Green's relations for hypergroupoids

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Abstract. We give some information concerning the Green's relations $\mathcal{R}$ and $\mathcal{L}$ in hypergroupoids extending the concepts of right (left) consistent or intra-consistent groupoids in case of hypergroupoids. We prove, for example, that if an hypergroupoid H is right (left) consistent or intra-consistent, then the Green's relations $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations on H and give some conditions under which in consistent commutative hypergroupoids the relation $\mathcal{R}$ $(= \mathcal{L})$ is a semilattice congruence. A commutative hypergroupoid is right consistent if and only if it is left consistent and if an hypergroupoid is commutative and right (left) consistent, then it is intra-consistent. A characterization of right (left) consistent (or intra-consistent) right (left) simple hypergroupoids has been also given. Illustrative examples are given.

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1. Introduction

It is well known that if S is a semigroup then the Green's relations $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations on S. Kenneth Kapp studied the Green's relations in case of groupoids [1]. He introduced the concepts of consistent (weakly consistent) and intra-consistent (weakly intra-consistent) groupoids in which these concepts play the role of the associativity of semigroups. He first characterized the relations $\mathcal{R}$ and $\mathcal{L}$ in this type of groupoids and proved, among others, that if G is either a weakly consistent or a weakly intra-consistent groupoid, then $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations on G. Moreover, if G is weakly consistent then $\mathcal{R}$ is a left congruence and $\mathcal{L}$ is a right congruence on G. He studied the case of commutative consistent or commutative weakly consistent groupoids as well.

For an hypergroupoid H, we denote by $\mathcal{R}$ the relation on H defined by $a\mathcal{R}b$ if $a = b$ or there exist $x, y \in H$ such that $a \in b \circ x$ and $b \in a \circ y$ and by $\mathcal{L}$ the relation on H defined by $a\mathcal{L}b$ if $a = b$ or there exist $x, y \in H$ such that $a \in x \circ b$ and $b \in y \circ a$. First we observe that if H is an hypersemigroup and $a, b \in H$, then we have $a\mathcal{R}b$ if and only if $(a * H) \cup \{a\} = (b * H) \cup \{b\}$ and $a\mathcal{L}b$ if and only if $(H * a) \cup \{a\} = (H * b) \cup \{b\}$. As a consequence if H is an hypersemigroup, then the relations $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations on H. It is interesting to know under what conditions the relations

$\mathcal{R}$ and $\mathcal{L}$ are equivalence relations, congruences or even semilattice congruences in case of hypergroupoids for which the associativity condition does not hold in general. In this respect, following Kapp [1], we introduce the concepts of right (left) consistent and intra-consistent hypergroupoids. Every hypersemigroup is a right consistent, left consistent and intra-consistent hypergroupoid. And from every right consistent, left consistent or intra-consistent groupoid, a right consistent, left consistent or intra-consistent hypergroupoid can be constructed. We prove that if an hypergroupoid H is right consistent or intra-consistent, then we have $a\mathcal{R}b$ if and only if $(a * H) \cup \{a\} = (b * H) \cup \{b\}$ and if it is left consistent, then we have $a\mathcal{L}b$ if and only if $(H * a) \cup \{a\} = (H * b) \cup \{b\}$. We also have $a\mathcal{R}b$ if and only if the right ideals generated by the elements a and b coincide and we have $a\mathcal{L}b$ if and only if the left ideals generated by the elements a and b are the same. As a consequence, if an hypergroupoid H is right (left) consistent or intra-consistent, then the relations $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations on H. We prove that a commutative hypergroupoid H is right consistent if and only if it is left consistent and therefore consistent; and if a commutative hypergroupoid H is right or left consistent, then it is intra-consistent. In addition, we give some conditions under which in consistent commutative hypergroupoids the Green's relation $\mathcal{R}$ $(= \mathcal{L})$ is a semilattice congruence that certainly leads to further investigation. Following [2, 3], some further related results that characterize the right (left) consistent right (left) simple hypergroupoids have been given. A similar characterization of intra-consistent right (left) simple hypergroupoids has been also given. We use the term right (left) consistent, intra-consistent instead of weakly right (left) consistent, weakly intra-consistent introduced by Kapp. In fact, Kapp calls right (left) consistent if the corresponding property holds for any subgroupoid of the groupoid and he uses the term "weakly" if it holds for the groupoid itself. Illustrative example are given.

An *hypergroupoid* is a nonempty set H with an hyperoperation "∘" on H and an operation "∗" on the set $\mathcal{P}^*(H)$ of nonempty subsets of H induced by "∘" such that $A*B = \bigcup_{(a,b)\in A \times B} (a \circ b)$ for every $A, B \in \mathcal{P}^*(H)$. An hypergroupoid H is called *hypersemigroup* if $\{x\} * (y \circ z) = (x \circ y) * \{z\}$ for every $x, y, z \in H$. If H is an hypergroupoid then, for every $x, y \in H$, we have $\{x\} * \{y\} = x \circ y$. The following two properties, though clear, play an essential role in the investigation: If A and B are two nonempty subsets of an hypergroupoid H, then we have the following

(1) if $x \in a \circ b$ for some $a \in A$, $b \in B$, then $x \in A * B$ and

(2) if $x \in A * B$, then there exist $a \in A$ and $b \in B$ such that $x \in a \circ b$.

Moroever, for nonempty subsets A, B, C of H such that $A \subseteq B$, we have $A * C \subseteq B * C$ and $C * A \subseteq C * B$, we also have $H * A \subseteq H$ and $A * H \subseteq H$. A nonempty subset A of H is called a *right* (resp. *left*) *ideal* of H if $A * H \subseteq A$ (resp. $H * A \subseteq A$), equivalently if for every $a \in A$ and every $h \in H$, we have $a \circ h \subseteq A$ (resp. $h \circ a \subseteq A$). A nonempty subset T of an hypergroupoid H is called a *subgroupoid* of H if $a, b \in T$ implies $a \circ b \subseteq T$, equivalently if $T * T \subseteq T$. Clearly, every right ideal or left ideal of H is a subgroupoid of H. An hypergroupoid H is called *commutative* if, for any $a, b \in H$, we have $a \circ b = b \circ a$. For further information we refer to [4–8].

2. Green's relations for hypersemigroups

Definition 2.1. For an hypergroupoid H, we define the relations $\mathcal{R}$ and $\mathcal{L}$ on H as follows:

$a\mathcal{R}b$ if $a = b$ or there exist $x, y \in H$ such that $a \in b \circ x$ and $b \in a \circ y$ and

$a\mathcal{L}b$ if $a = b$ or there exist $x, y \in H$ such that $a \in x \circ b$ and $b \in y \circ a$.

When is convenient, we write for short, $a * H$ instead of $\{a\} * H$ and $H * a$ instead of $H * \{a\}$.

Proposition 2.2. *Let H be an hypergroupoid and $a, b \in H$. Then we have the following:*
  (1) *if $(a * H) \cup \{a\} = (b * H) \cup \{b\}$, then $a\mathcal{R}b$;*
  (2) *if $(H * a) \cup \{a\} = (H * b) \cup \{b\}$, then $a\mathcal{L}b$.*

Proof. If $a = b$, then clearly $a\mathcal{R}b$ and $a\mathcal{L}b$. Suppose $a \neq b$. Then

(1) Let $(a * H) \cup \{a\} = (b * H) \cup \{b\}$. Since $a \in (b * H) \cup \{b\}$ and $a \neq b$, there exists $x \in H$ such that $a \in b \circ x$. Since $b \in (a * H) \cup \{a\}$ and $b \neq a$, there exists $y \in H$ such that $b \in a \circ y$. Since $x, y \in H$ such that $a \in b \circ x$ and $b \in a \circ y$, we have $a\mathcal{R}b$. The proof of (2) is similar. □

Proposition 2.3. *Let H be an hypersemigroup and $a\mathcal{R}b$. Then we have the following*
  (1) *if $a\mathcal{R}b$, then $(a * H) \cup \{a\} = (b * H) \cup \{b\}$ and*
  (2) *if $a\mathcal{L}b$, then $(H * a) \cup \{a\} = (H * b) \cup \{b\}$.*

Proof. (1) Let $a\mathcal{R}b$ and $u \in (a * H) \cup \{a\}$. Since $a\mathcal{R}b$, we have $a = b$ or there exist $x, y \in H$ such that $a \in b \circ x$ and $b \in a \circ y$. If $a = b$, then (1) holds. Suppose $a \neq b$ and $a \in b \circ x$, $b \in a \circ y$ for some $x, y \in H$. Since $u \in (a * H) \cup \{a\}$, we have $u \in a \circ t$ for some $t \in H$ or $u = a$. We consider the cases:

(A) $a \in b \circ x$, $b \in a \circ y$ and $u \in a \circ t$. Then we have

$$u \in a \circ t \subseteq (b \circ x) * \{t\} = \{b\} * (x \circ t) \text{ (since } H \text{ is an hypersemigroup)}.$$

Since $x, t \in H$, we have $x \circ t \subseteq H$, and then $u \in \{b\} * H \subseteq (b * H) \cup \{b\}$.

(B) $a \in b \circ x$, $b \in a \circ y$ and $u = a$. Then we have

$$u = a \in b \circ x \subseteq \{b\} * H \subseteq (b * H) \cup \{b\}.$$

Thus we have $(a*H) \cup \{a\} \subseteq (b*H) \cup \{b\}$. By symmetry, we get $(b*H) \cup \{b\} \subseteq (a*H) \cup \{a\}$, and (1) holds. The property (2) can be proved in a similar way. □

By Propositions 2.2 and 2.3 we have the following corollary

Corollary 2.4. *If H is an hypersemigroup and $a, b \in H$, then we have*
  $a\mathcal{R}b$ if and only if $(a * H) \cup \{a\} = (b * H) \cup \{b\}$ and
  $a\mathcal{L}b$ if and only if $(H * a) \cup \{a\} = (H * b) \cup \{b\}$.

By Corollary 2.4 we have the following

Corollary 2.5. *If H is an hypersemigroup, then the relations $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations on H.*

3. Green's relations for hypergroupoids

Definition 3.1. An hypergroupoid H is said to be *right consistent* if, for every $x, y \in H$, we have
$$(x \circ y) * H = \{x\} * \Big(\{y\} * H\Big).$$

It is called *left consistent* if, for every $x, y \in H$, we have
$$H * (x \circ y) = \Big(H * \{x\}\Big) * \{y\}.$$

An hypergroupoid H is said to be *intra-consistent* if, for any $x, y \in H$, we have
$$\Big(\{x\} * H\Big) * \{y\} = \{x\} * \Big(H * \{y\}\Big).$$

If H is both left and right consistent, then it is called *consistent.*

Example 3.2. We consider the hypergroupoid $H = \{a, b\}$ with the hyperoperation "$\circ$" on G given by the following table

$\circ$	a	b
a	$\{a\}$	$\{a, b\}$
b	$\{a\}$	$\{b\}$

Table 1.

This is a right consistent, left consistent and intra-consistent hypergroupoid.

Example 3.3. We consider the hypergroupoid $H = \{a, b, c\}$ with the hyperoperation defined in the following table

$\circ$	a	b	c
a	$\{a\}$	$\{a\}$	$\{c\}$
b	$\{c\}$	$\{b\}$	$\{a\}$
c	$\{c\}$	$\{b\}$	$\{c\}$

Table 2.

This is not right consistent because $(c \circ b) * H = H$ and $\{c\} * \Big(\{b\} * H\Big) = \{b, c\}$, not left consistent since $H * (a \circ b) = \{a, c\}$ and $\Big(H * \{a\}\Big) * \{b\} = \{a, b\}$ and not intra-consistent as $\Big(\{a\} * H\Big) * \{b\} = \{a, b\}$ and $\{a\} * \Big(H * \{b\}\Big) = \{a\}$.

Example 3.4. The hypergroupoid defined by Table 3 is right consistent, left consistent and intra-consistent.

$\circ$	a	b	c	d
a	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a, d\}$
b	$\{a, b\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, d\}$
c	$\{a, c\}$	$\{a, b, c\}$	$\{a, c\}$	$\{b, c, d\}$
d	$\{a, d\}$	$\{a, b, d\}$	$\{b, c, d\}$	$\{c, d\}$

Table 3.

Proposition 3.5. *Every hypersemigroup* $(H, \circ)$ *is a right consistent, left consistent and intra-consistent hypergroupoid.*

Proof. Let $x, y \in H$ and $t \in (x \circ y) * H$. Then $t \in u \circ h$ for some $u \in x \circ y$, $h \in H$. Then $t \in u \circ h \subseteq (x \circ y) * \{h\}$. Since H is an hypersemigroup, we have $(x \circ y) * \{h\} = \{x\} * (y \circ h)$. Since $y \circ h \subseteq \{y\} * H$, we have $\{x\} * (y \circ h) \subseteq \{x\} * (y * H)$ and so $t \in \{x\} * (y * H)$. Let now $t \in \{x\} * (y * H)$. Then $t \in x \circ u$ for some $u \in y * H$ and $u \in y \circ h$ for some $h \in H$. We have

$$
\begin{aligned}
t \in x \circ u \;\; &\subseteq \;\; \{x\} * (y \circ h) = (x \circ y) * \{h\} \text{ (since } H \text{ is an hypersemigroup)} \\
&\subseteq \;\; (x \circ y) * H,
\end{aligned}
$$

then $t \in (x \circ y) * H$ and so H is right consistent. In a similar way we can prove that H is left consistent.

Let now $x, y \in H$ such that $t \in (x * H) * \{y\}$. Then $t \in u \circ y$ for some $u \in x * H$ and $u \in x \circ h$ for some $h \in H$. We have

$$
\begin{aligned}
t \in u \circ y \;\; &\subseteq \;\; (x \circ h) * \{y\} = \{x\} * (h \circ y) \text{ (since } H \text{ is an hypersemigroup)} \\
&\subseteq \;\; \{x\} * (H * y),
\end{aligned}
$$

and so $(x * H) * \{y\} \subseteq \{x\} * (H * y)$. Let $t \in \{x\} * (H * y)$. Then $t \in x \circ u$ for some $u \in H * y$ and $u \in h \circ y$ for some $h \in H$. Then we get

$$
t \in x \circ u \subseteq \{x\} * (h \circ y) = (x \circ h) * \{y\} \subseteq (x * H) * \{y\},
$$

then $\{x\} * (H * y) \subseteq (x * H) * \{y\}$ and so H is intra-consistent. $\square$

We apply Proposition 3.5 to the following example

Example 3.6. The hypergroupoid defined by Table 4 is an hypersemigroup. So, by Proposition 3.5, it is right consistent, left consistent and intra-consistent.

$\circ$	a	b	c	d	e
a	$\{c\}$	$\{a, b, c\}$	$\{c\}$	$\{a, c\}$	$\{a, c\}$
b	$\{c\}$	$\{a, b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{a, b, c\}$
c	$\{c\}$	$\{a, b, c\}$	$\{c\}$	$\{c\}$	$\{c\}$
d	$\{a, c\}$	$\{a, b, c\}$	$\{c\}$	$\{d, e\}$	$\{d, e\}$
e	$\{a, c\}$	$\{a, b, c\}$	$\{c\}$	$\{d, e\}$	$\{e\}$

Table 4.

A groupoid $(G, \cdot)$ is said to be right (resp. left) consistent if, for any $x, y \in G$, we have $(xy)G = x(yG)$ (resp. $G(xy) = (Gx)y$); it is called intra-consistent if $(xG)y = x(Gy)$ for every $x, y \in G$ (cf. also [1]).

Proposition 3.7. *Let* $(G, \cdot)$ *be a right (resp. left) consistent or intra-consistent groupoid and "$\circ$" the hyperoperation on* G *defined by* $a \circ b := \{ab\}$. *Then* $(G, \circ)$ *is a right (resp. left) consistent or intra-consistent hypergroupoid.*

Proof. Let $(G, \cdot)$ be right consistent and $x, y \in G$. Then $(x \circ y) * G = \{x\} * (y * G)$. Indeed: If $t \in (x \circ y) * G$, then $t \in u \circ h$ for some $u \in x \circ y$, $h \in G$. Then $t = uh$ and $u = xy$ and so $t = (xy)h \in (xy)G = x(yG)$ since G is right consistent. Then $t = x(yk)$ for some $k \in G$. Then we have

$$t \in \{x(yk)\} \quad = \quad x \circ (yk) = \{x\} * \{yk\} = \{x\} * (y \circ k)$$
$$\subseteq \quad \{x\} * (y * G),$$

and so $(x \circ y) * G \subseteq \{x\} * (y * G)$. Similarly $\{x\} * (y * G) \subseteq (x \circ y) * G$ and equality holds; thus $(G, \circ)$ is right consistent. If $(G, \cdot)$ is a left consistent, then in a similar way we prove that $(G, \circ)$ is left consistent as well.

Let now $(G, \cdot)$ be an intra-consistent groupoid and $x, y \in G$. Then $(G, \circ)$ is intra-consistent, that is $(x * G) * \{y\} = \{x\} * (G * y)$. In fact: Let $t \in (x * G) * \{y\}$. Then $t \in u \circ y$ for some $u \in x * G$ and $u \in x \circ v$ for some $v \in G$. Then $t = uy = (xv)y \in (xG)y = x(Gy)$ since G is intra-consistent. Then there exists $h \in G$ such that $t = x(hy)$. Hence we obtain

$$t \in \{x(hy)\} \quad = \quad x \circ (hy) = \{x\} * \{hy\} = \{x\} * (h \circ y)$$
$$\subseteq \quad \{x\} * (G * y),$$

and $(x * G) * \{y\} \subseteq \{x\} * (G * y)$. In a similar way we prove that $\{x\} * (G * y) \subseteq (x * G) * \{y\}$, and equality holds. □

We apply Proposition 3.7 to the following example

Example 3.8. We consider the groupoid $G = \{a, b, c, d\}$ with the multiplication defined by the following table.

$\cdot$	a	b	c	d
a	a	a	b	b
b	b	b	a	a
c	c	c	d	d
d	d	d	c	c

Table 5.

As we have seen in [1; (1.9) Example] this is a left consistent groupoid. If we remark that $Gx = G$ for any $x \in G$ then, for any $x, y \in G$, we have $G(xy) = G$ and $(Gx)y = Gy = G$, so $G(xy) = (Gx)y$ and $(G, \cdot)$ is left consistent. Applying Proposition 3.7, the set G with the hyperoperation defined by Table 6 is a left consistent hypergroupoid.

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{b\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{a\}$	$\{a\}$
c	$\{c\}$	$\{c\}$	$\{d\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{c\}$	$\{c\}$

Table 6.

In addition, this is an example of a left consistent hypergroupoid which is not right consistent. In fact, we have $(b \circ d) * G = \{a, b\}$ but $\{b\} * (d * G) = \{a\}$.

By interchanging rows and columns in Table 5 we get the groupoid $(G, \cdot)$ given by the following Table 7.

$\cdot$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	b	a	d	c
d	b	a	d	c

Table 7.

We have $xG = G$ for any $x \in G$. Thus, for any $x, y \in G$, we have $(xy)G = G$ and $x(yG) = xG = G$. Then we have $(xy)G = x(yG)$, and G is right consistent (cf. also [1]). By Proposition 3.7, the hypergroupoid $(G, \circ)$ given by Table 8 is right consistent.

$\circ$	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
c	$\{b\}$	$\{a\}$	$\{d\}$	$\{c\}$
d	$\{b\}$	$\{a\}$	$\{d\}$	$\{c\}$

Table 8.

But this is not left consistent as $G * (b \circ d) = \{c, d\}$ and $(G * b) * \{d\} = \{d\}$.

The reader can check 16 cases (or better write a program for similar cases) to see that the groupoids given by Tables 5 and 7 are intra-consistent (cf. [1]). By Proposition 3.7, the hypergroupoids defined by Tables 6 and 8 are intra-consistent as well.

Proposition 3.9. *Let H be a right consistent hypergroupoid and $a\mathcal{R}b$. Then*
$$(a * H) \cup \{a\} = (b * H) \cup \{b\} \tag{$*$}$$

Proof. For $a = b$ the relation $(*)$ holds. If $a \neq b$, then there exist $x, y \in H$ such that $a \in b \circ x$ and $b \in a \circ y$. Then we have $a * H = b * H$. Indeed:

$$
\begin{aligned}
a * H &\subseteq (b \circ x) * H = \{b\} * (x * H) \text{ (since } H \text{ is right consistent)} \\
&\subseteq \{b\} * H \text{ (since } x \in H \text{ and } H * H \subseteq H) \\
&\subseteq (a \circ y) * H = \{a\} * (y * H) \text{ (since } H \text{ is right consistent)} \\
&\subseteq \{a\} * H \text{ (since } y \in H \text{ and } H * H \subseteq H).
\end{aligned}
$$

and so $a * H = b * H$. Hence we obtain

$$
\begin{aligned}
(a * H) \cup \{a\} &= (b * H) \cup \{a\} \subseteq (b * H) \cup (b * H) \text{ (since } a \in b \circ x) \\
&= b * H \subseteq (b * H) \cup \{b\}
\end{aligned}
$$

and

$$(b * H) \cup \{b\} \;=\; (a * H) \cup \{b\} \subseteq (a * H) \cup (a * H) = a * H$$
$$\subseteq \; (a * H) \cup \{a\},$$

and then $(a * H) \cup \{a\} = (b * H) \cup \{b\}$. $\square$

In a similar way the following proposition holds

Proposition 3.10. *If H is a left consistent hypergroupoid and $a\mathcal{L}b$, then*
$$(H * a) \cup \{a\} = (H * b) \cup \{b\} \tag{$*$}$$

By Propositions 2.2, 3.9 and 3.10 we have the following

Corollary 3.11. *If H is a right (resp. left) consistent hypergroupoid then, for every $a, b \in H$, we have*
$a\mathcal{R}b$ if and only if $(a * H) \cup \{a\} = (b * H) \cup \{b\}$ (resp.
$a\mathcal{L}b$ if and only if $(H * a) \cup \{a\} = (H * b) \cup \{b\}$).

Proposition 3.12. *Let H be an intra-consistent hypergroupoid and $a\mathcal{R}b$. Then*
$$(a * H) \cup \{a\} = (b * H) \cup \{b\} \tag{$*$}$$

Proof. If $a = b$, the result holds. If $a \neq b$, then there exist $x, y \in H$ such that $a \in b \circ x$ and $b \in a \circ y$. Then $a * H = b * H$. Indeed: If $t \in a * H$, then $t \in a \circ u$ for some $u \in H$, then

$$t \in a \circ u \;\subseteq\; (b \circ x) * \{u\} \subseteq (b * H) * \{u\}$$
$$=\; \{b\} * (H * u) \text{ (since } H \text{ is intra-consistent)}$$
$$\subseteq\; \{b\} * (H * H) \subseteq b * H$$

so $a * H \subseteq b * H$. Similarly $b * H \subseteq a * H$ and so $a * H = b * H$. Then, exactly as in the proof of Proposition 3.9, we have $(a * H) \cup \{a\} = (b * H) \cup \{b\}$. $\square$

By Propositions 2.2 and 3.12 we have the following

Proposition 3.13. *Let H be an intra-consistent hypergroupoid. Then we have*
$a\mathcal{R}b$ if and only if $(a * H) \cup \{a\} = (b * H) \cup \{b\}$.

Proposition 3.14. *If H is a right (resp. left) consistent hypergroupoid then, for every $a \in H$, the set $a * H$ (resp. $H * a$) is a subgroupoid of H.*

Proof. Let H be a right consistent hypergroupoid and $x, y \in a * H$. Then $x \in a \circ u$ and $y \in a \circ v$ for some $u, v \in H$. Then we have

$$x \circ y \;\subseteq\; (a \circ u) * (a \circ v) \subseteq (a \circ u) * (H * H) \subseteq (a \circ u) * H$$
$$=\; \{a\} * (u * H) \text{ (since } H \text{ is right consistent)}$$
$$\subseteq\; \{a\} * (H * H) \subseteq a * H,$$

so $a * H$ is a subgroupoid of H. $\square$

Proposition 3.15. *If an hypergroupoid H is right consistent (resp. left consistent) then, for any nonempty subset A of H, the set $A \cup (A * H)$ (resp. $A \cup (H * A)$) is the right (resp. left) ideal of H generated by A.*

Proof. Let H be right consistent. Clearly, the set $A \cup (A * H)$ is a nonempty subset of H containing A. Moreover, $\Big(A \cup (A * H)\Big) * H \subseteq A \cup (A * H)$. Indeed: Let $t \in \Big(A \cup (A * H)\Big) * H$. Then $t \in u \circ h$ for some $u \in A \cup (A * H)$, $h \in H$. If $u \in A$, then $t \in u \circ h \subseteq A * H \subseteq A \cup (A * H)$. Let $u \in A * H$. Then $u \in a \circ v$ for some $a \in A$, $v \in H$. Then we have

$$
\begin{aligned}
t \in u \circ h &\subseteq (a \circ v) * H = \{a\} * (v * H) \text{ (since } H \text{ is right consistent)} \\
&\subseteq A * (H * H) \subseteq A * H \subseteq A \cup (A * H),
\end{aligned}
$$

thus $A \cup (A * H)$ is a right ideal of H. Let now T be a right ideal of H such that $T \supseteq A$. Then we have $A \cup (A * H) \subseteq T \cup (T * H) \subseteq T$ and so the set $A \cup (A * H)$ is the right ideal of H generated by A. If H is left consistent, then in a similar way we prove that the set $A \cup (H * A)$ is the left ideal of H generated by A. $\qquad \square$

Corollary 3.16. *If an hypergroupoid H is right consistent (resp. left consistent) then, for any $a \in H$, the set $\{a\} \cup (a * H)$ (resp. $\{a\} \cup (H * a)$) is the right (resp. left) ideal of H generated by A.*

As a result, if H is a right or left consistent hypergroupoid then, for every $a \in H$, the sets $\{a\} \cup (a * H)$ and $\{a\} \cup (H * a)$ are subgroupoids of H.

Notation 3.17. We denote by $R(A)$ (resp. $L(A)$) the right (resp. left) ideal of H generated by A. For $A = \{a\}$, we write $R(a), L(a)$ instead of $R(\{a\}), L(\{a\})$.

By Proposition 3.15 we have the following

Corollary 3.18. *If an hypergroupoid H is right consistent (resp. left consistent) then, for every nonempty subset A of H, we have $R(A) = A \cup (A * H)$ (resp. $L(A) = A \cup (H * A)$).*

By Corollaries 3.11, 3.16 and Proposition 3.13 we have the following

Corollary 3.19. *If H is a right consistent or intra-consistent hypergroupoid, then $a\mathcal{R}b$ if and only if $R(a) = R(b)$. If H is a left consistent hypergroupoid, then $a\mathcal{L}b$ if and only if $L(a) = L(b)$.*

By Propositions 3.9, 3.10 and 3.12 or by Corollary 3.19. we have the following

Theorem 3.20. *If an hypergroupoid H is right (left) consistent or intra-consistent, then the relations $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations on H.*

Proposition 3.21. *A commutative hypergroupoid H is right consistent if and only if it is left consistent and therefore consistent.*

Proof. $\Longrightarrow$. Let H be right consistent and $x, y \in H$. Then we have

$$
\begin{aligned}
H * (x \circ y) &= (x \circ y) * H = (y \circ x) * H \text{ (since } H \text{ is commutative)} \\
&= \{y\} * (x * H) \text{ (since } H \text{ is right consistent)} \\
&= \{y\} * (H * x) = (H * x) * \{y\} \text{ (since } H \text{ is commutative)}
\end{aligned}
$$

so $H * (x \circ y) = (H * x) * \{y\}$, and H is left consistent.
$\Longleftarrow$. Let H be left consistent and $x, y \in H$. Then we have

$$(x \circ y) * H = H * (y \circ x) = (H * y) * \{x\} = \{x\} * (y * H),$$

so $(x \circ y) * H = \{x\} * (y * H)$, and H is right consistent. $\qquad\qquad\square$

Proposition 3.22. *Let H be a commutative hypergroupoid. If H is right (resp. left) consistent, then H is intra-consistent. The converse statement does not hold in general. However, there are commutative intra-consistent hypergroupoids that are consistent.*

Proof. Let H be right consistent and $x, y \in H$. Then we have

$$
\begin{aligned}
(x * H) * \{y\} &= \{y\} * (x * H) \text{ (since } H \text{ is commutative)} \\
&= (y \circ x) * H \text{ (since } H \text{ is right consistent)} \\
&= (x \circ y) * H \text{ (since } H \text{ is commutative)} \\
&= \{x\} * (y * H) \text{ (since } H \text{ is right consistent)} \\
&= \{x\} * (H * y) \text{ (since } H \text{ is commutative),}
\end{aligned}
$$

so $(x * H) * \{y\} = \{x\} * (H * y)$, and H is intra-consistent.

Let now H be left consistent and $x, y \in H$. Then we have

$$
\begin{aligned}
(x * H) * \{y\} &= (H * x) * \{y\} = H * (x \circ y) = H * (y \circ x) \\
&= (H * y) * \{x\} = \{x\} * (H * y),
\end{aligned}
$$

and again H is intra-consistent.

For the converse statement we give the following example. We consider the commutative groupoid $G = \{a, b, c\}$ given by Table 9 (cf. also [1; (1.7) Example]).

$\cdot$	a	b	c
a	c	b	c
b	b	b	c
c	c	c	c

Table 9.

One can check 9 cases to see that this is an intra-consistent groupoid. According to Proposition 3.7, the set G with the hyperoperation defined by Table 10 is an intra-consistent commutative hypergroupoid.

$\circ$	a	b	c
a	$\{c\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$

Table 10.

This is not right regular because $(a \circ a) * H = \{c\}$ and $\{a\} * (a * H) = \{b, c\}$. As $(G, \circ)$ is commutative, this is not left consistent as well.

We prove the last part of the proposition by the following example.

Applying Proposition 3.7 in the groupoid given in [1; (1.8) Example] we get the hypergroupopoid $G = \{a, b, c\}$ defined by Table 11.

∘	a	b	c
a	$\{c\}$	$\{c\}$	$\{a\}$
b	$\{c\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{c\}$

Table 11.

This is a commutative, intra-consistent and consistent hypergroupoid. □

Lemma 3.23. *If H is an hypergroupoid then, for any nonempty subsets A, B, C of H, we have*

(1) $A * (B \cup C) = (A * B) \cup (A * C)$ *and*

(2) $(A \cup B) * C = (A * C) \cup (B * C)$.

Proof. (1) Since $B \cup C \supseteq B, C$, we have $A * (B \cup C) \supseteq A * B, A * C$, so $A * (B \cup C) \supseteq (A * B) \cup (A * C)$. Let now $x \in (A * B) \cup (A * C)$. If $x \in A * B$, then $x \in a \circ b$ for some $a \in A$, $b \in B \subseteq B \cup C$, so $x \in a \circ b \subseteq A * (B \cup C)$. If $x \in A * C$, then $x \in a \circ c$ for some $a \in A$, $c \in C$, then again $x \in A * (B \cup C)$ and (1) holds. The proof of condition (2) is similar. □

Proposition 3.24. *If an hypergroupoid H is right (resp. left) consistent, $(a, b) \in \mathcal{R}$ (resp. $(a, b) \in \mathcal{L}$) and $c \in H$, then we have $R(c \circ a) = R(c \circ b)$ (resp. $L(a \circ c) = L(b \circ c)$).*

Proof. Let H be right consistent, $(a, b) \in \mathcal{R}$ and $c \in H$. Then $R(c \circ a) = R(c \circ b)$. Indeed:

$$
\begin{aligned}
R(c \circ a) &= (c \circ a) \cup \Big((c \circ a) * H\Big) \text{ (by Corollary 3.18)} \\
&= \Big(\{c\} * \{a\}\Big) \cup \Big(\{c\} * (a * H)\Big) \text{ (since } H \text{ is right consistent)} \\
&= \{c\} * \Big(\{a\} \cup (a * H)\Big) \text{ (by Lemma 3.23)} \\
&= \{c\} * \Big(\{b\} \cup (b * H)\Big) \text{ (by Proposition 3.9)} \\
&= \Big(\{c\} * \{b\}\Big) \cup \Big(\{c\} * (b * H)\Big) \text{ (by Lemma 3.23)} \\
&= (c \circ b) \cup \Big((c \circ b) * H\Big) \text{ (since } H \text{ is right consistent)} \\
&= R(c \circ b) \text{ (by Corollary 3.18).}
\end{aligned}
$$

□

Proposition 3.25. *If H is a right consistent hypergroupoid such that $x \in x \circ x$ for every $x \in H$ then, for every $a \in H$, we have $R(a) = R(a \circ a)$.*

Proof. Let $a \in H$. Then, we have

$$
\begin{aligned}
R(a \circ a) &= (a \circ a) \cup \Big((a \circ a) * H\Big) \text{ (by Corollary 3.18)} \\
&= \Big(\{a\} * \{a\}\Big) \cup \Big(\{a\} * (a * H)\Big) \text{ (since } H \text{ is right consistent)} \\
&\subseteq (a * H) \cup \Big(\{a\} * (H * H)\Big)
\end{aligned}
$$

$$\subseteq \ (a * H) \cup (a * H) = (a * H) \subseteq \{a\} \cup (a * H) = R(a)$$

and

$$R(a) = \{a\} \cup (a * H) \subseteq (a \circ a) \cup \Big((a \circ a) * H\Big) = R(a \circ a),$$

so $R(a) = R(a \circ a)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In a similar way we have the following

Proposition 3.26. *If H is a left consistent hypergroupoid such that $x \in x \circ x$ for every $x \in H$ then, for every $a \in H$, we have $L(a) = L(a \circ a)$.*

Remark 3.27. If H is a commutative hypergroupoid and $a, b \in H$, then we have
  $R(a) = L(a)$ and $R(a \circ b) = (a \circ b) \cup (a \circ b) * H = (b \circ a) \cup (b \circ a) * H = R(b \circ a)$.

The concepts of right (left) congruences and semilattice congruences on groupoids can be naturally transferred to hypergroupoids as follows: An equivalence relation σ on an hypergroupoid H is called *right* (resp. *left*) *congruence* if $(a, b) \in \sigma$ implies $(a \circ c, b \circ c) \in \sigma$ (resp. $(c \circ a, c \circ b) \in \sigma$) for every $c \in H$; it is called a *congruence* on H if it is both a right congruence and a left congruence on H. A congruence σ on H is called *semilattice congruence* if, for any $a, b \in H$, we have $(a, a \circ a) \in \sigma$ and $(a \circ b, b \circ a) \in \sigma$.

By Propositions 3.25, 3.26 and Remark 3.27 we have the following

Proposition 3.28. *Let H be a consistent commutative hypergroupoid such that*

(1) *for every $a \in H$, we have $a \in a \circ a$ and*

(2) *for any nonempty subsets A, B of H, $R(A) = R(B)$ implies $(A, B) \in \mathcal{R}$.*

Then the relation $\mathcal{R}$ $(= \mathcal{L})$ is a semilattice congruence on H.

For an hypergroupoid H and a subset A of H, we denote by σ_A the equivalence relation on H defined by

$$\sigma_A := \{(x, y) \mid x, y \in A \text{ or } x, y \notin A\}.$$

Proposition 3.29. *If an hypergroupoid H is right consistent (resp. left consistent), $\mathcal{A}$ is the set of right ideals and $\mathcal{B}$ is the set of left ideals of H, then we have*

$$\mathcal{R} = \bigcap\{\sigma_I \mid I \in \mathcal{A}\} \ and \ \mathcal{L} = \bigcap\{\sigma_{\mathcal{I}} \mid \mathcal{I} \in \mathcal{B}\},$$

respectively.

Proof. Let H be right consistent, $(x, y) \in \mathcal{R}$ and $I \in \mathcal{A}$. If $x \in I$, then we have

$$\begin{aligned}
y \in R(y) \ &= \ R(x) = \{x\} \cup (x * H) \text{ (by Corollary 3.16)} \\
&\subseteq \ I \cup (I * H) = I,
\end{aligned}$$

so $y \in I$. Then $x, y \in I$ and so $(x, y) \in \sigma_I$. If $x \notin I$, then $y \notin I$ as well. This is because if $y \in I$ then, by symmetry to the previous case, we get $x \in I$ which is impossible. Thus we have $x, y \notin I$ and so $(x, y) \in \sigma_I$. Let now $(x, y) \in \sigma_I$ for every $I \in \mathcal{A}$. Since $R(x)$ is a right ideal of H, we have $(x, y) \in \sigma_{R(x)}$. Since $x \in R(x)$, we have $y \in R(x)$, then $R(y) \subseteq R(x)$.

Since $y \in R(y)$ and $(x, y) \in \sigma_{R(y)}$, we have $x \in R(y)$, then $R(x) \subseteq R(y)$. Thus we get $R(x) = R(y)$ and so $(x, y) \in \mathcal{R}$. If H is left consistent, the proof is analogous. $\square$

Corollary 3.30. *If an hypergroupoid H is right consistent (resp. left consistent), A is a right ideal and B is a left ideal of H, then we have*

(1) $A = \bigcap\{(x)_\mathcal{R} \mid x \in A\}$ *and*
(2) $B = \bigcap\{(x)_\mathcal{L} \mid x \in B\}$, *respectively.*

Proof. (1) Let H be right consistent and A be a right ideal of H. If $t \in A$, then clearly $t \in (t)_\mathcal{R} \subseteq \bigcap\{(x)_\mathcal{R} \mid x \in A\}$. Conversely, let $t \in (x)_\mathcal{R}$ for every $x \in A$. Take an element $a \in A$ ($A \neq \emptyset$). Since $t \in (a)_\mathcal{R}$, we have $(t, a) \in \mathcal{R}$. On the other hand, by Proposition 3.29, we have $\mathcal{R} = \bigcap \sigma_I \mid I \in \mathcal{A}\}$. Since $(t, a) \in \mathcal{R}$ and $A \in \mathcal{A}$, we have $(t, a) \in \sigma_A$; since $a \in A$, we have $t \in A$ and property (1) is satisfied. If H is left consistent, then property (2) holds at a similar way. $\square$

In what follows, following [2, 3], we characterize the right (left) consistent right (left) simple hypergroupoids and the intra-consistent right (left) simple hypergroupoids.

An hypergroupoid H is called *right* (resp. *left*) *simple* if H is the only right (resp. left) ideal of H; in other words, if T is a right (resp. left) ideal of H, then $T = H$.

Proposition 3.31. *If H is an hypergroupoid such that $a * H = H$ (resp. $H * a = H$) for every $a \in H$, then H right (resp. left) simple.*

Proof. Let $a * H = H$ for every $a \in H$ and let T be a right ideal of H. Then $T = H$. Indeed: let $b \in H$. Take an element $t \in T$ ($T \neq \emptyset$). Since $t \in H$, by hypothesis, we have $t * H = H$. Since $b \in H$, we have $b \in t \circ h$ for some $h \in H$. Then we have $b \in t \circ h \subseteq T * H \subseteq T$ and so $T = H$. Similarly if H is left simple, then $H * a = H$ for every $a \in H$. $\square$

Proposition 3.32. *Let H be a right (resp. left) consistent hypergroupoid. If H is right (resp. left) simple then, for every $a \in H$, we have $a * H = H$ (resp. $H * a = H$).*

Proof. Suppose H is right simple and let $a \in H$. The set $a * H$ is a right ideal of H. Indeed: Let $t \in (a * H) * H$. Then $t \in u \circ v$ for some $u \in a * H$, $v \in H$. Since $u \in a * H$, we have $u \in a \circ w$ for some $w \in H$. Then we have

$$
\begin{aligned}
t \in u \circ v \ &\subseteq \ (a \circ w) * \{v\} \subseteq (a \circ w) * H \\
&= \ \{a\} * (w * H) \ \text{(since } H \text{ is right consistent)} \\
&\subseteq \ \{a\} * (H * H) \subseteq a * H,
\end{aligned}
$$

then $t \in a * H$ and so $a * H$ is a right ideal of H. Since H is right simple, we have $a * H = H$. If H is left consistent then, in a similar way we get $H * a = H$. $\square$

By Propositions 3.31 and 3.32 we have the following

Proposition 3.33. *A right (resp. left) consistent hypergroupoid H is right (resp. left) simple if and only if, for any $a \in H$, we have $a * H = H$ (resp. $H * a = H$).*

Proposition 3.34. *Let H be an intra-consistent hypergroupoid. If H is right (resp. left) simple then, for every $a \in H$, we have $a * H = H$ (resp. $H * a = H$).*

Proof. Let H be right simple and $a \in H$. The set $a * H$ is a right ideal of H. Indeed: Let $t \in (a * H) * H$. Then $t \in u \circ v$ for some $u \in a * H$ and $u \in a \circ w$ for some $w \in H$. Then we have

$$
\begin{aligned}
t \in u \circ v \quad &\subseteq \quad (a \circ w) * \{v\} = \Big(\{a\} * \{w\}\Big) * \{v\} \subseteq \Big(\{a\} * H\Big) * \{v\} \\
&= \quad \{a\} * \Big(H * \{v\}\Big) \text{ (since } H \text{ is intra-consistent)} \\
&\subseteq \quad \{a\} * (H * H) \subseteq a * H,
\end{aligned}
$$

so $t \in a * H$ and H is a right ideal of H. Since H is right simple, we have $a * H = H$. The "dual" case can be proved similarly. $\square$

By Propositions 3.31 and 3.34 we have the following

Proposition 3.35. *An intra-consistent hypergroupoid H is right (rep. left) simple if and only if, for any $a \in H$, we have $a * H = H$ (resp. $H * a = H$).*

Problem. Find conditions under which for a right (left) consistent or intra-consistent hypergroupoid H the equivalence relations $\mathcal{R}$ and $\mathcal{L}$ are semilattice congruences on H.

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