EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 3, 2018, 751-761
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Generalization of Schur's Lemma in Ring Representations on Modules over a Commutative Ring

Na'imah Hijriati[1,2,*], Sri Wahyuni[1], Indah Emilia Wijayanti[1]

[1] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Yogyakarta, Indonesia*
[2] *Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lambung Mangkurat, Banjarmasin, Indonesia*

Abstract. Let R, S be rings with unity, M a module over S, where S a commutative ring, and $f\colon R \to S$ a ring homomorphism. A ring representation of R on M via f is a ring homomorphism $\mu\colon R \to End_S(M)$, where $End_S(M)$ is a ring of all S-module homomorphisms on M. One of the important properties in representation of ring is the Schur's Lemma. The main result of this paper is partly the generalization of Schur's Lemma in representations of ring on modules over a commutative ring

2010 Mathematics Subject Classifications: 16G10, 16W20, 12E15

Key Words and Phrases: Representation of ring on module, Generalized Schur's Lemma, Ring homomorphism

---

1. Introduction

In [2], an abelian group M is called an R-module if there is a ring homomorphism $\varphi\colon R \to End_{\mathbb{Z}}(M)$ where $End_{\mathbb{Z}}(M)$ is a ring of all group endomorphism of M. We know that if there is a ring homomorphism $f\colon R \to S$, then every S-module is also R-module, where the scalar multiplication on R defined by $rm = f(r)m$ [2]. We generalize the codomain of the ring homomorphism φ to the ring of endomorphisms of a module over any ring S. Due to this aim, we need a connection between the ring R and the S-module M. Moreover, We need the commutativity of S to guarantee $\mu(r) \in End_S(M)$, where $\mu(r) := \mu_r\colon M \to M, m \mapsto f(r)m$.

Recall the definition of a representation of a ring R on a vector space V over a field F i.e a ring homomorphism $\rho\colon r \to End_F(V)$, where $End_F(V)$ is a ring of all linear transformations of V [7]. Analog to the definition of representations of rings on vector spaces, we define representations of rings on modules over a commutative ring as follow :

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v11i3.3285

Email addresses: naimah.hijriati@mail.ugm.ac.id (N. Hijriati), swahyuni@ugm.ac.id (S. Wahyuni), ind_wijayanti@ugm.ac.id (I.E Wijayanti)

Definition 1. *Let R be a ring with unity, S a commutative ring with unity and $f\colon R \to S$ a ring homomorphism. A representation of a ring R on an S-module M is a ring homomorphism*

$$\mu\colon R \to End_S(M), \quad r \mapsto \mu_r, \tag{1}$$

where $\mu_r \in End_S(M)$ is defined by $\mu_r(m) = f(r)m$ for every $r \in R$ and $m \in M$. Furthermore, this representation μ of ring R on module M is called an f-representation of R and M is called an f-representation module of R.

Note that prefix f in "f-representation" depends on a ring homomorphism $f\colon R \to S$ we choose. These are some examples of representation of ring.

Example 1. *Every ring commutative R has a 1_d-representation on R-module M, since there exists the identity ring homomorphism $1_d\colon R \to R$.*

Example 2. *Let $R[G]$ be a group ring, where R is a commutative ring and G is a finite group. Since there is a ring homomorphism*

$$h\colon R[G] \to R, \quad h(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g, \tag{2}$$

a ring homomorphism $\mu\colon R[G] \to End_R(M)$ defined by

$$\mu(\sum_{g\in G} a_g g) = \mu_{\sum_{g\in G} a_g g} \in End_R(M) \tag{3}$$

where

$$\mu_{\sum_{g\in G} a_g g}(m) = h(\sum_{g\in G} a_g g)m = \sum_{g\in G} a_g m \tag{4}$$

is an h-representation of $R[G]$.

Example 3. *Let $M_2'(\mathbb{Z})$ be a ring of all 2×2 lower triangle matrices and let $\mathbb{Z}^2$ be a module over itself, where the scalar multiplication defined by $(a, b)(x, y) = (ax, by)$ for every $(a, b) \in \mathbb{Z}^2$ and $(x, y) \in \mathbb{Z}^2$. The function $\theta\colon M_2'(\mathbb{Z}) \to End_{\mathbb{Z}^2}(\mathbb{Z}^2)$ defined by $\theta_r(m) = f(r)m$ is an f-representation of ring $M_2'(\mathbb{Z})$ by a ring homomorphism $f\colon M_2'(\mathbb{Z}) \to \mathbb{Z}^2$ defined by $f(\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}) = (a, b)$.*

Example 4. *Let $M_2(\mathbb{Z})$ be an additive group of all 2×2 matrices, where entries in $\mathbb{Z}$. We define the scalar multiplication in $M_2(\mathbb{Z})$ over $\mathbb{Z}^2$ as $(a, b)\begin{bmatrix} u & r \\ s & t \end{bmatrix} := \begin{bmatrix} au & br \\ bs & at \end{bmatrix}$. Hence $M_2(\mathbb{Z})$ is a $\mathbb{Z}^2$-module. Since $g\colon M_2'(\mathbb{Z}) \to \mathbb{Z}^2$ defined by $g(\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}) = (a, 0)$ is a ring homomorphism, $\varphi\colon M_2'(\mathbb{Z}) \to End_{\mathbb{Z}^2}(M_2(\mathbb{Z}))$ defined by $\varphi_(r)(m) = g(r)m$ is a g-representation of ring $M_2'(\mathbb{Z})$.*

Example 5. *Let $M_2(\mathbb{Z})$ be a $\mathbb{Z}^2$-module defined as Example 4. If $f, g\colon \mathbb{Z}^3 \to Z^2$ are ring homomorphisms defined by $f(a,b,c) = (a,b)$ and $g(a,b,c) = (0,c)$ respectively, then we have an f-representation $\mu\colon \mathbb{Z}^3 \to End_{\mathbb{Z}^2}(M_2(\mathbb{Z}))$ of $\mathbb{Z}^3$ and a g-representation $\varphi\colon \mathbb{Z}^3 \to End_{\mathbb{Z}^2}(M_2(\mathbb{Z}))$ of $\mathbb{Z}^3$ where $\mu_r(m) = f(r)m$ and $\varphi_r(m) = g(r)m$ for all $r \in R$, and $m \in M$.*

Example 3, Example 4, and Example 5 can be generalized to any commutative ring R. Furthermore, since μ is a ring homomorphism, so we can find the kernel of μ, i.e.

$$\ker(\mu) = \{r \in R \mid f(r) \in \mathrm{Ann}(M)\}. \tag{5}$$

We know that the element of $\ker(\mu)$ depend on the properties of f and $Ann(M)$. So we have the following properties:

Proposition 1. *Let $\mu\colon R \to End_S(M)$ is an f-representation of ring R. If f is injective and M is faithful ($\mathrm{Ann}(M) = \{0_S\}$), then μ is injective.*

Proof. Let r be any element of $\ker(\mu)$. Then $\mu_r = O \in End_S(M)$. For any $m \in M$, we have $\mu_r(m) = O(m) = 0$ if and only if $f(r)m = 0$. So we have $f(r) \in \mathrm{Ann}(M)$. Since $\mathrm{Ann}(M) = \{0\}$, $f(r) = 0$ and $r \in \ker(f)$. Furthermore since f is injective, $r = 0$. Thus μ is injective.

The converse of Proposition 1 is not always true. This is the counterexample.

Example 6. *Let*

$$f\colon \mathbb{Z} \to \mathbb{Z}^2, a \mapsto (a, 0) \tag{6}$$

be a ring homomorphism and $M_2(\mathbb{Z})$ a module over $\mathbb{Z}^2$. The scalar multiplication is defined by

$$(a,b)\begin{bmatrix} u & v \\ w & x \end{bmatrix} = \begin{bmatrix} au & av \\ aw & ax \end{bmatrix} \tag{7}$$

The f-representation $\mu\colon \mathbb{Z} \to End_{\mathbb{Z}^2}(M_2(\mathbb{Z}))$ is injective, since $\ker(\mu) = \{0\}$. But $M_2(\mathbb{Z})$ is not faithful, since $\mathrm{Ann}(M) = \{(0,b) \mid b \in \mathbb{Z}\}$.

Remark 1. *If S is an integral domain and M is a free torsian S-module, then an f-representation μ of a ring R on a module M is injective if and only if f is injective.*

There is an important result in representation of rings on vector spaces i.e Schur's Lemma. Schur's Lemma showed that the set all morphism of an irreducible representation ρ of a ring R ($Hom_R(\rho,\rho)$) on a vector space V over a field F is a skew field, and the necessary condition when ρ is irreducible([7],[8]). In linear algebra, the ring $End(V)$, where V is a finite dimension vector space over F ($\dim(V) = n$), is isomorphic to the ring of all $n \times n$ matrices $M_n(F)$[1]. So the definition of the representation of a ring R on a vector spaces V over F can be denoted as a ring homomorphism from $\rho\colon R \to M_n(F)$. If we generalize the ring of all $n \times n$ matrices to the ring of all $m \times n$ matrices, Schur's Lemma still considered [12].

Let S be an R-algebra. Then there is a ring homomorphism $g\colon R \to S$ such that $sg(r) = g(r)s$ for all $r \in R$, and $s \in S$ ($g(r) \in Z(S)$, for all $r \in R$). If M is an S-module, then M is an R-module with scalar multiplication defined by $r.m = g(r)m$ for every $r \in R$ and $m \in M$, such that we can defined a ring homomorphism

$$\varphi\colon R \to End_S(M), r \mapsto \varphi_r \tag{8}$$

where $\varphi_r\colon M \to M, m \mapsto r.m$ [8]. Since ring S in Definition 1 is a commutative ring then $Z(S) = S$ and S is an R-algebra and a representation module M is a module over R-algebra S.

Auslander introduces the concept of modules over an R-algebra, where the ring R is Artinian by using categorical approach [3],[4], and used the result to study the representation of finite dimension of algebra [5]. This concept is used in the representation theory of finite F-algebras on vector spaces over a field F, such as the representation theory of quiver [15], and the representation theory of a group ring $F[G]$ where G is a finite group([6]. There are many mathematicians is developed the concept that Auslander given, such as Iyama [10],[11], and Oppermann [14]. From the previous paragraph, we have been knowing that the f-representation module of ring R is a module over R-algebra S. However, base on Example 5 and Proposition 1, the properties of the f-representation do not only depend on f-representation module of R but also on the properties of a ring homomorphism f.

In the case of the representation of ring on the module over a commutative ring, the generalization of Schur's Lemma is considered. In this paper, we investigate how to generalize Schur's Lemma in representations of rings on modules over a commutative ring. The proof of generalization of Schur's Lemma in this paper is analog with the proof of Schur's Lemma in module theory. However, the properties of the representation module M (M is an S-module) of a ring R depend on the properties of a ring S and M as S-module.

2. The Main Result

To generalize Schur's Lemma, we need to see some properties of representation of ring on module over commutative ring such as equivalence of two representations, and morphism between two representations.

Let μ be a representation of ring R on an S-module M. A submodule U of a representation module M is called R-invariant if for any $r \in R$, $\mu_r(U) \subseteq U$. Since every f-representation module M is an R-module, every submodule of M is an R-invariant. So we have this properties.

Remark 2.

(i) If an S-module M is an f-representation module of an f-representation μ of ring R, then every submodule of M is R-invariant.

(ii) If U is an R-invariant submodule, then we can construct a new f-representa-tion of R, that is a ring homomorphism $\mu'\colon R \to End_S(U)$ defined by $\mu'_r(a) = \mu_r(a) = f(r)a$ for any $r \in R$ and $a \in U$.

A non zero R-module M is called irreducible if it has only trivial submodule. The definition of an irreducible representation depend on properties of a representation module that we give as the following :

Definition 2. *A non-zero f-representation $\mu\colon R \to End_S(M)$ of a ring R is said to be irreducible if the only R-invariant submodules of M are zero and M.*

Let μ be an f-representation of R on an S-module M and φ a g-representation of R on S-module N. If there is a module homomorphism $T\colon M \to N$, then we obtain $T\mu_r\colon M \to N$ and $\varphi_r T\colon M \to N$ for every $r \in R$. But it is not necessary $T\mu_r = \varphi_r T$. We give an example to show this fact.

Example 7. *We consider Example 3 and Example 4. Let $T\colon \mathbb{Z}^2 \to M_2(\mathbb{Z})$ be a $\mathbb{Z}^2$-module homomorphism defined by $T(a,b) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Since for any $A = \begin{bmatrix} u & 0 \\ w & v \end{bmatrix} \in M_2'(\mathbb{Z})$ and $(a,b) \in \mathbb{Z}^2$ we have*

$$T\mu_A(a,b) = \begin{bmatrix} ua & bv \\ bv & ua \end{bmatrix},$$

$$\varphi_A T(a,b) = \begin{bmatrix} ua & 0 \\ 0 & av \end{bmatrix}.$$

We conclude $T\mu_A \neq \varphi_A T$.

Furthermore, if there is a module isomorphism $T\colon M \to N$ such that $T\mu_r = \varphi_r T$ for all $r \in R$, then μ and φ are called equivalent. The definition of the equivalent of two representations is given in the following definition.

Definition 3. *Let $\mu\colon R \to End_S(M)$ be an f-representation of R and let $\varphi\colon R \to End_S(N)$ be a g-representations of R. Representations μ and φ are called equivalent if there is an S-module isomorphism $T\colon M \to N$, such that $T\mu_r = \varphi_r T$ for any $r \in R$. Furthermore μ equivalent to φ denoted by $\mu \sim \varphi$.*

From Definition 3, we know that two representations μ and φ of a ring R are equivalent if there is an S-module isomorphism $T\colon M \to N$, such that the diagram

$$\begin{tikzcd} M \arrow[r, "\mu_r"'] \arrow[d, "T"'] & M \arrow[d, "T"] \\ N \arrow[r, "\varphi_r"'] & N \end{tikzcd}$$

commutes. The following proposition is the sufficient condition two representations of a ring on modules over a commutative ring are equivalent.

Proposition 2. *Let $\mu\colon R \to End_S(M)$ be an f-representation of ring R and $\varphi\colon R \to End_S(N)$ a g-representation of ring R. Representation μ and φ are equivalent ($\mu \sim \varphi$) if and only if there is an S-module isomorphism $T\colon M \to N$ and satisfy $f(r)-g(r) \in \mathrm{Ann}(N)$ for any $r \in R$.*

Proof. Suppose that $T\colon M \to N$ is an S-module isomorphism and $f(r) - g(r) \in \mathrm{Ann}(N)$. Then $\mathrm{Im}(T) = N$, and for any $r \in R$, $n \in N$

$$(f(r) - g(r))n = 0_N \Leftrightarrow f(r)n - g(r)n = 0_N \Leftrightarrow f(r)n = g(r)n \tag{9}$$

So for any $m \in M$

$$\begin{aligned}(T\mu_r)(m) &= T\mu_r(m) = T(f(r)m)\\ &= f(r)T(m) = g(r)T(m)\\ &= \varphi_r(T(m)) = (\varphi_r T)(m).\end{aligned} \tag{10}$$

Thus μ and φ are equivalent.
Conversely, if μ is equivalent to φ, then there is an S-module isomorphism $T\colon M \to N$, such that $T\mu_r = \varphi_r T$ for any $r \in R$. Then we have for any $n \in N$ there is $m \in M$ such that $T(m) = n$. So for any $r \in R$,

$$\begin{aligned}g(r)T(m) &= \varphi_r(T(m))\\ &= (\varphi_r T)(m) = (T\mu_r)(m)\\ &= T(\mu_r(m)) = T(f(r)m)\\ &= f(r)T(m).\end{aligned} \tag{11}$$

Hence we have $g(r)T(m) = f(r)T(m) \Leftrightarrow (f(r)-g(r))T(m) = 0$. Since $T \neq 0$, $f(r)-g(r) \in \mathrm{Ann}(T(m)) = \mathrm{Ann}(n)$ for any $n \in N$. Thus $f(r) - g(r) \in \mathrm{Ann}(N)$.

Example 8. *Let μ be an f-representation of $M_2'(\mathbb{Z})$ defined in Example 3. Let $M_2^*(\mathbb{Z})$ be an abelian group of all 2×2 diagonal matrices and it also a $\mathbb{Z}^2$-module, where scalar multiplication defined as Example 4. We define*

$$\varphi\colon M_2'(\mathbb{Z}) \to End_{\mathbb{Z}^2}(M_2^*(\mathbb{Z})) \tag{12}$$

as a g-representation of $M_2'(\mathbb{Z})$ where g is a ring homomorphism in Example 4. For any $A = \begin{bmatrix} u & 0 \\ w & v \end{bmatrix} \in M_2'(\mathbb{Z})$, we have $f(A) - g(A) = (0, v) \in \mathrm{Ann}(M_2^*(\mathbb{Z}))$ and there is a $\mathbb{Z}^2$-module isomorphism

$$T\colon \mathbb{Z}^2 \to M_2^*(\mathbb{Z}), (a, b) \mapsto \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}. \tag{13}$$

So we conclude $\mu \sim \varphi$.

From Definition 3, if $f = g$, then for any $m \in M$ and $r \in R$ we always have $T\mu_r(m) = T(f(r)m) = f(r)T(m) = \varphi_r T(m)$. Furthermore, if N is a torsian S-module, then $\mathrm{Ann}(N) = 0$. Based on this facts we have this following :

Corollary 1. *Let $\mu\colon R \to End_S(M)$ be an f-representation of R and $\varphi\colon R \to End_S(N)$ a g-representation of R.*

(i) If $f = g$, then $\mu \sim \varphi$ if and only if there is an S-module isomorphism $T\colon M \to N$.

(ii) If N is a free torsian S-module, then $\mu \sim \varphi$ if and only if there is a ring isomorphism from M to N and $f = g$.

Proof.

(i) If $f = g$, then $f(r) - g(r) = 0$ for any $r \in R$. Hence by Proposition 2 $\mu \sim \varphi$ if and only if there is an S-module isomorphism $T\colon M \to N$.

(ii) If N is a free module, then $\mathrm{Ann}(N) = \{0\}$. So by Proposition 2, $\mu \sim \varphi$ if and only if there is a ring isomorphism from M to N and $f(r) - g(r) \in \mathrm{Ann}(N) = \{0\} \Leftrightarrow f(r) = g(r)$ for any $r \in R$ i.e $f = g$.

In the paragraph before, we have explained that not every S-module homomorphism $T\colon M \to N$ satisfy $T\mu_r = \varphi_r T$ where μ is an f-representation of R on S-module M and φ is a g-representation of R on S-module N. If T satisfy $T\mu_r = \varphi_r T$ for all $r \in R$, then T is called a morphism from μ to φ.

Definition 4. *Let $\mu\colon R \to End_S(M)$ be an f-representation of R and $\varphi\colon R \to End_S(N)$ a g-representation of ring R. A morphism from μ to φ is an S-module homomorphism $T\colon M \to N$, such that $T\mu_r = \varphi_r T$ for all $r \in R$.*

Proposition 3. *Let $\mu\colon R \to End_S(M)$ be an f-representation of R and $\varphi\colon R \to End_S(N)$ a g-representation of ring R. An S-module homomorphism $T\colon M \to N$ is a morphism from μ to φ if $f(r) - g(r) \in \mathrm{Ann}(\mathrm{Im}(T))$.*

Proof. Suppose $f(r) - g(r) \in \mathrm{Ann}(\mathrm{Im}(T))$. Then for any $n \in \mathrm{Im}(T)$ there is $m \in M$ such that $T(m) = n$, and $(f(r) - g(r))n = 0 \Leftrightarrow f(r)n = g(r)n$. So for any $m \in M$

$$\begin{aligned}(T\mu_r)(m) &= T(\mu_r(m)) = T(f(r)m) \\ &= f(r)T(m) = g(r)T(m) \\ &= \varphi_r(T(m)) = (\varphi_r T)(m). \end{aligned} \tag{14}$$

Hence T is a morphism from μ to φ.

Example 9. *Let μ be an f-representation of $M_2'(\mathbb{Z})$ defined in Example 3 and φ a g-representation of $M_2'(\mathbb{Z})$ defined in Example 4. For any $\begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \in M_2'(\mathbb{Z})$ we have*

$$f(\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}) - g(\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}) = (0, b) \tag{15}$$

(i) Let $T\colon \mathbb{R}^2 \to M_2(\mathbb{R})$ *defined by* $T(a,b) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ *be an* S*-module homomorphism. Since an annihilator of* S*-module homomorphism is a set* $\{(0,x) \in \mathbb{Z}^2 \mid x \in \mathbb{Z}\}$*, then* T *is a morphism from* μ *to* φ*.*

(ii) Let $T'\colon \mathbb{R}^2 \to M_2(\mathbb{R})$ *defined by* $T'(a,b) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ *is an* S*-module homomorphism. Because an annihilator of* T' *is only* $(0,0)$ *and there is* $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in M_2'(\mathbb{Z})$*, where* $b \neq 0$ *such that*

$$f(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}) - g(\begin{bmatrix} 0 & 0 \\ 0 & b, \end{bmatrix}) = (0,b) \neq (0,0) \tag{16}$$

then T' *is not a morphism from* μ *to* φ*.*

Let $\mu\colon R \to End_S(M)$ be an f-representation of R and let $\varphi\colon R \to End_S(N)$ be a g-representation of R. The set of all morphism from μ to φ is denoted $Hom_R(\mu,\varphi)$. From Definition 4, we know that every $T \in Hom_R(\mu,\varphi)$ is an element in $Hom_S(M,N)$.

Remark 3. *Let* $\mu\colon R \to End_S(M)$ *be an* f*-representation of* R *and* $\varphi\colon R \to End_S(N)$ *a* g*-representation of* R*.*

(i) if $f = g$*, then every* S*-module homomorphism* $T\colon M \to N$ *is morphism from* μ *to* φ

(ii) if $T \in Hom_R(\mu,\varphi)$ *is an* S*-module isomorphism, then* $\mu \sim \varphi$*.*

(iii) The identity map $i_d\colon M \to M$ *is always an element in* $Hom_R(\mu,\mu)$*.*

Proposition 4. *Let* $\mu\colon R \to End_S(M)$ *be* f*-representation of* R *and* $\varphi\colon R \to End_S(N)$ *a* g*-representation of ring* R*. Then* $Hom_R(\mu,\varphi)$ *is a module over* S*, and it is a submodule of* $Hom_S(M,N)$*.*

Proof. To prove $Hom_R(\mu,\varphi)$ is an S-module, we must show $Hom_R(\mu,\varphi)$ is an abelian additive group and it is closed under scalar multiplication over S. Let T, T_1 and T_2 be any elements of $Hom_R(\mu,\varphi)$, and let s be any element of S, then we have $T, T_1, T_2 \in Hom_S(M,N)$, and from [1] $sT, T_1 + T_2 \in Hom_S(M,N)$. So By Proposition 3 to prove $Hom_R(\mu,\varphi)$ is an S-module, it is enough to prove $f(r) - g(r) \in \mathrm{Ann}(\mathrm{Im}(T_1 + T_2))$ and $f(r) - g(r) \in \mathrm{Ann}(\mathrm{Im}(sT))$ for any $r \in R$.

(i) Since $T_1, T_2 \in Hom_R(\mu,\varphi)$, then by Proposition 3 we have $f(r) - g(r) \in \mathrm{Ann}(\mathrm{Im}(T_i))$ i.e $(f(r) - g(r))T_i(m) = 0$ for any $m \in M$, $i = 1, 2$.
Let n be any element in $\mathrm{Im}(T_1 + T_2)$, then there is $m \in M$ such that $(T_1 + T_2)(m) = n$ if and only if $T_1(m) + T_2(m) = n$. Hence we have

$$\begin{aligned}(f(r) - g(r))n &= (f(r) - g(r))(T_1(m) + T_2(m)) \\ &= (f(r) - g(r))T_1(m) + (f(r) - g(r))T_2(m) = 0\end{aligned} \tag{17}$$

(ii) Analog to (1), if $T \in Hom_R(\mu, \varphi)$, then for any $m \in M$ $f(r) - g(r)T(m) = 0$. So we have for any $n \in \mathrm{Im}(sT)$, there is $m \in M$ such that $(sT)(m) = sT(m) = n$ and we have

$$(f(r) - g(r))sT(m) = s(f(r) - g(r))T(m) = 0 \tag{18}$$

From (1) and (2) so we have $f(r)-g(r) \in \mathrm{Ann}(\mathrm{Im}(T_1+T_2))$ and $f(r)-g(r) \in \mathrm{Ann}(\mathrm{Im}(sT))$. In other words $Hom_R(\mu, \varphi)$ is an S-module.

Furthermore, since $Hom_R(\mu, \varphi) \subseteq Hom_S(M, N)$, then $Hom_R(\mu, \varphi)$ is an S-submodule of $Hom_S(M, N)$.

Now we give the generalization of Schur's theorem which is the main result of this paper as follows

Proposition 5. *Let $\mu\colon R \to End_S(M)$ be an irreducible f-representation of R and $\varphi\colon R \to End_S(N)$ an irreducible g-representation of ring R. We have the following :*

(i) If $\mu \nsim \varphi$, then $Hom_R(\mu, \varphi) = 0$.

(ii) If $\mu = \varphi$, then $Hom_R(\mu, \mu)$ is a skew field. Furthermore, if S is a principle ideal domain (PID), where K is a fractional field of S and K' is an extension field of K, and M is a free S-module with finite dimension, then there is scalar $\alpha \in K'$, such that $\mu = \alpha I$.

Proof.

(i) Suppose that $Hom_R(\mu, \varphi) \neq 0$. Then there is $T \neq 0$ in $Hom_R(\mu, \varphi)$. Since $\ker T$ is a submodule of M, by Remarks 2 $\ker(T)$ is an R-invariant submodule of M and hence either $\ker(T) = M$ or $\ker(T) = 0$. Because μ is irreducible and $T \neq 0$, then $\ker T = 0$. So T is injective. Furthermore, since $\mathrm{Im}(T)$ is also submodule of N, by Remarks 2 $\mathrm{Im}(T)$ is an R-invariant submodule of N, so $\mathrm{Im}(T) = 0$ or $\mathrm{Im}(T) = N$. If $\mathrm{Im}(T) = 0$, then $T = 0$. So It must be $\mathrm{Im}(T) = N$, i.e T is surjective. Thus T is invertible. Hence by Corollary 1 $\mu \sim \varphi$. This is the contrapositive of what we want to prove.

(ii) By using Proposition 4 and Schur's Lemma in [13], then $Hom_R(\mu, \mu)$ is a skew field. Let $T \in Hom_R(\mu.\mu)$ and $T \neq 0$. If M is a free S-module with finite dimension, then there is $\alpha \in K'$ which is eigenvalue of T. Since S is a principle ideal domain, S is Dedekind domain and by Proposition 16.3.14 in [9], S is integrally closed. Hence, if $\alpha \in K$, then $\alpha \in R$. By definition eigenvalue $\alpha I - T$ is not invertible. Consider that $I \in Hom_R(\mu, \mu)$, then by Proposition 4 $\alpha I - T \in Hom_R(\mu, \mu)$. We have that $Hom_R(\mu, \mu)$ is a skew field, then $\alpha I - T = 0 \Leftrightarrow T = \alpha I$. Furthermore, if $\alpha \notin K$ then $T = \alpha I$ with $\alpha \in K'$.

Acknowledgements

This paper is one of the result of the Graduate Research Grant (Hibah Penelitian Pascasarjana) from the Ministry of Research and Higher Education by Contract No. 7167/UN1.P.III/DIT-LIT/LT/2017.

References

[1] W A Adkins and S H Weintraub. *Algebra : An Approach Via Module Theory.* Springer-Verlag, New York, 1992.

[2] F W Anderson and F K Fuller. *Rings and Categories of Modules.* Springer-Verlag, New York, 2nd ed. edition, 1992.

[3] M Auslander. Representation Theory of Artin Algebras I. *Communications In Algebra*, 1(3):177–268, 1974a.

[4] M Auslander. Representation Theory of Artin Algebras II. *Communications In Algebra*, 1(4):269–310, 1974b.

[5] M Auslander. *A Functorial Approach to Representation Theory*, volume 944 of *Lecture Notes in Mathematics*, pages 105–179. Springer-Verlag, New York, 1982.

[6] M Auslander, I Reiten, and S O Smalo. *Representation Theory of Artin Algebras.* Cambridge University Press, United Kingdom, 1st ed edition, 1997.

[7] M Burrow. *Representation Theory of Finite.* Academic Press, New York, 1st ed. edition, 1965.

[8] C W Curtis and I Reiner. *Representation Theory of Finite Groups And Associative Algebras.* John Wiley and Sons.Inc, New York, 1962.

[9] D S Dummit and R M Foote. *Abstract Algebra.* John Wiley and Sons Inc, New York, third edition, 2004.

[10] O Iyama. Finiteness of Representation Dimension. In *Proceeding of The American Mathematics Society*, volume 131, pages 1011–1014, 2002.

[11] O Iyama. Auslander-Reiten Theory Revisited. pages 1–47, 2008. arXiv:0803.2841v2.

[12] D Lahat and C Jutten. A generalization to schurs lemma with an application to joint independent subspace analysis. *HAL Id: hal-01247899, https://hal.archives-ouvertes.fr/hal-01247899v2*, 2016.

[13] T Y Lam. *A First Course in Noncommutative Rings.* Springer-Verlag, New York, 1991.

[14] S Oppermann. Representation Dimension of Artin Algebras. *São Paulo Journal of Mathematics Science*, 3:479–498, 2010.

[15] C M Ringel. Representation theory of finite dimensional algebras. In *Representations of algebras: proceedings of the Durham symposium 1985*, volume 116 of *London Mathematical Society lecture note series*, pages 7–79. Cambridge: Univ, 1985.