A General Family of the Srivastava-Gupta Operators Preserving Linear Functions

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Abstract. The general sequence of positive linear operators containing some well-known operators as special cases were introduced in the earlier work by Srivastava and Gupta [9], which reproduce only the constant functions. In the present sequel, we provide a general sequence of operators which preserve not only the constant functions, but also linear functions.

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1. Introduction and Preliminaries

In the year 2003, Srivastava and Gupta [9] introduced a general sequence of positive linear operators defined by

\[ V_{n,c}(f, x) = \sum_{k=1}^{\infty} p_{n,k}(x, c) \int_0^\infty p_{n+c,k-1}(t, c)f(t)dt \]

\[ + p_{n,0}(x, c)f(0) \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots \}), \]

(1)

where

\[ p_{n,k}(x, c) = \frac{(-x)^k}{k!} \phi^{(k)}_{n,c}(x). \]

The following special cases of the operator defined by (1) are worthy of mention here:

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• If \( c = 0 \) and \( \phi_{n,c}(x) = e^{-nx} \), then we get
  \[ p_{n,k}(x,0) = e^{-nx} \frac{(nx)^k}{k!}; \]

• If \( c \in \mathbb{N} \) and \( \phi_{n,c}(x) = (1 + cx)^{-n} \), then we obtain
  \[ p_{n,k}(x,c) = \frac{1}{(cx)^n} \frac{(nx)^k}{k!} \]
  where, and in what follows, \((\lambda)_n\) denotes the Pochhammer symbol (or the shifted factorial) defined, for \( \lambda \in \mathbb{C} \), by
  \[(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \lambda(\lambda + 1) \cdots (\lambda + n - 1) \quad (n \in \mathbb{N});\]

• If \( c = -1 \) and \( \phi_{n,c}(x) = (1 - x)^n \), then
  \[ p_{n,k}(x,-1) = \binom{n}{k} x^k(1-x)^{n-k}. \]

Here, for the last case when \( c = -1 \), we have \( x \in [0,1] \) whereas, for \( c \in \mathbb{N} \cup \{0\} \), we have \( x \in [0,\infty) \). These operators were further discussed by Ispir and Yüksel [6], Atakuta and Büyükyazıcı [2], Deo [3], Gupta and Tachev [5], Kumar [7] and Yadav [13], and other authors (see also the closely-related works by Acar et al. [1], Maheshwari [8], Srivastava and Gupta [10], Srivastava and Zeng [11], and Verma and Agrawal [12]).

The moments of the above-defined operator \( V_{n,c}(f,x) \) of order \( r \) (\( r \in \mathbb{N} \)) are given, in terms of the Gauss hypergeometric function \( \,_{2}F_{1} \) and Kummer’s confluent hypergeometric function \( \,_{1}F_{1} \), as follows:

\[ V_{n,c}(e_r,x) = \begin{cases} \frac{r! \cdot (nx)}{(n-c)(n-2c) \cdots (n-rc)} \cdot \,_{2}F_{1}\left( \frac{n}{c+1},1-r;2;-cx \right) & \text{for } c \in \mathbb{N} \cup \{-1\} \end{cases} \]  
\[ 2 \]  
Here, \( \ell \) is the number of numerator parameters \( \alpha_1, \ldots, \alpha_\ell \) and \( m \) is the number of denominator parameters \( \beta_1, \ldots, \beta_m \) defined by

\[ \ell F_m(\alpha_1, \ldots, \alpha_\ell; \beta_1, \ldots, \beta_m; x) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\ell} (\alpha_j)_n}{\prod_{j=1}^{m} (\beta_j)_n} x^n \]
\[ 3 \]
\( (\ell, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \ \ell \leq m + 1; \ \ell \leq m \text{ and } |x| < \infty; \)
\[ \ell = m + 1 \text{ and } |x| < 1; \ \ell = m + 1, |x| = 1 \text{ and } \Re(\omega) > 0), \]
where
\[ \omega := \sum_{j=1}^{m} \beta_j - \sum_{j=1}^{\ell} \alpha_j \]
\[ (\alpha_j \in \mathbb{C} \ (j = 1, \cdots, \ell); \ \beta_j \in \mathbb{C} \setminus \mathbb{Z}^- \ (j = 1, \cdots, m)). \]

The operators \( V_{n,c}(f, x) \) are known to preserve the constant functions only except for the case \( c = 0 \). We observe that in (1) the suffix \( n \) in the basis function \( p_{n,k}(x, c) \) has a difference of \( c \) under summation and integration. But, if we have a difference of \( 2c \) in place of \( c \) under summation and integration, we may get the modified operators which preserve the constant functions as well as linear functions.

### 2. Modified Operators Preserving Linear Functions

Here, in this section, we introduce a modification of the operators \( V_{n,c}(f, x) \) which do preserve linear functions as well. For \( m \) an integer and \( c \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), we define
\[ G_{n,c}(f, x) = [n + (m + 1)c] \sum_{k=1}^{\infty} p_{n+mc,k}(x, c) \]
\[ \cdot \int_{0}^{\infty} p_{n+(m+2)c,k-1}(t, c)f(t)dt + p_{n+mc,0}(x, c)f(0), \]  
(4)
where \( p_{n,k}(x, c) \) is defined by (1). The above-mentioned two cases lead to the familiar Phillips operators and the genuine Baskakov-Durrmeyer type operators for \( c = 0 \) and \( c \in \mathbb{N} \), respectively. In the special case when \( m = 0 \) and \( c = 1 \), the operators \( G_{n,c}(f, x) \) were considered by Finta [4], who also estimated some converse results. Moreover, for \( c = -1 \), the operators \( G_{n,c}(f, x) \) take the following form:
\[ G_{n,-1}(f, x) = (n - m - 1) \sum_{k=1}^{n-m-1} p_{n-m,k}(x, -1) \]
\[ \cdot \int_{0}^{1} p_{n-m-2,k-1}(t, -1)f(t)dt \]
\[ + p_{n-m,0}(x, -1)f(0) + p_{n-m,n-m}(x, -1)f(1), \]  
(5)

The moments of the operators \( G_{n,c}(f, x) \) of order \( r \ (r \in \mathbb{N}) \) are given, in terms of the Gauss hypergeometric function \( {}_2F_1 \), as follows:
Finally, by applying this last result (6), it can easily be verified that the operators $G_{n,c}(f, x)$ preserve not only the constant functions, but also linear functions.

3. Concluding Remarks and Observations

The present investigation was motivated essentially by the fact that the widely-studied Srivastava-Gupta operator $V_{n,c}(f, x)$ preserves only the constant functions, but not linear functions. Here, in this paper, we have successfully provided a general sequence $G_{n,c}(f, x)$ of positive linear operators which preserve not only the constant functions, but also linear functions. We have also considered several recent developments on the subject of positive linear operators.

References


REFERENCES


