EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 4, 2018, 1003-1013
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

# Double Lusin Condition for the Itô-Henstock Integrable Operator-Valued Stochastic Process

Mhelmar A. Labendia[1,*], Jayrold P. Arcede[2]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines*
[2] *Department of Mathematics, College of Arts and Sciences, Caraga State University, 8600 Butuan City, Philippines*

**Abstract.** In this paper, using double Lusin condition, we give an equivalent definition of the Itô-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued Wiener process.

**2010 Mathematics Subject Classifications**: 60H30, 60H05
**Key Words and Phrases**: Itô-Henstock integral, $Q$-Wiener process, double Lusin condition

## 1. Introduction

The Henstock integral, which was studied independently by Henstock and Kurzweil in the 1950s and later known as the Henstock-Kurzweil integral, is one of the notable integrals that was introduced which in some sense is more general than the Lebesgue integral. To avoid an extensive study of measure theory, Henstock-Kurzweil integration had been deeply studied and investigated by numerous authors, see [3–5, 8–10]. The Henstock-Kurzweil integral is a Riemann-type definition of an integral which is more explicit and minimizes the technicalities in the classical approach of the Lebesgue integral. This approach to integration is known as the generalized Riemann approach or Henstock approach.

In the classical approach to stochastic integration, the Itô integral of a real-valued stochastic process, which is adapted to a filtration, is attained from a limit of Itô integrals of simple processes. To give a more explicit definition and reduce the technicalities in the classical way of defining the Itô integral in the real-valued case, Henstock approach to stochastic integration had already been studied in several papers, see [12, 13, 17–19].

In infinite dimensional spaces, the Itô integral of an operator-valued stochastic process, adapted to a normal filtration, is obtained by extending an isometry from the space of

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v11i4.3310

*Email addresses:* mhelmar.labendia@g.msuiit.edu.ph (M. Labendia),
jparcede@carsu.edu.ph (J. Arcede)

 

elementary processes to the space of continuous square-integrable martingales. In this case, the value of the integrand is a Hilbert-Schmidt operator and the integrator is a $Q$-Wiener process, a Hilbert space-valued Wiener process which is dependent on a symmetric nonnegative definite trace-class operator $Q$. In [7], the authors defined the Itô-Henstock integral of an operator-valued stochastic process with respect to a $Q$-Wiener process and formulated a version of Itô's formula, the stochastic counterpart of the classical chain rule of differentiation.

In this paper, we revisit the concept of Itô-Henstock integral for the operator-valued stochastic process with respect to a $Q$-Wiener process and characterize Itô-Henstock integrability by using the concept of double Lusin condition and $AC^2[0,T]$-property, a version of absolute continuity.

## 2. Preliminaries

Throughout this paper, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a *filtered probability space*, $\mathcal{B}(H)$ be the Borel $\sigma$-field of a separable Banach space $H$, and $\mathcal{L}(h)$ be the *probability distribution* or the *law* of a random variable $h : \Omega \to H$.

A *stochastic process* $f : [0,T] \times \Omega \to H$, or simply a *process* $\{f_t\}_{0 \leq t \leq T}$, is said to be *adapted* to a filtration $\{\mathcal{F}_t\}$ if $f_t$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$. When no confusion arises, we may refer to a process adapted to $\{\mathcal{F}_t\}$ as simply an adapted process.

Let $U$ and $V$ be separable Hilbert spaces. Denote by $L(U,V)$ the space of all bounded linear operators from $U$ to $V$, $L(U) := L(U,U)$, $Qu := Q(u)$ if $Q \in L(U,V)$, and $L^2(\Omega, V)$ the space of all square-integrable random variables from $\Omega$ to $V$. An operator $Q \in L(U)$ is said to be *self-adjoint* or *symmetric* if for all $u, u' \in U$, $\langle Qu, u' \rangle_U = \langle u, Qu' \rangle_U$ and is said to be *nonnegative definite* if for every $u \in U$, $\langle Qu, u \rangle_U \geq 0$. Using the Square-root Lemma [16, p.196], if $Q \in L(U)$ is nonnegative definite, then there exists a unique operator $Q^{\frac{1}{2}} \in L(U)$ such that $Q^{\frac{1}{2}}$ is nonnegative definite and $(Q^{\frac{1}{2}})^2 = Q$.

Let $\{e_j\}_{j=1}^{\infty}$, or simply $\{e_j\}$, be an orthonormal basis (abbrev. as ONB) in $U$. If $Q \in L(U)$ is nonnegative definite, then the trace of $Q$ is defined by $\text{tr } Q = \sum_{j=1}^{\infty} \langle Qe_j, e_j \rangle_U$. It is shown in [16, p.206] that tr $Q$ is well-defined and may be defined in terms of an arbitrary ONB. An operator $Q : U \to U$ is said to be *trace-class* if $\text{tr } [Q] := \text{tr } (QQ^*)^{\frac{1}{2}} < \infty$. Denote by $L_1(U)$ the space of all trace-class operators on $U$, which is known [16, p.209] to be a Banach space with norm $\|Q\|_1 = \text{tr } [Q]$. If $Q \in L(U)$ is a symmetric nonnegative definite trace-class operator, then there exists an ONB $\{e_j\} \subset U$ and a sequence of nonnegative real numbers $\{\lambda_j\}$ such that $Qe_j = \lambda_j e_j$ for all $j \in \mathbb{N}$, $\{\lambda_j\} \in \ell^1$, and $\lambda_j \to 0$ as $j \to \infty$ [16, p.203]. We shall call the sequence of pairs $\{\lambda_j, e_j\}$ an *eigensequence defined by $Q$*.

Let $Q : U \to U$ be a symmetric nonnegative definite trace-class operator. Let $\{\lambda_j, e_j\}$ be an eigensequence defined by $Q$. Then the subspace $U_Q := Q^{\frac{1}{2}} U$ of $U$ equipped with the inner product $\langle u, v \rangle_{U_Q} = \left\langle Q^{-1/2} u, Q^{-1/2} v \right\rangle_U$, where $Q^{1/2}$ is being restricted to $[\text{Ker} Q^{1/2}]^{\perp}$ is a separable Hilbert space with $\left\{ \sqrt{\lambda_j} e_j \right\}$ as its ONB, see [15, p.90], [2, p.23].

Let $\{f_j\}$ be an ONB in $U_Q$. An operator $S \in L(U_Q, V)$ is said to be *Hilbert-Schmidt* if $\sum_{j=1}^{\infty} \|Sf_j\|_V^2 = \sum_{j=1}^{\infty} \langle Sf_j, Sf_j \rangle_V < \infty$. Denote by $L_2(U_Q, V)$ the space of all Hilbert-

Schmidt operators from $U_Q$ to $V$, which is known [14, p.112] to be a separable Hilbert space with norm $\|S\|_{L_2(U_Q,V)} = \sqrt{\sum_{j=1}^{\infty} \|Sf_j\|_V^2}$. The Hilbert-Schmidt operator $S \in L_2(U_Q, V)$ and the norm $\|S\|_{L_2(U_Q,V)}$ may be defined in terms of an arbitrary ONB, see [15, p.418], [14, p.111]. It is shown in [2, p.25] that $L(U, V)$ is properly contained in $L_2(U_Q, V)$. We also note that $L_2(U_Q, V)$ contains genuinely unbounded linear operators from $U$ to $V$.

Let $Q : U \to U$ be a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ be an eigensequence defined by $Q$, and $\{B_j\}$ be a sequence of independent *Brownian motions* (abbrev. as $BM$) defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. The process

$$\tilde{W}_t := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) e_j \tag{1}$$

is called a *$Q$-Wiener process in $U$*. The series in (1) converges in $L^2(\Omega, U)$. For each $u \in U$, denote $\tilde{W}_t(u) := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) \langle e_j, u \rangle_U$, with the series converging in $L^2(\Omega, \mathbb{R})$. Since the operator $Q$ is assumed to be symmetric nonnegative definite trace-class, there exists a $U$-valued process $W$ such that

$$\tilde{W}_t(u)(\omega) = \langle W_t(\omega), u \rangle_U \quad \mathbb{P}\text{-almost surely (abbrev. as } \mathbb{P}\text{-a.s.).} \tag{2}$$

We call the process $W$ a *$U$-valued $Q$-Wiener process.* This process is a multidimentional $BM$. It should be noted that if we assume that $\lambda_j > 0$ for all $j$, $\frac{W_t(e_j)}{\sqrt{\lambda_j}}$, $j = 1, 2, \ldots$, is a sequence of real-valued $BM$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, see [15, p.87].

A filtration $\{\mathcal{F}_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *normal* if (i) $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$, and (ii) $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0, T]$. A $Q$-Wiener process $W_t$, $t \in [0, T]$ is called a *$Q$-Wiener process with respect to a filtration* $\{\mathcal{F}_t\}$ if (i) $W_t$ is adapted to $\{\mathcal{F}_t\}$, $t \in [0, T]$ and (ii) $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t \leq T$. It is shown in [14, p.16] that a $U$-valued $Q$-Wiener process $W(t)$, $t \in [0, T]$, is a $Q$-Wiener process with respect to a normal filtration. From now onwards, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ shall mean a probability space equipped with a normal filtration.

## 3. Itô-Hentock Integral and Double Lusin Condition

In [19], Chew et al. introduced the Itô-Henstock integral of a real-valued process with respect to a Brownian motion. We shall use the same definition of belated partial division employed by the authors in [19] to define the Itô-Henstock integral of an $L(U, V)$-valued stochastic process with respect to a $U$-valued $Q$-Wiener process. We note that the given closed and bounded interval $[0, T]$ is *nondegenerate*, i.e. $0 < T$, which can be replaced with any interval $[a, b]$. If no confusion arises, we may write $(D) \sum$ instead of $\sum_{i=1}^{n}$ for the given finite collection $D$.

**Definition 1.** Let $\delta$ be a positive function on $[0,T]$. A finite collection $D$ of interval-point pairs $\{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ is a *$\delta$-fine belated partial division* of $[0,T]$ if

(i) $(\xi_i, v_i], i = 1, 2, \ldots, n$, are disjoint subintervals of $[0,T]$; and

(ii) each $(\xi_i, v_i]$ is $\delta$-fine belated, that is, $(\xi_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i))$.

The term *partial* is used in Definition 1 since the finite collection of disjoint left-open subintervals of $[0,T]$ may not cover the entire interval $[0,T]$. Using the Vitali covering lemma, the following concept can be defined.

**Definition 2.** Given $\eta > 0$, a given $\delta$-fine belated partial division $D = \{((\xi, v], \xi)\}$ is said to be a *$(\delta, \eta)$-fine belated partial division* of $[0,T]$ if it fails to cover $[0,T]$ by at most length $\eta$, that is,

$$\left| T - (D)\sum (v - \xi) \right| \leq \eta.$$

This type of partial division is the basis to which we define the Itô-Henstock integral.

Throughout the succeeding discussions, assume that $U$ and $V$ are separable Hilbert spaces, $Q : U \to U$ is a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ is an eigensequence defined by $Q$, and $W$ is a $U$-valued $Q$-Wiener process. A stochastic process $f : [0,T] \times \Omega \to L(U,V)$ means a process measurable as mappings from $([0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F})$ to $(L_2(U_Q, V), \mathcal{B}(L_2(U_Q, V)))$.

**Definition 3.** Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process. Then $f$ is said to be *Itô-Henstock integrable*, or $\mathcal{IH}$-integrable, on $[0,T]$ with respect to $W$ if there exists $A \in L^2(\Omega, V)$ such that for every $\epsilon > 0$, there is a positive function $\delta$ on $[0,T]$ and a positive number $\eta > 0$ such that for any $(\delta,\eta)$-fine belated partial division $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ of $[0,T]$, we have

$$\mathbb{E}\left[ \|S(f, D, \delta, \eta) - A\|_V^2 \right] < \epsilon,$$

where

$$S(f, D, \delta, \eta) := (D)\sum f_\xi (W_v - W_\xi) := \sum_{i=1}^n f_{\xi_i}(W_{v_i} - W_{\xi_i}).$$

In this case, $f$ is $\mathcal{IH}$-integrable to $A$ on $[0,T]$ and $A$ is called the $\mathcal{IH}$-integral of $f$ which will be denoted by $(\mathcal{IH})\int_0^T f_t \, dW_t$ or $(\mathcal{IH})\int_0^T f \, dW$. For convenience, we shall denote $(\mathcal{IH})\int_0^0 f_t \, dW_t$ by the zero random variable $\mathbf{0} \in L^2(\Omega, V)$.

**Example 1.** *$f : [0,T] \times \Omega \to L(U,V)$ be an adapted process such that $\mathbb{E}\left[ \|f_t\|_{L_2(U_Q,V)}^2 \right] = 0$ for all $t \in [0,T]$ except on a set of Lebesgue measure zero. Then $f$ is $\mathcal{IH}$-integrable to $\mathbf{0}$ on $[0,T]$.*

The following statements show that the Itô-Henstock integral possesses the standard properties of an integral. Refer to [8] for the proofs.

(1) The Itô-Henstock integral is uniquely determined, in the sense that if $A_1$ and $A_2$ are two Itô-Henstock integrals of $f$ in Definition 3, then $\|A_1 - A_2\|_{L^2(\Omega,V)} = 0$.

(2) Let $\alpha \in \mathbb{R}$. If $f$ and $g$ are $\mathcal{IH}$-integrable on $[0,T]$, then

(i) $f+g$ is $\mathcal{IH}$-integrable on $[0,T]$, and

$$(\mathcal{IH}) \int_0^T (f+g)\ dW = (\mathcal{IH}) \int_0^T f\ dW + (\mathcal{IH}) \int_0^T g\ dW;$$

(ii) $\alpha f$ is $\mathcal{IH}$-integrable on $[0,T]$, and

$$(\mathcal{IH}) \int_0^T (\alpha f)\ dW = \alpha \cdot (\mathcal{IH}) \int_0^T f\ dW.$$

(3) If $f : [0,T] \times \Omega \to L(U,V)$ is $\mathcal{IH}$-integrable on $[0,c]$ and $[c,T]$ where $c \in (0,T)$, then $f$ is $\mathcal{IH}$-integrable on $[0,T]$ and

$$(\mathcal{IH}) \int_0^T f\ dW = (\mathcal{IH}) \int_0^c f\ dW + (\mathcal{IH}) \int_c^T f\ dW.$$

(4) If $f : [0,T] \times \Omega \to L(U,V)$ is $\mathcal{IH}$-integrable on $[0,T]$, then $f$ is also $\mathcal{IH}$-integrable on every subinteval $[c,d]$ of $[0,T]$.

(5) A process $f : [0,T] \times \Omega \to L(U,V)$ is $\mathcal{IH}$-integrable on $[0,T]$ if and only if there exist $A \in L^2(\Omega,V)$, a decreasing sequence $\{\delta_n\}$ of positive functions defined on $[0,T]$, and a decreasing sequence of positive numbers $\{\eta_n\}$ such that for any $(\delta_n, \eta_n)$-fine belated partial division $D_n$ of $[0,T]$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\|S(f, D_n, \delta_n, \eta_n) - A\|_V^2\right] = 0.$$

In this case,

$$A = (\mathcal{IH}) \int_0^T f_t\ dW_t.$$

(6) (Cauchy Criterion). A process $f : [0,T] \times \Omega \to L(U,V)$ is $\mathcal{IH}$-integrable on $[0,T]$ if and only if for every $\epsilon > 0$, there exist a positive function $\delta$ on $[0,T]$ and a positive number $\eta$ such that for any two $(\delta,\eta)$-fine belated partial divisions $D$ and $D'$ of $[0,T]$, we have

$$\mathbb{E}\left[\left\|S(f, D, \delta, \eta) - S(f, D', \delta, \eta)\right\|_V^2\right] < \epsilon.$$

(7) (Weak Version of Saks-Henstock Lemma). Let $f$ be $\mathcal{IH}$-integrable on $[0,T]$ and $F(u,v] := (\mathcal{IH}) \displaystyle\int_u^v f_t\ dW_t$ for any $(u,v] \subset [0,T]$. Then for every $\epsilon > 0$, there exist a positive function $\delta$ on $[0,T]$ such that for any $\delta$-fine belated partial division $D = \{((\xi,v],\xi)\}$ of $[0,T]$, we have

$$\mathbb{E}\left[\left\|(D)\sum \{f_\xi(W_v - W_\xi) - F(\xi,v]\}\right\|_V^2\right] < \epsilon.$$

(8) (Itô Isometry). Let $f$ be $\mathcal{IH}$-integrable on $[0,T]$. Then $\mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]$ is Lebesgue integrable on $[0,T]$ and

$$\mathbb{E}\left[\left\|(\mathcal{IH})\int_0^T f_t\ dW_t\right\|_V^2\right] = (\mathcal{L})\int_0^T \mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right]\ dt < \infty.$$

In [6], the Itô-Henstock integral has been characterized using $AC^2[0,T]$-property, a version of absolute continuity. Throughout the following, denote by $\mathcal{J}$, the collection of all closed intervals $(u,v] \subset [0,T]$. In the following definition, when no confusion arises, we may refer to $F((u,v],\cdot)$ or $F((u,v],\omega)$ as simply $F(u,v]$.

**Definition 4.** A function $F : \mathcal{J}\times\Omega \to V$ is said to be $AC^2[0,T]$ if for every $\epsilon > 0$, there exists $\eta > 0$ such that for any finite collection $D = \{(\xi,v]\}$ of non-overlapping subintervals of $[0,T]$ with $(D)\sum(v-\xi) < \eta$, we have $\mathbb{E}\left[\left\|(D)\sum F(\xi,v]\right\|_V^2\right] < \epsilon$.

**Theorem 1.** [6, Theorem 3.4] *Let $f : [0,T]\times\Omega \to L(U,V)$ be an adapted process. Then $f$ is $\mathcal{IH}$-integrable on $[0,T]$ if and only if there exists a function $F : \mathcal{J}\times\Omega \to V$ such that*

*(i) $F$ is $AC^2[0,T]$ and*

*(ii) for every $\epsilon > 0$, there exist a positive function $\delta$ on $[0,T]$ such that whenever $D = \{((\xi,v],\xi)\}$ is a $\delta$-fine belated partial division of $[0,T]$, we have*

$$\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_v - W_\xi) - F(\xi,v]\}\right\|_V^2\right] < \epsilon.$$

We remark that in Theorem 1 if $f$ is $\mathcal{IH}$-integrable on $[0,T]$, then the existing function $F$ that satisfies conditions $(i)$ and $(ii)$ is given by $F(u,v] := (\mathcal{IH})\displaystyle\int_u^v f_t\ dW_t$ for each $(u,v]\in\mathcal{J}$, see [6, proof of Theorem 3.4].

Next, we present the *double Lusin condition*-property for a process $f : [0,T]\times\Omega \to L(U,V)$ and a function $F : \mathcal{J}\times\Omega \to V$. This property is analogous to the double Lusin condition used in [1, 11].

**Definition 5.** Let $f : [0,T]\times\Omega \to L(U,V)$ be an adapted process and $F : \mathcal{J}\times\Omega \to V$ be a function. For any given $\epsilon > 0$, let $\Gamma_\epsilon$ be the set of all interval-point pairs $\{((\xi,v],\xi)\}$ such that

$$\begin{aligned}\mathbb{E}\left[\|f_\xi(W_v - W_\xi) - F(\xi,v]\|_V^2\right] &\geq \epsilon\mathbb{E}\left[\|W_v - W_\xi\|_U^2\right]\\ &= \epsilon(v-\xi)\mathrm{tr}\ Q.\end{aligned}$$

**Definition 6.** A process $f : [0,T]\times\Omega \to L(U,V)$ and a function $F : \mathcal{J}\times\Omega \to V$ are said to satisfy the double Lusin condition if for every $\epsilon > 0$, there exists a positive function $\delta$ on $[0,T]$ such that for any $\delta$-fine belated partial division $D = \{((\xi,v],\xi)\}\subseteq\Gamma_\epsilon$ of $[0,T]$,

$$\mathbb{E}\left[\|(D)\textstyle\sum f_\xi(W_v - W_\xi)\|_V^2\right] < \epsilon \text{ and } \mathbb{E}\left[\|(D)\textstyle\sum F(\xi,v]\|_V^2\right] < \epsilon.$$

**Definition 7.** A function $F : \mathcal{J} \times \Omega \to V$ is said to satisfy the double Lusin condition if for every $\epsilon > 0$, there exists a positive function $\delta$ on $[0, T]$ such that for any $\delta$-fine belated partial division $D = \{((\xi, v], \xi)\} \subseteq \Gamma_\epsilon$ of $[0, T]$,

$$\mathbb{E}\left[\|(D)\sum(W_v - W_\xi)\|_V^2\right] < \epsilon \text{ and } \mathbb{E}\left[\|(D)\sum F(\xi, v]\|_V^2\right] < \epsilon.$$

Before giving an equivalent definition of $\mathcal{IH}$-integrable operator-valued process, we need to consider the following known results:

**Lemma 1.** [7, Lemma 3.6] *Let $f : [0, T] \times \Omega \to L(U, V)$ be an adapted process and $\{(\xi_i, v_i]\}_{i=1}^n$ be a finite collection of disjoint subintervals of $[0, T]$. Then*

$$\begin{aligned}\mathbb{E}\left[\left\|\sum_{i=1}^n f_{\xi_i}(W_{v_i} - W_{\xi_i})\right\|_V^2\right] &= \sum_{i=1}^n \mathbb{E}\left[\|f_{\xi_i}(W_{v_i} - W_{\xi_i})\|_V^2\right] \\ &= \sum_{i=1}^n (v_i - \xi_i)\mathbb{E}\left[\|f_{\xi_i}\|_{L_2(U_Q,V)}^2\right].\end{aligned}$$

**Lemma 2.** (Strong Version of Saks-Henstock Lemma). *Let $f$ be $\mathcal{IH}$-integrable on $[0, T]$ and $F(u, v] := (\mathcal{IH})\int_u^v f_t \, dW_t$ for any $(u, v] \subset [0, T]$. Then for every $\epsilon > 0$, there exist a positive function $\delta$ on $[0, T]$ such that for any $\delta$-fine belated partial division $D = \{((\xi, v], \xi)\}$ of $[0, T]$, we have*

$$(D)\sum \mathbb{E}\left[\|f_\xi(W_v - W_\xi) - F(\xi, v]\|_V^2\right] < \epsilon.$$

We shall now characterize the Itô-Henstock integral using the double Lusin condition.

**Theorem 2.** *Let $f : [0, T] \times \Omega \to L(U, V)$ be an adapted process. Then $f$ is $\mathcal{IH}$-integrable on $[0, T]$ if and only if there exists an $AC^2[0, T]$ function $F : \mathcal{J} \times \Omega \to V$ and that $f$ and $F$ satisfy the double Lusin condition.*

*Proof.* Suppose that $f$ is $\mathcal{IH}$-integrable on $[0, T]$ and let $F(u, v] = (\mathcal{IH})\int_u^v f_t \, dW_t$ for each $(u, v] \in \mathcal{J}$. By Theorem 1, $F$ is $AC^2[0, T]$. Let $\epsilon > 0$. By Theorem 1 and the strong version of Saks-Henstock Lemma, for each $k \in \mathbb{N}$, there exists a positive function $\delta_k$ on $[0, T]$ such that for any $\delta_k$-fine belated partial division $D_k = \{((\xi, v], \xi)\}$ of $[0, T]$, we have

$$\begin{aligned}(D_k)\sum \mathbb{E}\left[\|f_\xi(W_v - W_\xi) - F(\xi, v]\|_V^2\right] &= \mathbb{E}\left[\left\|(D_k)\sum\{f_\xi(W_v - W_\xi) - F(\xi, v]\}\right\|_V^2\right] \\ &< \frac{\epsilon^2(\text{tr } Q)}{k \cdot 2^{k+2}}.\end{aligned}$$

Moreover, there exists a positive function $\delta'$ on $[0, T]$ such that for any $\delta'$-fine belated partial division $D' = \{((\xi, v], \xi)\}$ of $[0, T]$, we have

$$\mathbb{E}\left[\left\|(D')\sum\{f_\xi(W_v - W_\xi) - F(\xi, v]\}\right\|_V^2\right] < \frac{\epsilon}{4}.$$

For each $k \in \mathbb{N}$, let $G_k := \left\{t \in [0,T] : k-1 \leq \mathbb{E}\left[\|f_t\|^2_{L_2(U_Q,V)}\right] < k\right\}$. Choose $\delta(\xi) = \min\{\delta'(\xi), \delta_k(\xi)\}$ if $\xi \in G_k$ for some $k \in \mathbb{N}$. Let $D = \{((\xi,v],\xi)\} \subseteq \Gamma_\epsilon$ be a $\delta$-fine belated partial division of $[0,T]$. For each $k \in \mathbb{N}$, let $D_k \subseteq D$ such that each tag in $D_k$ is in $G_k$. Then by Lemma 1,

$$\begin{aligned}
\mathbb{E}\left[\left\|(D)\sum f_\xi(W_v - W_\xi)\right\|_V^2\right] &= (D)\sum (v-\xi)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right] \\
&\leq \sum_{k\in\mathbb{N}}\left((D_k)\sum(v-\xi)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right]\right) \\
&\leq \sum_{k\in\mathbb{N}}\left(k\cdot(D_k)\sum(v-\xi)\right) \\
&\leq \sum_{k\in\mathbb{N}}\left(\frac{k}{\epsilon(\mathrm{tr}\ Q)}\mathbb{E}\left[\left\|(D_k)\sum\{f_\xi(W_v-W_\xi) - F(\xi,v]\}\right\|_V^2\right]\right) \\
&\leq \sum_{k\in\mathbb{N}}\frac{k}{\epsilon(\mathrm{tr}\ Q)}\cdot\frac{\epsilon^2(\mathrm{tr}\ Q)}{k\cdot 2^{k+2}} = \frac{\epsilon}{4}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left[\left\|(D)\sum F(\xi,v]\right\|_V^2\right] &\leq 2\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_v-W_\xi)-F(\xi,v]\}\right\|_V^2\right] \\
&\quad +2\mathbb{E}\left[\left\|(D)\sum f_\xi(W_v-W_\xi)\right\|_V^2\right] \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Conversely, suppose that there exists an $AC^2[0,T]$ function $F : \mathcal{J}\times\Omega \to V$ and that $f$ and $F$ satisfy the double Lusin condition. Let $\epsilon > 0$. Then there exists a positive function $\delta$ on $[0,T]$ such that for any $\delta$-fine belated partial division $D' = \{((\xi,v],\xi)\} \subseteq \Gamma_\epsilon$ of $[0,T]$, we have

$$\mathbb{E}\left[\|(D')\sum f_\xi(W_v-W_\xi)\|_V^2\right] < \epsilon \text{ and } \mathbb{E}\left[\|(D')\sum F(\xi,v]\|_V^2\right] < \epsilon.$$

Let $D = \{((\xi,v],\xi)\}$ be $\delta$-fine belated partial division of $[0,T]$. Then

$$\begin{aligned}
&\mathbb{E}\left[\|(D)\sum f_\xi(W_v-W_\xi)-F(\xi,v]\|_V^2\right] \\
&\leq 2\left((D\setminus\Gamma_\epsilon)\sum\sqrt{\mathbb{E}\left[\|f_\xi(W_v-W_\xi)-F(\xi,v]\|_V^2\right]}\right)^2 \\
&\quad +4\mathbb{E}\left[\left\|(D\cap\Gamma_\epsilon)\sum f_\xi(W_v-W_\xi)\right\|_V^2\right] \\
&\quad +4\mathbb{E}\left[\|(D\cap\Gamma_\epsilon)F(\xi,v]\|_V^2\right] \\
&< 2\left((D\setminus\Gamma_\epsilon)\sum\sqrt{\epsilon(v-\xi)\mathrm{tr}\ Q}\right)^2 + 4\epsilon+4\epsilon = \epsilon(2T\cdot\mathrm{tr}\ Q+8).
\end{aligned}$$

By Theorem 1, $f$ is $\mathcal{IH}$-integrable on $[0,T]$. $\square$

**Theorem 3.** *Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process. Then $f$ is $\mathcal{IH}$-integrable on $[0,T]$ if and only if there exists an $AC^2[0,T]$ function $F : \mathcal{J} \times \Omega \to V$ that satisfies the double Lusin condition.*

*Proof.* Suppose that $f$ is $\mathcal{IH}$-integrable on $[0,T]$ and let $F(u,v] = (\mathcal{IH}) \int_u^v f_t \, dW_t$ for each $(u,v] \in \mathcal{J}$. By Theorem 1, $F$ is $AC^2[0,T]$. Let $\epsilon > 0$. By Theorem 1 and the strong version of Saks-Henstock Lemma, there exists a positive function $\delta$ on $[0,T]$ such that for any $\delta$-fine belated partial division $D' = \{((\xi, v], \xi)\}$ of $[0,T]$, we have

$$(D') \sum \mathbb{E}\left[\|f_\xi(W_v - W_\xi) - F(\xi, v]\|_V^2\right] = \mathbb{E}\left[\left\|(D') \sum \{f_\xi(W_v - W_\xi) - F(\xi, v]\}\right\|_V^2\right] < \epsilon^2.$$

Let $D = \{((\xi, v], \xi)\} \subseteq \Gamma_\epsilon$ be a $\delta$-fine belated partial division of $[0,T]$. Then by Lemma 1,

$$\begin{aligned}
\mathbb{E}\left[\left\|(D) \sum (W_v - W_\xi)\right\|_V^2\right] &= (D) \sum (v - \xi) \text{tr } Q \\
&\le \frac{1}{\epsilon} (D) \sum \mathbb{E}\left[\|f_\xi(W_v - W_\xi) - F(\xi, v]\|_V^2\right] \\
&< \frac{1}{\epsilon} \cdot \epsilon^2 = \epsilon.
\end{aligned}$$

Conversely, suppose that there exists an $AC^2[0,T]$ function $F : \mathcal{J} \times \Omega \to V$ that satisfies the double Lusin condition. Let $\epsilon > 0$. By Theorem 1 and the strong version of Henstock Lemma, for each $k \in \mathbb{N}$, there exists a positive function $\delta_k$ on $[0,T]$ such that for any $\delta_k$-fine belated partial division $D_k = \{((\xi, v], \xi)\}$ of $[0,T]$, we have

$$\begin{aligned}
(D_k) \sum \mathbb{E}\left[\|f_\xi(W_v - W_\xi) - F(\xi, v]\|_V^2\right] &= \mathbb{E}\left[\left\|(D_k) \sum \{f_\xi(W_v - W_\xi) - F(\xi, v]\}\right\|_V^2\right] \\
&< \frac{\epsilon^2 (\text{tr } Q)}{k \cdot 2^{k+1}}.
\end{aligned}$$

For each $k \in \mathbb{N}$, let $G_k := \left\{t \in [0,T] : k - 1 \le \mathbb{E}\left[\|f_t\|_{L_2(U_Q,V)}^2\right] < k\right\}$. Choose $\delta(\xi) \le \delta_k(\xi)$ if $\xi \in G_k$ for some $k \in \mathbb{N}$. Let $D = \{((\xi, v], \xi)\} \subseteq \Gamma_\epsilon$ be a $\delta$-fine belated partial division of $[0,T]$. For each $k \in \mathbb{N}$, let $D_k \subseteq D$ such that each tag in $D_k$ is in $G_k$. Then by Lemma 1,

$$\begin{aligned}
\mathbb{E}\left[\left\|(D) \sum f_\xi(W_v - W_\xi)\right\|_V^2\right] &= (D) \sum (v - \xi) \mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right] \\
&\le \sum_{k \in \mathbb{N}} \left((D_k) \sum (v - \xi) \mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right]\right) \\
&\le \sum_{k \in \mathbb{N}} \left(k \cdot (D_k) \sum (v - \xi)\right)
\end{aligned}$$

$$\leq \sum_{k\in\mathbb{N}} \left( \frac{k}{\epsilon(\text{tr } Q)} \mathbb{E}\left[ \left\| (D_k) \sum \{ f_\xi(W_v - W_\xi) - F(\xi, v] \} \right\|_V^2 \right] \right)$$
$$\leq \sum_{k\in\mathbb{N}} \frac{k}{\epsilon(\text{tr } Q)} \cdot \frac{\epsilon^2(\text{tr } Q)}{k \cdot 2^{k+1}} < \epsilon.$$

By Theorem 2, $f$ is $\mathcal{IH}$-integrable on $[0, T]$. □

We remark that in Theorem 2, the double Lusin condition involving the process $f$ and the function $F$ may be restated as the double Lusin condition involving the function $F$ only.

## 4. Conclusion and Recommendation

In this paper, we formulate an equivalent definition of the Itô-Henstock integral of an operator-valued stochastic process with respect to a Hilbert space-valued $Q$-Wiener process. To attain this objective, we use the concept of the double Lusin condition and $AC^2[0, T]$-property, a version of absolute continuity. A worthwhile direction for further investigation is to use Henstock-Kurzweil approach to define the stochastic integral with respect to a cylindrical Wiener process.


## Acknowledgements

The first author would like to express his sincerest gratitude to his wife for the inspiration in completing this paper. Also, the authors would like to acknowledge the financial support from the National Research Council of the Philippines (NRCP) and to thank the unknown referee for the helpful comments for the improvement of this paper.


## References


[1] E. Cabral and P. Y. Lee. A fundamental theorem of calculus for the Kuzweil-Henstock integral in $R^m$. *Real Anal. Exchange*, 26:867–876, 2000-2001.

[2] L. Gawarecki and V. Mandrekar. *Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations.* Springer, Berlin, 2011.

[3] R. A. Gordon. *The Integrals of Lebesgue, Denjoy, Perron and Henstock.* American Mathematical Society, 1994.

[4] R. Henstock. *Lectures on the Theory of Integration.* World Scientific, Singapore, 1988.

[5] J. Kurzweil. *Henstock-Kurzweil Integration: Its Relation to Topological Vector Spaces.* World Scientific, Singapore, 2000.

[6] M. Labendia and J. Benitez. Convergence theorems for the Itô-Henstock integrable operator-valued stochastic process. *Malaysian Journal of Mathematical Sciences*, to appear.

[7] M. Labendia E. De Lara-Tuprio and T. R. Teng. Itô-Henstock integral and Itô's formula for the operator-valued stochastic process. *Mathematica Bohemica*, 143:135–160, 2018.

[8] P. Y. Lee. *Lanzhou Lectures on Henstock Integration.* World Scientific, Singapore, 1989.

[9] P. Y. Lee and R. Výborný. *The Integral: An Easy Approach after Kurzweil and Henstock.* Cambridge University Press, Cambridge, 2000.

[10] T. Y. Lee. *Henstock-Kurzweil Integration on Euclidean Spaces.* World Scientific, Singapore, 2011.

[11] J. T. Lu and P. Y. Lee. The primitives of Henstock integrable functions in Euclidean space. *Bull. London Math. Society*, 31:173–180, 1999.

[12] E. J. McShane. Stochastic integrals and stochastic functional equations. *SIAM J. Appl. Math.*, 17:287–306, 1969.

[13] Z. R. Pop-Stojanovic. On Mcshane's belated stochastci integral. *SIAM J. Appl. Math.*, 22:87–92, 1972.

[14] C. Prévôt and M. Röckner. A concise course on stochastic partial differential equations. 2007.

[15] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, Cambridge, 1992.

[16] M. Reed and B. Simon. Methods of modern mathematical physics i: Functional analysis. 1980.

[17] T. L. Toh and T. S. Chew. The Riemann approach to stochastic integration using non-uniform meshes. *J. Math. Anal. Appl.*, 280:133–147, 2003.

[18] T. L. Toh and T. S. Chew. On the Henstock-Fubini theorem for multiple stochastic integrals. *Real Anal. Exchange*, 30:295–310, 2004-2005.

[19] T. S. Chew T. L. Toh and J. Y. Tay. The non-uniform riemann approach to Itô's integral. *Real Anal. Exchange*, 27:495–514, 2002-2003.