{ad} - B^{adh}B_{hac} - \frac{5}{3}C_{ac}^{ad} - \frac{1}{2n}r_c^a\delta_a^d = 0 \tag{3.9}$$

Since M is an Einstein manifold. So according to the Theorem 3.2, the equation (3.9) reduced to

$$-B^{adh}B_{hac} = 0 \tag{3.10}$$

Contracting the equation (3.10) by the induces (d, c) , it follows that

$$-B^{adh}B_{had} = 0$$

Since B^{abh} and B_{hab} are antisymmetric tensors, then we get

$$\sum_{a,d,h} |B_{had}|^2 = 0$$

Consequently, we deduce that $B_{had} = 0$.

Theorem 3.4. *Let M be NC-manifold, then the classes CR_3 and PR_3 are coincide if and only if, M is Φ -invariant Ricci tensor.*

Proof: Suppose that CR_3 and PR_3 are coincide, then we have

$$\frac{1}{2n}r_{bc}\delta_d^a \tag{3.11}$$

Contracting the equation (3.11) by the induces (a, b) , we get

$$r_{dc} = 0$$

Suppose that M is Φ -invariant Ricci tensor.

Making use of Lemmas 3.1 and 2.2, it follows that

$$P_{abcd} = R_{abcd}$$

Therefore, CR_3 and PR_3 are coincide.

Theorem 3.5. *Suppose that M is P_r -recurrent NC-manifold. Then M is either projective symmetrical manifold or vanishing first structure tensor.*

Proof: Let M be P_r -recurrent NC-manifold. According to the Definition 2.11, M is P -recurrent and r -recurrent NC-manifold. From Definition 2.10, we have

$$\nabla P = \rho \otimes P$$

Which has the following coordinate form

$$P_{ijk\ell,h} = \rho_h P_{ijk\ell} \tag{3.12}$$

Consider the equation (3.12) in the G -adjoined structure space, so we have

$$P_{\hat{a}\hat{b}\hat{c}\hat{d},k} = \rho_k P_{\hat{a}\hat{b}\hat{c}\hat{d},k} \tag{3.13}$$

According to the Lemma 3.1, the equation (3.13) becomes

$$-2B^{abh}B_{hcd,k} - \frac{1}{2n}[r_{c,k}^a\delta_d^b - r_{c,k}^b\delta_d^a] = \rho_k[-2B^{abh}B_{hcd} - \frac{1}{2n}[r_c^a\delta_d^b - r_c^b\delta_d^a]]$$

Making use of the Definition 2.10, it follows that

$$B^{abh}B_{hcd,k} = -\rho_k B^{abh}B_{hcd}$$

Symmetrization and antisymmetrization by induces (a, b) , we obtain

$$\rho_k B^{abh}B_{hcd} = 0$$

Contracting the last equation by induces (a, c) and (b, d) ,we get

$$\rho_k B^{abh}B_{hab} = 0$$

Consequently, either $\rho_k = 0$, i.e $\nabla P = 0$ which means that M is projective symmetrical manifold. Or, $B^{abh}B_{hab} = 0$ so by using the same technique in proof of the Theorem 3.1, we have

$$B^{abh} = 0$$

Therefore, the first structure tensor is vanishing.

Theorem 3.6. *Suppose that M is P_r -recurrent NC-manifold. Then the sectional curvature tensor is recurrent if and only if, the second structure tensor is recurrent.*

Proof: Let M be P_r -recurrent NC-manifold. According to the Definition 2.11, M is P -recurrent and r -recurrent NC-manifold. Now the Definition 2.10 implies,

$$\nabla P = \rho \otimes P$$

The previous tensor has the following coordinate form

$$P_{ijk\ell,h} = \rho_h P_{ijk\ell} \tag{3.14}$$

Consider the equation (3.14) in the G -adjoined structure space, it follows that

$$P_{\hat{a}bcd,k} = \rho_k P_{\hat{a}bcd,k} \tag{3.15}$$

By using Lemma 3.1, then the equation (3.15) becomes

$$A_{bc,k}^{ad} - B^{adh}B_{hbc,k} - \frac{5}{3}C_{bc,k}^{ad} - \frac{1}{2n}r_{c,k}^a\delta_b^d = \rho_k[A_{bc}^{ad} - B^{adh}B_{hbc} - \frac{5}{3}C_{bc}^{ad} - \frac{1}{2n}r_c^a\delta_b^d]$$

According to the Definition 2.10, we have

$$A_{bc,k}^{ad} - B^{adh}B_{hbc,k} - \frac{5}{3}C_{bc,k}^{ad} = \rho_k[A_{bc}^{ad} - B^{adh}B_{hbc} - \frac{5}{3}C_{bc}^{ad}] \tag{3.16}$$

Symmetrization and antisymmetrization the equation (3.16) by the induces (h, b) , we get

$$A_{bc,k}^{ad} - \frac{5}{3}C_{bc,k}^{ad} = \rho_k[A_{bc}^{ad} - \frac{5}{3}C_{bc}^{ad}] \tag{3.17}$$

Now, If the structure tensor is recurrent so (3.17) becomes

$$A_{bc,k}^{ad} = \rho_k A_{bc}^{ad}$$

Conversely, If the sectional curvature tensor is recurrent, the the equation (3.17) gives the following desired

$$C_{bc,k}^{ad} = \rho_k C_{bc}^{ad}$$

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