EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 3, 2018, 834-843
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

On ω-Connectedness and ω-Continuity in the Product Space

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Abstract. In this paper, the concepts of ω-open and ω-closed functions between topological spaces will be introduced and characterized. Moreover, related concepts such as ω-connectedness and ω-continuity from an arbitarary topological space into the product space will also be characterized.

2010 Mathematics Subject Classifications: 54A05

Key Words and Phrases: ω-open, ω-closed, ω-connected, ω-continuous

1. Introduction

One of the attempts to substitute numerous concepts in topology with concepts possessing either weaker or stronger properties was done by N. Levine [4]. He introduced the concepts of semi-open, semi-closed set and semi-continuity of a function, which generated new results, some of which are generalization of existing ones. After this noteworthy work of Levine several mathematicians became attracted in presenting other topological concepts which can substitute the concepts of open sets.

In [5], Velicko introduced the concepts of θ-continuity between topological spaces and subsequently defined the concepts of θ-closure and θ-interior of a subset of topological space. In [1], Al-Hawary characterized θ-continuity and the other well-known variations of continuity such as strong continuity, semi-continuity and closure-continuity.

Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. The *θ-closure* and *θ-interior* of A are, respectively, denoted and defined by

$$Cl_s(A) = \{x \in X : Cl(U) \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x\}$$

and

$$Int_s(A) = \{x \in X : Cl(U) \subseteq A \text{ for some open set } U \text{ containing } x\},$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v11i3.3293

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where $Cl(U)$ is the closure of U in X. A subset A of X is *θ-closed* if $Cl_s(A) = A$ and *θ-open* if $Int_s(A) = A$. Equivalently, A is θ-open if and only if $X\backslash A$ is θ-closed.

In [2], Hdeib introduced the concepts of ω-open and ω-closed sets and ω-closed mappings on a topological space. He showed that ω-closed mappings are strictly weaker than closed mappings and also showed that the Lindelöf property is preserved by counter images of ω-closed mappings with Lindelöf counter image of points. In 2010, Ekici et al. [3] introduced the concepts of ω_θ-open and ω_θ-closed sets on a topological space. They showed that the family of all ω_θ-open sets in a topological space X forms a topology on X. They also introduced the notions of ω_θ-interior and ω_θ-closure of a subset of a topological space.

A point x of a topological space X is called a *condensation point* of $A \subseteq X$ if for each open set G containing x, $G \cap A$ is uncountable. A subset B of X is *ω-closed* if it contains all of its condensation points. The complement of B is *ω-open*. Equivalently, a subset U of X is ω-open (resp., ω_θ-open [3]) if and only if for each $x \in U$, there exists an open set O containing x such that $O\backslash U$ (resp., $O\backslash Int_s(U)$) is countable. A subset B of X is *ω_θ-closed* [3] if its complement $X\backslash B$ is ω_θ-open. The *ω-closure* (resp., *ω_θ-closure* [3]) and *ω-interior* (resp., *ω_θ-interior* [3]) of $A \subseteq X$ are, respectively, denoted and defined by

$$Cl_\omega(A) = \cap\{F : F \text{ is an } \omega\text{-closed set containing } A\}$$

$$(\text{resp., } Cl_{\omega_\theta}(A) = \cap\{F : F \text{ is an } \omega_\theta\text{-closed set containing } A\})$$

and

$$Int_\omega(A) = \cup\{G : G \text{ is an } \omega\text{-open set contained in } A\}$$

$$(\text{resp., } Int_{\omega_\theta}(A) = \cup\{G : G \text{ is an } \omega_\theta\text{-open set contained in } A\}).$$

It is worth noting that $A \subseteq Cl_\omega(A)$ (resp., $A \subseteq Cl_{\omega_\theta}(A)$ [3]) and $Int_\omega(A) \subseteq A$ (resp., $Int_{\omega_\theta}(A) \subseteq A$ [3]). Let $\mathcal{T}_\omega$ (resp., $\mathcal{T}_{\omega_\theta}$) be the family of all ω-open (resp., ω_θ-open) subsets of a topological space X. Since $\mathcal{T}_\omega$ (resp., $\mathcal{T}_{\omega_\theta}$) is a topology on X, for any set $A \subseteq X$, $Int_\omega(A)$ (resp., $Int_{\omega_\theta}(A)$) is ω-open (resp., ω_θ-open) and the largest ω-open (resp., ω_θ-open set) contained in A. Moreover, for any set $A \subseteq X$, $Cl_\omega(A)$ (resp., $Cl_{\omega_\theta}(A)$) is ω-closed (resp., ω_θ-closed) and the smallest ω-closed (resp., ω_θ-closed) set containing A.

A topological space X is said to be ω-connected (resp., θ-connected, ω_θ-connected [3]) if X cannot be written as the union of two nonempty disjoint ω-open (resp., θ-open, ω_θ-open) sets. Otherwise, X is ω-disconnected (resp., θ-disconnected, ω_θ-disconnected [3]). A subset B of X is ω-connected (resp., θ-connected, ω_θ-connected) if it is ω-connected (resp., θ-connected, ω_θ-connected) as a subspace of X. Throughout the paper, related results of ω_θ-open, ω_θ-closed, ω_θ-closure, ω_θ-interior, and ω_θ-connectedness are due to [3].

A function f from a topological space X to another topological space Y is said to be

(i) ω-open (resp., θ-open, ω_θ-open) if $f(G)$ is ω-open (resp., θ-open, ω_θ-open) in Y for every open set G in X;

(ii) ω-closed (resp., θ-closed, ω_θ-closed) if $f(G)$ is ω-closed (resp., θ-closed, ω_θ-closed) in Y for every closed set G in X;

(iii) ω-continuous if $f^{-1}(V)$ is ω-open (resp., θ-open, ω_θ-open) in X for every open subset V of Y;

(iv) ω-irresolute if for every $x \in X$ and every ω_θ-open set A containing $f(x)$, there exists an ω-open set U containing x such that $f(U) \subseteq A$.

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathfrak{T}_\alpha$ be the topology on Y_α. The *Tychonoff topology* on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathfrak{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, ..., k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

2. ω-Open and ω-Closed Functions

In this section, we investigate the connection of ω-open (resp., ω-closed) function to the other well-known functions such as open, θ-open, and ω_θ-open (resp., closed, θ-closed, ω_θ-closed) functions. We also give some characterizations of ω-open and ω-closed functions. Throughout, if no confusion arises, let X and Y be topological spaces.

We shall be using the following lemma later.

Lemma 1. *Let $f : X \to Y$ be a bijective function. Then f is ω-open on X if and only if it is ω-closed on X.*

Remark 1. [3, Remark 4] *Let $A \subseteq X$. Then*

(i) If A is open, then A is ω-open;

(ii) If A is θ-open, then A is open;

(iii) If A is θ-open, then A is ω_θ-open; and

(iv) If A is ω_θ-open, then A is ω-open.

It is shown in [3, p.295] that the implications above are not reversible.

Remark 2. *Let $f : X \to Y$ be a function. Then*

(i) *If f is open (resp., closed), then f is ω-open (resp., ω-closed).*

(ii) *If f is ω_θ-open (resp., ω_θ-closed), then f is ω-open (resp., ω-closed).*

Remark 3. *The converses of Remark 2 (i) and (ii) do not necessarily hold.*

(i): First, consider the topological spaces $X = Z = Y = \{a, b, c, d\}$ with respective topologies $\mathcal{T}_X = \{\varnothing, X, \{a, b, c\}\}$, $\mathcal{T}_Z = \{\varnothing, Z, \{a, b\}\}$, and $\mathcal{T}_Y = \{\varnothing, Y, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$. Define $f : X \to Y$ by $f(a) = a$, $f(b) = b$, $f(c) = c$, and $f(d) = a$. Define $g : Z \to Y$ by $g(a) = g(b) = g(c) = g(d) = d$. Since Y is countable, $f(A)$ is ω-open in Y for every open set A of X. However, $f(\{a, b, c\}) = \{a, b, c\}$ is not open in Y. Thus, f is ω-open but not open on X. Since Y is countable, $g(A)$ is ω-closed in Y for every closed set A in X. However, $g(\{c, d\}) = \{d\}$ is not closed in Y. Thus, g is ω-closed but not closed on Z.

(ii): Next, consider $\mathbb{R}$ with topologies $\mathcal{T}_1 = \{\varnothing, \mathbb{R}, \mathbb{Q}^c \cup \{0\}\}$ and $\mathcal{T}_2 = \{\varnothing, \mathbb{R}, \mathbb{Q}^c\}$. Define $f : (\mathbb{R}, \mathcal{T}_1) \to (\mathbb{R}, \mathcal{T}_2)$ by $f(x) = x$ for all $x \in \mathbb{R}$. We will show first that $\mathbb{Q}^c \cup \{0\}$ is ω-open in $\mathcal{T}_2$. Let $x \in \mathbb{Q}^c \cup \{0\}$. Then $x \in \mathbb{R}$ and $\mathbb{R} \backslash (\mathbb{Q}^c \cup \{0\}) = \mathbb{Q} \backslash \{0\}$ is countable. Hence, $\mathbb{Q}^c \cup \{0\}$ is ω-open in $\mathcal{T}_2$. Since $f(\mathbb{Q}^c \cup \{0\}) = \mathbb{Q}^c \cup \{0\}$, f is ω-open on $(\mathbb{R}, \mathcal{T}_1)$. Next we show that every nonempty proper subset A of $\mathbb{R}$ is not ω_θ-open in $(\mathbb{R}, \mathcal{T}_2)$. Suppose that $Int_s(A) \neq \varnothing$. Note first that the only nonempty open sets in $\mathcal{T}_2$ are $\mathbb{R}$ and $\mathbb{Q}^c$ with $Cl(\mathbb{R}) = Cl(\mathbb{Q}^c) = \mathbb{R}$. Let $y \in Int_s(A)$. Then there exists an open set O containing y such that $Cl(O) = \mathbb{R} \subseteq A$, a contradiction. Hence, $Int_s(A) = \varnothing$. If follows that $\mathbb{R} \backslash Int_s(A) = \mathbb{R}$ and $\mathbb{Q}^c \backslash Int_s(A) = \mathbb{Q}^c$, which are uncountable. Thus, A is not ω_θ-open in $(\mathbb{R}, \mathcal{T}_2)$. This means that $f(\mathbb{Q}^c \cup \{0\}) = \mathbb{Q}^c \cup \{0\}$ is not ω_θ-open in $(\mathbb{R}, \mathcal{T}_2)$. Thus, f is not ω_θ-open on $(\mathbb{R}, \mathcal{T}_1)$. Since f is bijective, f is ω-closed but not ω_θ-closed on $(\mathbb{R}, \mathcal{T}_1)$.

Lemma 2. *Let $A \subseteq X$. Then*

(i) *$x \in Int_\omega(A)$ if and only if there exists an ω-open set U containing x such that $U \subseteq A$;*

(ii) *A is ω-open if and only if $A = Int_\omega(A)$;*

(iii) *$x \in Cl_\omega(A)$ if and only if for every ω-open set U containing x, $U \cap A \neq \varnothing$;*

(iv) *A is ω-closed if and only if $A = Cl_\omega(A)$; and*

(v) *$Cl_\omega(X \setminus A) = X \setminus Int_\omega(A)$.*

We shall now give some characterizations of ω-open and ω-closed functions.

Theorem 1. *Let $f : X \to Y$ be a function. Then the following statements are equivalent.*

(i) *f is ω-open on X.*

(ii) *$f(Int(A)) \subseteq Int_\omega(f(A))$ for every $A \subseteq X$.*

(iii) *$f(B)$ is ω-open for every basic open set B in X.*

(iv) *For each $p \in X$ and every open set O in X containing p, there exists an open set U in Y containing $f(p)$ and a countable subset V of Y such that $U \backslash V \subseteq f(O)$.*

Proof. (i) $\Rightarrow$ (ii): Let $A \subseteq X$. Then $f(Int(A)) \subseteq f(A)$. Since $Int(A)$ is open in X, $f(Int(A))$ is ω-open in Y. Then $f(Int(A)) \subseteq Int_\omega(f(A))$ since $Int_\omega(f(A))$ is the largest ω-open set contained in $f(A)$.

(ii) $\Rightarrow$ (iii): Let B be a basic open set in X. Then $f(B) = f(Int(B))$. By assumption,

$$f(B) = f(Int(B)) \subseteq Int_\omega(f(B)) \subseteq f(B).$$

Hence, $f(B) = Int_\omega(f(B))$. Thus, $f(B)$ is ω-open in Y.

(iii) $\Rightarrow$ (iv): Let $p \in X$ and O be an open set containing p. Then there exists a basic open set B containing p such that $B \subseteq O$. This implies that $f(p) \in f(B) \subseteq f(O)$. By assumption, there exists an open set U in Y containing $f(p)$ such that $U \backslash f(B)$ is countable. Take $V = U \backslash f(B)$. Then $U \backslash V = U \backslash (U \backslash f(B)) = U \cap f(B) \subseteq f(B) \subseteq f(O)$.

(iv) $\Rightarrow$ (i): Let O be open in X and $y \in f(O)$. Then there exists $x \in O$ such that $f(x) = y$. By assumption, there exists an open set U in Y containing y and a countable subset V of Y such that $U \backslash V \subseteq f(O)$. Then $U \backslash f(O) \subseteq U \backslash (U \backslash V) = U \cap V \subseteq V$ so that $U \backslash f(O)$ is countable. Thus, $f(O)$ is ω-open in Y.

Theorem 2. *The function $f : X \to Y$ is ω-closed if and only if*

$$Cl_\omega(f(A)) \subseteq f(Cl(A)) \text{ for any } A \subseteq X.$$

Proof. Suppose that f is ω-closed on X. Now, $f(A) \subseteq f(Cl(A))$. Since $Cl(A)$ is closed in X, $f(Cl(A))$ is ω-closed in Y. Then $Cl_\omega(f(A)) \subseteq f(Cl(A))$ since $Cl_\omega(f(A))$ is the smallest ω-closed set containing $f(A)$.

Conversely, assume that $Cl_\omega(f(A)) \subseteq f(Cl(A))$ for any $A \subseteq X$. Let B be closed in X. Then $f(B) = f(Cl(B))$. By assumption,

$$f(B) \subseteq Cl_\omega(f(B)) \subseteq f(Cl(B)) = f(B).$$

Thus, $f(B) = Cl_\omega(f(B))$. Accordingly, $f(B)$ is ω-closed in Y.

3. ω-Connectedness

In this section, we study the relationship of ω-connected topological spaces to connected, θ-connected, and ω_θ-connected topological spaces and characterize the concept of ω-connectedness. Denote by $\mathcal{D}$, the topological space $\{0, 1\}$ with the discrete topology.

The proof of the following lemma is straightforward, hence omitted.

Lemma 3. *Let X be any topological space and $\chi_A : X \to \mathcal{D}$ the characteristic function of a subset A of X. Then χ_A is ω-continuous if and only if A is both ω-open and ω-closed.*

Theorem 3. *Let X be a topological space. Then the following statements are equivalent:*

(i) *X is ω-connected.*

(ii) *The only subsets of X that are both ω-open and ω-closed are $\varnothing$ and X.*

(iii) *No ω-continuous function from X to $\mathcal{D}$ is surjective.*

Proof. (i) $\Rightarrow$ (ii): Let $G \subseteq X$ which is both ω-open and ω-closed. Then $X \backslash G$ is also both ω-open and ω-closed. Moreover, $X = G \cup (X \backslash G)$. Since X is ω-connected, either $G = \varnothing$ or $G = X$.

(ii) $\Rightarrow$ (iii): Suppose that $f : X \to \mathcal{D}$ is an ω-continuous surjection. Then $f^{-1}(\{0\}) \neq \varnothing, X$. Since $\{0\}$ is open and closed in $\mathcal{D}$, $f^{-1}(\{0\})$ is both ω-open and ω-closed in X. This is a contradiction.

(iii) $\Rightarrow$ (i): If $X = A \cup B$, where A and B are disjoint nonempty ω-open sets, then A and B are also ω-closed sets. Consider the characteristic function $\chi_A : X \to \mathcal{D}$ of $A \subseteq X$. By Lemma 3, χ_A is ω-continuous. This is a contradiction. Thus X is ω-connected.

In view of Remark 1, we have the following consequences.

Remark 4. *Let X be a topological space.*

(i) *If X is ω-connected, then X is ω_θ-connected;*

(ii) *If X is ω_θ-connected, then X is θ-connected; and*

(iii) *If X is ω-connected, then X is connected.*

Remark 5. *The converse of Remark 4 (i) is not necessarily true.*

To see this, consider $\mathbb{R}$ with topology $\mathcal{T} = \{\varnothing, \mathbb{R}, \mathbb{Q}\}$. We will show first that $Int_s(A) = \varnothing$ for every nonempty proper subset A of $\mathbb{R}$. Suppose that $Int_s(A) \neq \varnothing$. Note first that the only nonempty open sets in $\mathcal{T}$ are $\mathbb{R}$ and $\mathbb{Q}$ with $Cl(\mathbb{R}) = Cl(\mathbb{Q}) = \mathbb{R}$. Let $y \in Int_s(A)$. Then there exists an open set O containing y such that $Cl(O) = \mathbb{R} \subseteq A$, a contradiction. Hence, $Int_s(A) = \varnothing$. Next we show that all $A \subseteq \mathbb{Q}$ are the only ω_θ-open subsets of $\mathbb{R}$. Let $A \subseteq \mathbb{Q}$. Then for all $x \in A$, there exists an open set $O = \mathbb{Q}$ containing x such that $Q \backslash Int_s(A) = \mathbb{Q}$ is countable. Hence, $A \subseteq \mathbb{Q}$ is ω_θ-open. Let $B \not\subseteq \mathbb{Q}$. Then there exists $y \in B$ such that $y \notin \mathbb{Q}$. Hence, the only open set containing y is $\mathbb{R}$. But $\mathbb{R} \backslash Int_s(B) = \mathbb{R}$ is uncountable. Thus, B is not ω_θ-open. Accordingly, $(\mathbb{R}, \mathcal{T})$ is ω_θ-connected. Using the similar argument in Remark 3, $Q^c \cup \{0\}$ is ω-open in $\mathbb{R}$. Also, for all $x \in \mathbb{Q} \backslash \{0\}$, there exists an open set $O = \mathbb{Q}$ containing x such that $\mathbb{Q} \backslash (\mathbb{Q} \backslash \{0\})$ is countable. Thus, $\mathbb{Q} \backslash \{0\}$ is ω-open in $\mathbb{R}$. Moreover, $\mathbb{Q}^c \cup \{0\}$ and $\mathbb{Q} \backslash \{0\}$ are disjoint and $\mathbb{R} = (\mathbb{Q}^c \cup \{0\}) \cup (\mathbb{Q} \backslash \{0\})$. This means that $(\mathbb{R}, \mathcal{T})$ is ω-disconnected.

Next, we show that a surjective ω-irresolute function sends an ω-connected space to an ω-connected space. We shall consider first the following characterization of ω-irresolute functions.

Theorem 4. *Let $f : X \to Y$ be a function. Then the following statements are equivalent.*

(i) *f is ω-irresolute on X.*

(ii) *$f^{-1}(A)$ is ω-open in X for each ω-open subset A of Y.*

(iii) *$f^{-1}(F)$ is ω-closed in X for each ω-closed subset F of Y.*

(iv) *$Cl_\omega(f^{-1}(A)) \subseteq f^{-1}(Cl_\omega(A))$ for each subset A of Y.*

(v) *$f^{-1}(Int_\omega(A)) \subseteq Int_\omega(f^{-1}(A))$ for each subset A of Y.*

Proof. $(i) \Leftrightarrow (ii)$: Let A be ω-open in Y and $x \in f^{-1}(A)$. By (i), there exists an ω-open subset U containing x such that $f(U) \subseteq A$, so that $U \subseteq f^{-1}(A)$. Thus, $f^{-1}(A)$ is ω-open in X.

Conversely, let $x \in X$ and V be an ω-open subset of Y containing $f(x)$. By assumption, $f^{-1}(V)$ is ω-open in X containing x. Let $U := f^{-1}(V)$. Hence, $f(U) \subseteq V$.

$(ii) \Leftrightarrow (iii)$: Let F be an ω-closed subset of Y. By assumption, $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$ is ω-open in X. Thus, $f^{-1}(F)$ is ω-closed.

The converse is proved similarly.

$(ii) \Rightarrow (iv)$: Let $A \subseteq Y$. Let $x \in X \setminus f^{-1}(Cl_\omega(A))$. Then $f(x) \in Y \setminus Cl_\omega(A)$. Then there exists an ω-open subset V of Y containing $f(x)$ such that $V \cap A = \varnothing$. By assumption, $f^{-1}(V)$ is ω-open in X containing x such that $f^{-1}(V) \cap f^{-1}(A) = \varnothing$. Hence, $x \in X \setminus Cl_\omega(f^{-1}(A))$. Thus, $Cl_\omega(f^{-1}(A)) \subseteq f^{-1}(Cl_\omega(A))$.

$(iv) \Rightarrow (v)$: Let $A \subseteq Y$. By assumption and Lemma 2 (v),

$$\begin{aligned} X \setminus Int_\omega(f^{-1}(A)) &= Cl_\omega(f^{-1}(Y \setminus A)) \\ &\subseteq f^{-1}(Cl_\omega(Y \setminus A)) \\ &= f^{-1}(Y \setminus Int_\omega(A)) \\ &= X \setminus f^{-1}(Int_\omega(A)). \end{aligned}$$

$(v) \Rightarrow (i)$: Let $x \in X$ and A be an ω-open subset of Y containing $f(x)$. Then $x \in f^{-1}(A) = f^{-1}(Int_\omega(A)) \subseteq Int_\omega(f^{-1}(A))$. This means that $B := f^{-1}(A)$ is ω-open in X containing x such that $f(B) \subseteq A$.

The following result shows that a surjective ω-irresolute function sends an ω-connected space to an ω-connected space.

Theorem 5. *If $f : X \to Y$ is a surjective ω-irresolute function and X is ω-connected, then Y is ω-connected.*

Proof. Suppose that Y is ω-disconnected. Then there exist disjoint nonempty ω-open sets U and V such that $Y = U \cup V$. Since f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. By Theorem 4, $f^{-1}(U)$ and $f^{-1}(V)$ are ω-open and $X = f^{-1}(U) \cup f^{-1}(V)$. This implies that X is ω-disconnected, a contradiction.

4. ω-Continuity of Functions in the Product Space

This section gives a characterization of an ω-continuous function from an arbitrary topological space into the product space.

We shall give first the characterization of ω-continuous function.

Theorem 6. *Let $f : X \to Y$ be a function. Then the following statements are equivalent.*

(*i*) *f is ω-continuous on X.*

(*ii*) *$f^{-1}(F)$ is ω-closed in X for each closed subset F of Y.*

(*iii*) *$f^{-1}(B)$ is ω-open in X for each (subbasic) basic open set B in Y.*

(*iv*) *For every $p \in X$ and every open set V of Y containing $f(p)$, then exists an ω-open set U containing p such that $f(U) \subseteq V$.*

(*v*) *$f(Cl_\omega(A)) \subseteq Cl(f(A))$ for each $A \subseteq X$.*

(*vi*) *$Cl_\omega(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for each $B \subseteq Y$.*

Proof. $(i) \Leftrightarrow (ii)$: Let F be closed in Y. Then $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$ is ω-open in X. Thus, $f^{-1}(F)$ is ω-closed in X. The converse is proved similarly.

$(i) \Leftrightarrow (iii)$: (i) implies (iii) holds since (subbasic) basic open sets are open sets.

Conversely, suppose that $f^{-1}(B)$ is ω-open in X for each $B \in \mathcal{B}$ where $\mathcal{B}$ is a basis for the topology in Y. Let G be an open set in Y. Then $G = \cup\{B : B \in \mathcal{B}^*\}$, where $\mathcal{B}^* \subseteq \mathcal{B}$. It follows that $f^{-1}(G) = \cup\left\{f^{-1}(B) : B \in \mathcal{B}^*\right\}$. Since the collection of all ω-open sets forms a topology, $f^{-1}(G)$ is ω-open in X.

$(i) \Rightarrow (iv)$: Let $p \in X$ and V be an open set in Y containing $f(p)$. Since f is ω-open, $U := f^{-1}(V)$ is ω-open in X containing p. Also, $f(U) = f(f^{-1}(V)) \subseteq V$.

$(iv) \Rightarrow (v)$: Let $A \subseteq X$ and $p \in Cl_\omega(A)$. Let G be an open subset of Y containing $f(p)$. Since f is ω-continuous on X, there exists an ω-open subset O of X containing p such that $f(O) \subseteq G$. Since $p \in Cl_\omega(A)$, $O \cap A \neq \varnothing$. It follows that $\varnothing \neq f(O \cap A) \subseteq f(O) \cap f(A) \subseteq G \cap f(A)$. This implies that $f(p) \in Cl(f(A))$. Hence, $f(Cl_\omega(A)) \subseteq Cl(f(A))$.

$(v) \Rightarrow (vi)$: Let $B \subseteq Y$ and let $A = f^{-1}(B) \subseteq X$. By assumption, $f(Cl_\omega(A)) \subseteq Cl(f(A))$. Hence, $Cl_\omega(f^{-1}(B)) \subseteq f^{-1}(f(Cl_\omega(A))) \subseteq f^{-1}(Cl(f(A))) \subseteq f^{-1}(Cl(B))$.

$(vi) \Rightarrow (ii)$: Let F be a closed subset of Y. By assumption,

$$Cl_\omega(f^{-1}(F)) \subseteq f^{-1}(Cl(F)) = f^{-1}(F).$$

Hence, $f^{-1}(F) \subseteq Cl_\omega(f^{-1}(F))$. Then $Cl_\omega(f^{-1}(F)) = f^{-1}(F)$, which means that $f^{-1}(F)$ is ω-closed.

Theorem 7. *Let X be a topological space and $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ a product space. A function $f : X \to Y$ is ω-continuous on X if and only if each coordinate function $p_\alpha \circ f$ is ω-continuous on X.*

Proof. Suppose that f is ω-continuous on X. Let $\alpha \in \mathcal{A}$, and U_α be open in Y_α. Since p_α is continuous, $p_\alpha^{-1}(U_\alpha)$ is open in Y. Hence,

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$$

is an ω-open set in X. Thus, $p_\alpha \circ f$ is ω-continuous for every $\alpha \in \mathcal{A}$.

Conversely, suppose that each coordinate function $p_\alpha \circ f$ is ω-continuous. Let G_α be open in Y_α. Then $\langle G_\alpha \rangle$ is a subbasic open set in Y and $(p_\alpha \circ f)^{-1}(G_\alpha) = f^{-1}(p_\alpha^{-1}(G_\alpha)) = f^{-1}(\langle G_\alpha \rangle)$ is an ω-open set in X. Therefore, f is ω-continuous on X.

Corollary 1. *Let X be a topological space, $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ a product space, and $f_\alpha : X \to Y_\alpha$ a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then f is ω-continuous on X if and only if each f_α is ω-continuous for each $\alpha \in \mathcal{A}$.*

Proof. For each $\alpha \in \mathcal{A}$ and each $x \in X$, we have

$$(p_\alpha \circ f)(x) = p_\alpha(f(x)) = p_\alpha(\langle f_\beta(x) \rangle) = f_\alpha(x).$$

Thus, $p_\alpha \circ f = f_\alpha$ for every $\alpha \in \mathcal{A}$. The result now follows from Theorem 7.

Theorem 8. *Let $Y = \prod\{Y_{\alpha_i} : 1 \le i \le n\}$ be a product space and $\varnothing \neq O_{\alpha_i} \subseteq Y_{\alpha_i}$ for each $i \in \{1, 2, \ldots, n\}$. If $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$ is ω-open in Y, then each O_{α_i} is ω-open in Y_{α_i}.*

Proof. Suppose that $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$ is ω-open in Y. Let $a_{\alpha_i} \in O_{\alpha_i} = p_{\alpha_i}(O)$ for each $i \in \{1, 2, \ldots, n\}$. Then there exists $x = \langle a_{\alpha_i} \rangle \in O$ such that $p_{\alpha_i}(x) = a_{\alpha_i}$. Since O is ω-open, there exists a basic open set $U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle$ containing x such that $U \backslash O$ is countable. Note that $p_{\alpha_i}(U) \backslash p_{\alpha_i}(O) = U_{\alpha_i} \backslash O_{\alpha_i} \subseteq p_{\alpha_i}(U \backslash O)$ and since $U \backslash O$ is countable, $p_{\alpha_i}(U \backslash O)$ is countable. It follows that $U_{\alpha_i} \backslash O_{\alpha_i}$ is also countable. Thus, each O_{α_i} is ω-open in Y_{α_i}.

Theorem 9. *Let $X = \prod\{X_{\alpha_i} : 1 \le i \le n\}$ and $Y = \prod\{Y_{\alpha_i} : 1 \le i \le n\}$ be product spaces, and for each $i \in \{1, 2, \ldots, n\}$, let $f_{\alpha_i} : X_{\alpha_i} \to Y_{\alpha_i}$ be a function. If $f : X \to Y$ defined by $f(\langle x_{\alpha_i} \rangle) = \langle f_{\alpha_i}(x_{\alpha_i}) \rangle$, is ω-continuous on X, then each f_{α_i} is ω-continuous on X_{α_i}.*

Proof. Assume that $f : X \to Y$ is ω-continuous. Let O_{α_i} be an open set in Y_{α_i}. For each $i \in \{1, 2, \ldots, n\}$, let $a_{\alpha_i} \in f_{\alpha_i}^{-1}(O_{\alpha_i}) := G_{\alpha_i}$. Then

$$\begin{aligned} x := \langle a_{\alpha_1}, \ldots, a_{\alpha_n} \rangle \in \langle G_{\alpha_1}, \ldots, G_{\alpha_n} \rangle &= \left\langle f_{\alpha_1}^{-1}(O_{\alpha_1}), \ldots, f_{\alpha_n}^{-1}(O_{\alpha_n}) \right\rangle \\ &= f^{-1}(\langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle). \end{aligned}$$

Since each O_{α_i} is open in Y_{α_i}, $O := \langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$ is open in Y. Since f is ω-continuous, $f^{-1}(O) = \langle G_{\alpha_1}, \ldots G_{\alpha_n} \rangle$ is ω-open in X. Then there exists a basic open set $U = \langle U_{\alpha_1}, \ldots U_{\alpha_n} \rangle$ containing x such that $U \setminus \langle G_{\alpha_1}, \ldots, G_{\alpha_n} \rangle$ is countable. Note that

$$p_{\alpha_i}(U) \setminus p_{\alpha_i}(\langle G_{\alpha_1}, \ldots, G_{\alpha_n} \rangle) = U_{\alpha_i} \setminus G_{\alpha_i} \subseteq p_{\alpha_i}(U \setminus \langle G_{\alpha_1}, \ldots, G_{\alpha_n} \rangle).$$

Since $U \setminus \langle G_{\alpha_1}, \ldots, G_{\alpha_n} \rangle$ is countable, $p_{\alpha_i}(U \setminus \langle G_{\alpha_1}, \ldots, G_{\alpha_n} \rangle)$ is also countable. It follows that $U_{\alpha_i} \setminus G_{\alpha_i}$ is countable. This means that each $G_{\alpha_i} = f_{\alpha_i}^{-1}(O_{\alpha_i})$ is ω-open in X_{α_i}. Thus, each f_{α_i} is ω-continuous on X_{α_i}.

5. Conclusion

The paper has studied the relationships between connected, ω_θ-connected and ω-connected topological spaces and gave a characterization of an ω-continuous function from an arbitrary topological space into the product space via ω-open sets and ω_θ-open sets. The present paper is related to some good papers.

Acknowledgements

The main author would like to express his sincerest gratitude to his wife for the inspiration in completing this paper. Also, the authors would like to thank the referees for their helpful comments for the improvement of this paper.

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