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On Non-trivially Associated Tensor Categories

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Abstract. The purpose of this article is to provide mathematical formulas for some operations on the objects of a non-trivially associated tensor category constructed from a factorization of a group into a subgroup and a set of left coset representatives. A detailed example is provided.

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1. Introduction

In [4], Beggs form a set M of left coset representatives for the left action of a subgroup G of a group X on the group X. Moreover, he defined an operation on M which has a left identity and satisfies the right division property. This binary operation is not associative. However, associativity can be obtained by a "cocycle" $\tau : M \times M \longrightarrow G$. By using this cocycle, one can construct a non-trivial associator for a category $\mathcal{C}$ whose objects are the M-graded right representations of G. Every object in this category has a dual. Consequently, it is possible to define an evaluation and a coevaluation maps to make the category into a rigid tensor category. If we assume that the binary operation on M satisfies the left division property, then the grading and group action can be combined into the action of an algebra A on the objects in the category. It turns out that A itself is in $\mathcal{C}$, and that the multiplication is associative.

It is well known that for every factorization $X = GM$ of a group into two subgroups G and M, a Hopf algebra $H = KM \triangleright\!\blacktriangleleft K(G)$ can be constructed, where K is a field, KM is the group Hopf algebra of M and $K(G)$ is the Hopf algebra functions on G. In the symbol $KM \triangleright\!\blacktriangleleft K(G)$, the $\triangleright$ part means that KM acts on $K(G)$, and the $\blacktriangleleft$ part means that $K(G)$ coacts on KM, [3]. Moreover, if A is an algebra (resp. a coalgebra) in a rigid

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tensor category, then its dual A^* is a coalgebra (resp. an algebra) in the same category. In [1], Al-shomrani reproved this result by using specific definitions in terms of diagrams that had been used in [2], [5], [7]and [8] .

In this article we obtain mathematical formulas for some operations on the objects of a non-trivially associated tensor category constructed from a factorization of a group into a subgroup and a set of left coset representatives. We consider the same non-trivially associated tensor category $\mathcal{C}$ as defined in [4].

Throughout this article, we use the same formulas and ideas from [4] which is itself based on [3], [5] and [6], but is mostly self-contained in terms of notation and definitions. In addition, we assume that all groups mentioned, unless otherwise stated, are finite, and that all vector spaces are finite dimensional over a field k, which will be denoted by $\underline{1}$ as an object in the category. Moreover we are going to restrict ourselves to the finite case of algebras, coalgebras and Hopf algebras although many results are still true in the infinite case (see [10]).

2. Preliminaries

In this section, we include some definitions and results that will be used later in this article.

Definition 2.1. *[11] A K-algebra is a triple (A, μ_A, η_A) consisting of a vector space A over a field K and K-linear maps $\mu_A : A \otimes A \longrightarrow A$ and $\eta_A : K \longrightarrow A$ such that the following diagrams commute:*

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\mu_A\otimes I_A"] \arrow[d, "I_A\otimes\mu_A"'] & A\otimes A \arrow[d, "\mu_A"] \\ A\otimes A \arrow[r, "\mu_A"] & A \end{tikzcd} \qquad \begin{tikzcd} K\otimes A \arrow[r, "\eta_A\otimes I_A"] \arrow[d, "\cong"'] & A\otimes A \arrow[d, "\mu_A"] & A\otimes K \arrow[l, "I_A\otimes\eta_A"'] \arrow[d, "\cong"] \\ A \arrow[r, "I_A"] & A & A \arrow[l, "I_A"'] \end{tikzcd}$$

Figure 1: Unit and the associative property on A.

Here the map $I_A : A \longrightarrow A$ is the identity map and the maps $I_A \otimes \mu_A : A \otimes A \otimes A \longrightarrow A \otimes A$ and $\mu_A \otimes I_A : A \otimes A \otimes A \longrightarrow A \otimes A$ are defined by $a \otimes b \otimes c \longmapsto a \otimes \mu_A(b \otimes c)$ and $a \otimes b \otimes c \longmapsto \mu_A(a \otimes b) \otimes c$, respectively, for all $a, b, c \in A$. The maps $I_A \otimes \eta_A, \eta_A \otimes I_A$ are defined by $a \otimes k \longmapsto a \otimes \eta_A(k), k \otimes a \longmapsto \eta_A(k) \otimes a$ for all $k \in K, a \in A$, respectively. These commuted diagrams can be represented in terms of equations as follows for all $k \in K$ and $a, b, c \in A$:

$$\mu_A(I_A \otimes \mu_A)(a \otimes b \otimes c) = \mu_A(\mu_A \otimes I_A)(a \otimes b \otimes c) \tag{1}$$

and

$$\mu_A(I_A \otimes \eta_A)(a \otimes k) = ka = \mu_A(\eta_A \otimes I_A)(k \otimes a) \tag{2}$$

The map μ_A is the multiplication map and η_A is the unit map. The associative property follows from (1) and the unit property follows from (2).

We say that the K-algebra A is commutative if $\mu_A \tau = \mu_A$, where τ is the twist map which is defined by $\tau(a \otimes b) = b \otimes a$ for $a, b \in A$.

Definition 2.2. *[11] A K-coalgebra is a triple $(C, \Delta_C, \epsilon_C)$ consisting of a vector space C over a field K and K-linear maps $\Delta_C : C \longrightarrow C \otimes C$ and $\epsilon_C : C \longrightarrow K$ such that the following diagrams commute:*

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta_C \otimes I_C"'] \\
C \otimes C \arrow[u, "I_C \otimes \Delta_C"] & C \arrow[l, "\Delta_C"] \arrow[u, "\Delta_C"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\epsilon_C \otimes I_C"'] \arrow[r, "I_C \otimes \epsilon_C"] & C \otimes k \\
C \arrow[u, "1 \otimes -"] & C \arrow[l, "I_C"'] \arrow[r, "I_C"] \arrow[u, "\Delta_C"'] & C \arrow[u, "- \otimes 1"']
\end{tikzcd}$$

Figure 2: Counit and the coassociative property on C.

Here the map $I_C : C \longrightarrow C$ is the the identity map on C. Also, the maps $I_C \otimes \Delta_C : C \otimes C \longrightarrow C \otimes C \otimes C$ and $\Delta_C \otimes I_C : C \otimes C \longrightarrow C \otimes C \otimes C$ are defined by $a \otimes b \longmapsto a \otimes \Delta_C(b)$ and $a \otimes b \longmapsto \Delta_C(a) \otimes b$, for all $a, b \in C$, respectively. In addition, the maps $- \otimes 1$ and $1 \otimes -$ are defined by $c \longmapsto c \otimes 1$ and $c \longmapsto 1 \otimes c$, respectively.

These commuted diagrams can be represented in terms of equations as follows for all $c \in C$:

$$(I_C \otimes \Delta_C)\Delta_C(c) = (\Delta_C \otimes I_C)\Delta_C(c) \tag{3}$$

and

$$(\epsilon_C \otimes I_C)\Delta_C(c) = 1 \otimes c, \quad (I_C \otimes \epsilon_C)\Delta_C(c) = c \otimes 1\,. \tag{4}$$

The maps Δ_C and ϵ_C are called the comultiplication and counit maps on the coalgebra C, respectively. The coassociative property is presented by equation (3) and the counit property is presented by equation (4).

A K-coalgebra C is cocommutative if $\tau(\Delta_C(c)) = \Delta_C(c)$, for all $c \in C$. We use the notation of Sweedler [9] to write $\Delta_C(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$. Since $(1 \otimes c) = c = (c \otimes 1)$, equation (4) implies that $\sum_{(c)} \epsilon_C(c_{(1)})c_{(2)} = c = \sum_{(c)} \epsilon_C(c_{(2)})c_{(1)}$. Moreover, we have

$$(I_C \otimes \Delta_C)\Delta_C(c) = (I_C \otimes \Delta_C)\Big(\sum_{(c)} c_{(1)} \otimes c_{(2)}\Big) = \sum_{(c)} c_{(1)} \otimes \Delta_C(c_{(2)}) = \sum_{(c),(c_{(2)})} c_{(1)} \otimes c_{(2)_{(1)}} \otimes c_{(2)_{(2)}}$$

and

$$(\Delta_C \otimes I_C)\Delta_C(c) = (\Delta_C \otimes I_C)\big(\sum_{(c)} c_{(1)} \otimes c_{(2)}\big) = \sum_{(c)} \Delta_C(c_{(1)}) \otimes c_{(2)} = \sum_{(c),(c_{(1)})} c_{(1)_{(1)}} \otimes c_{(1)_{(2)}} \otimes c_{(2)}.$$

But, $(I_C \otimes \Delta_C)\Delta_C = (\Delta_C \otimes I_C)\Delta_C$ by (3). So, the expressions in both of the above equations are equal. The common value in both is denoted by

$$\sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}.$$

In general we write

$$\Delta_{n-1}(c) = \sum_{(c)} c_{(1)} \otimes \ .\ .\ .\ . \otimes c_{(n)}\,.$$

$\Delta_{n-1}(c)$ is the element obtained by applying the coassociativity $(n-1)$ times.

Definition 2.3. *[11] A K-vector space H over a field K is a bialgebra if (H, μ_H, η_H) is an algebra, $(H, \Delta_H, \epsilon_H)$ is a coalgebra and either of the following equivalent conditions holds:*
1) Δ_H and ϵ_H are algebra maps.
2) μ_H and η_H are coalgebra maps.

Corollary 2.4. *[11] Let K be a filed and let V_i, $1 \le i \le n$, be a finite set of vector spaces over K. Then*

$$V_1^* \otimes V_2^* \otimes ... \otimes V_n^* \subseteq (V_1 \otimes V_2 \otimes ... \otimes V_n)^*.$$

Definition 2.5. *[4] For a group X and a subgroup G, we call $M \subset X$ a set of left coset representatives if for every $x \in X$ there is a unique $s \in M$ such that $x \in Gs$. The decomposition $x = us$ is called the unique factorization of x where $u \in G$ and $s \in M$.*

In what follows, $M \subset X$ is assumed to be a set of left coset representatives for the subgroup $G \subset X$. In addition, the identity in X will be denoted by e.

Definition 2.6. *[4] For $s, t \in M$ we define $\tau(s,t) \in G$ and $s \cdot t \in M$ by the unique factorization $st = \tau(s,t)(s \cdot t)$ in X. The functions $\triangleright : M \times G \to G$ and $\triangleleft : M \times G \to M$ are also defined by the unique factorization $su = (s \triangleright u)(s \triangleleft u)$ for $s, s \triangleleft u \in M$ and $u, s \triangleright u \in G$.*

It was shown in [4] that the binary operation $(M, \cdot)$ has a unique left identity $e_m \in M$ and also has the right division property (i.e. there is a unique solution $p \in M$ to the equation $p \cdot s = t$ for all $s, t \in M$). If $e \in M$ then $e_m = e$ is also a right identity [4].

The next proposition will be used at many places in this article:

Proposition 2.7. *[4] For $t, s, p \in M$ and $u, v \in G$, the following identities between $(M, \cdot)$ and τ are satisfied:*

$$s \triangleright (t \triangleright u) = \tau(s,t)\big((s \cdot t) \triangleright u\big)\tau\big(s \triangleleft (t \triangleright u), t \triangleleft u\big)^{-1} \quad \text{and} \quad (s \cdot t) \triangleleft u = \big(s \triangleleft (t \triangleright u)\big) \cdot (t \triangleleft u)\,,$$

$$s \triangleright uv = (s \triangleright u)\big((s \triangleleft u) \triangleright v\big) \quad \text{and} \quad s \triangleleft uv = (s \triangleleft) \triangleleft v\,,$$

$$\tau(p,s)\tau(p \cdot s, t) = \big(p \triangleright \tau(s,t)\big)\tau\big(p \triangleleft \tau(s,t), s \cdot t\big) \quad \text{and} \quad \big(p \triangleleft \tau(s,t)\big) \cdot (s \cdot t) = (p \cdot s) \cdot t\,.$$

In what follows, unless otherwise stated, we assume that $e \in M$ for the sake of simplicity. In [4], it was proved that for all $t \in M$ and $v \in G$, the following identities hold:

$$e \triangleleft v = e\,,\ e \triangleright v = v\,,\ t \triangleright e = e\,,\ t \triangleleft e = t\,.$$

Let $X = GM$ be a factorization of a finite group as defined before, the category $\mathcal{C}$ is defined as the following [4]: Take a category $\mathcal{C}$ of finite dimensional vector spaces over a field K, whose objects are right representations of the group G and have M-gradings. The action for the representation is written as $\bar{\triangleleft} : V \times G \to V$. In addition it is supposed that the action and the grading satisfy the compatibility condition, i.e. $\langle \xi \bar{\triangleleft} u\rangle = \langle \xi\rangle \triangleleft u$ where $\xi \in V_s$ corresponds to $\langle \xi\rangle = s$. The morphisms in the category $\mathcal{C}$ is defined to be linear maps that preserve both of grading and action, i.e. for a morphism $\vartheta : V \to W$ we have $\langle \vartheta(\xi)\rangle = \langle \xi\rangle$ and $\vartheta(\xi)\bar{\triangleleft} u = \vartheta(\xi \bar{\triangleleft} u)$ for all $\xi \in V$ and $u \in G$. $\mathcal{C}$ can be made into a tensor category by taking $V \otimes W$ to be the usual vector space tensor product, with actions and gradings given by

$$\langle \xi \otimes \eta\rangle = \langle \xi\rangle \cdot \langle \eta\rangle \quad \text{and} \quad (\xi \otimes \eta)\bar{\triangleleft} u = \xi \bar{\triangleleft}(\langle \eta\rangle \triangleright u) \otimes \eta \bar{\triangleleft} u\,.$$

There is an associator $\Phi_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ given by

$$\Phi((\xi \otimes \eta) \otimes \zeta) = \xi \bar{\triangleleft} \tau(\langle \eta\rangle, \langle \zeta\rangle) \otimes (\eta \otimes \zeta)\,.$$

Now, for the rigidity of $\mathcal{C}$, suppose that $(M, \cdot)$ has right inverses, i.e. for every $s \in M$ there is an $s^R \in M$ so that $s \cdot s^R = e$ and consider $V = \bigoplus_{s\in M} V_s$, where $\xi \in V_s$ corresponds to $\langle \xi\rangle = s$. Now take the dual vector space V^*, and set $V^*_{s^L} = \{\alpha \in V^* : \alpha|_{V_t} = 0 \quad \forall t \neq s\}$. Then $V^* = \bigoplus_{s\in M} V^*_{s^L}$, and we define $\langle \alpha\rangle = s^L$ when $\alpha \in V^*_{s^L}$, where s^L is the left inverse of s in M. The evaluation map $\mathrm{ev} : V^* \otimes V \to K$ is defined by $\mathrm{ev}(\alpha, \xi) = \alpha(\xi)$. Considering the action $\bar{\triangleleft} u$, if we apply evaluation to $\alpha \bar{\triangleleft}(\langle \xi\rangle \triangleright u) \otimes \xi \bar{\triangleleft} u$ we should get $\alpha(\xi)\bar{\triangleleft} u = \alpha(\xi)$. So we define $\big(\alpha \bar{\triangleleft}(\langle \xi\rangle \triangleright u)\big)(\xi \bar{\triangleleft} u) = \alpha(\xi)$, or if we put $\eta = \xi \bar{\triangleleft} u$ we get $\big(\alpha \bar{\triangleleft}\big((\langle \eta\rangle \triangleleft u^{-1}) \triangleright u\big)\big)(\eta) = \alpha(\eta \bar{\triangleleft} u^{-1}) = \big(\alpha \bar{\triangleleft}(\langle \eta\rangle \triangleright u^{-1})^{-1}\big)(\eta)$. If this is rearranged to give $\alpha \triangleleft v$, we get the following formula:

$$(\alpha \bar{\triangleleft} v)(\eta) = \alpha\big(\eta \bar{\triangleleft} \tau(\langle \eta\rangle^L, \langle \eta\rangle)^{-1}(\langle \eta\rangle^L \triangleright v^{-1})\tau(\langle \eta\rangle^L \triangleleft v^{-1}, (\langle \eta\rangle^L \triangleleft v^{-1})^R)\big). \tag{5}$$

For the coevaluation map to be defined, a basis $\{\xi\}$ of each V_s is taken and a corresponding dual basis $\{\hat{\xi}\}$ of each $V^*_{s^L}$, i.e. $\hat{\eta}(\xi) = \delta_{\xi,\eta}$. Then these bases are put together for all $s \in M$ to get the following definition, which is a morphism in $\mathcal{C}$ [4]:

$$\mathrm{coev}(1) = \sum_{\xi \in basis} \xi \bar{\triangleleft} \tau(\langle \xi\rangle^L, \langle \xi\rangle)^{-1} \otimes \hat{\xi}\,.$$

The algebra A in the tensor category $\mathcal{C}$ is constructed such that the group action and the grading in the definition of $\mathcal{C}$ can be combined. We consider a single object A in $\mathcal{C}$, a vector space spanned by a basis $\delta_s \otimes u$ for $s \in M$ and $u \in G$. For any object V in $\mathcal{C}$ define a map $\bar{\triangleleft} : V \otimes A \to V$ by $\xi \bar{\triangleleft}(\delta_s \otimes u) = \delta_{s,\langle \xi\rangle} \xi \bar{\triangleleft} u$. This map is a morphism in $\mathcal{C}$ only if $\langle \xi\rangle \cdot \langle \delta_s \otimes u\rangle = \langle \xi \bar{\triangleleft} u\rangle$, i.e. $s \cdot \langle \delta_s \otimes u\rangle = s \triangleleft u$, where $\langle \xi\rangle = s$. If we put $a = \langle \delta_s \otimes u\rangle$, the action of $v \in G$ is given by $(\delta_s \otimes u)\bar{\triangleleft} v = \delta_{s \triangleleft (a \triangleright v)} \otimes (a \triangleright v)^{-1} u v$.

In the remaining of this article, when an algebra A in $\mathcal{C}$ is mentioned, it is meant to refer to this construction.

Proposition 2.8. *[4] The formula of the multiplication μ_A for A in $\mathcal{C}$ is given by*

$$(\delta_s \otimes u)(\delta_t \otimes v) = \delta_{t,s\triangleleft u}\delta_{s\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1}uv,$$

where $a = \langle \delta_s \otimes u \rangle$ and $b = \langle \delta_t \otimes v \rangle$.

Proposition 2.9. *[4] Multiplication $\mu_A : A \otimes A \longrightarrow A$ is a morphism and associative in $\mathcal{C}$. Also there are an identity I for the multiplication and an algebra map $\epsilon_A : A \longrightarrow K$ in the category given by*

$$I_A = \sum_t \delta_t \otimes e, \quad \epsilon_A(\delta_s \otimes u) = \delta_{s,e}.$$

The identity I_A has the trivial action on the objects of $\mathcal{C}$. Also the action of $h \in A$ on the object K is just multiplication by $\epsilon_A(h)$, and $\epsilon_A(I) = 1$, the identity element in K.

Proposition 2.10. *[1] Define a basis $s \otimes \delta_u$ of A^* with evaluation map given by*

$$\mathrm{ev}\big((s \otimes \delta_u) \otimes (\delta_t \otimes v)\big) = \delta_{s,t}\,\delta_{u,v}\,,$$

for $s, t \in M$ and $u, v \in G$. Then the M-grade and the G-action on A^* are defined as follows: $\langle s \otimes \delta_u \rangle = \langle \delta_s \otimes u \rangle^L$, and for any $w \in G$

$$(s \otimes \delta_u)\bar{\triangleleft}(\langle s \otimes \delta_u \rangle^R \triangleright w) = s\triangleleft(\langle s \otimes \delta_u \rangle^R \triangleright w) \otimes \delta_{(\langle s\otimes\delta_u\rangle^R \triangleright w)^{-1}uw}\,.$$

Proposition 2.11. *[1] If A is an algebra in a rigid tensor category, then its dual A^* is a coalgebra in the category using the following definitions:*

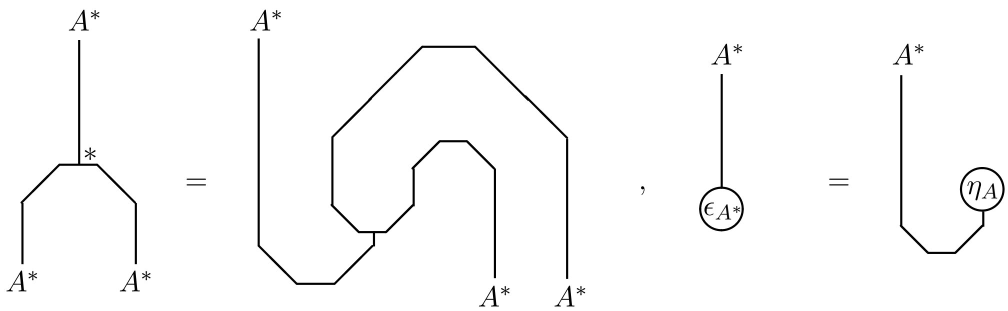


Figure 3: Comultiplication and counit on A^*.

Proposition 2.12. *[1] If C is a coalgebra in a rigid tensor category, then its dual C^* is an algebra in the category using the following definitions:*

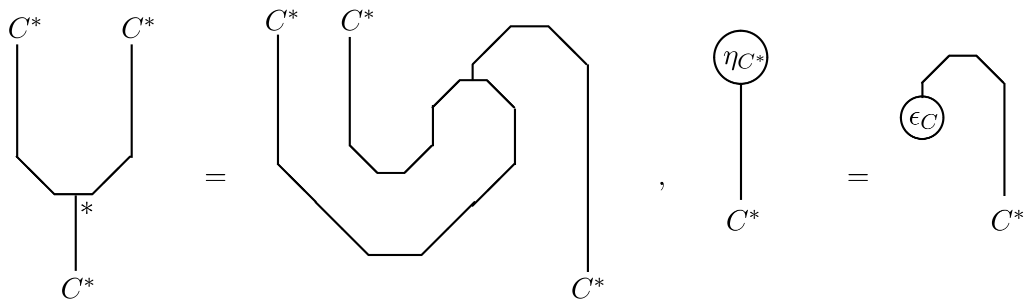


Figure 4: Multiplication and unit on C^*.

3. Results

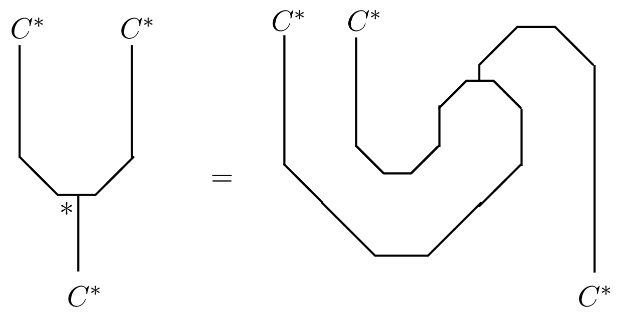
In this section, we consider an algebra A and a coalgebra C in the rigid tensor category $\mathcal{C}$ as defined before as well as their duals in the same category. We provide mathematical formulas for some operations on the dual objects of $\mathcal{C}$. Precisely, formulas for the multiplication μ_{C^*} on C^*, the counit ϵ_{A^*} on A^* and the unit η_{C^*} on C^* are obtained. Moreover, the unit property and the counit property for η_{C^*} and ϵ_{A^*}, respectively, are checked.

Proposition 3.1. *Let C be a coalgebra in the category $\mathcal{C}$. Then the multiplication μ_{C^*} on C^* for any elements $\alpha' = (t_1 \otimes \delta_{v_1})$ and $\alpha = (t_2 \otimes \delta_{v_2})$ in C^* for $v_1, v_2 \in G$ and $t_1, t_2 \in M$, can be given by*

$$\mu_{C^*}(\alpha \otimes \alpha') = \delta_{t_1 \triangleleft v_1, t_2} \ \big(t_1 \triangleleft \tau(a_1, a_2) \ \otimes \ \delta_{\tau(a_1,a_2)^{-1} v_1 v_2}\big),$$

with $\tau(a_2, a^L) = e$ where $a_1 = \langle \delta_{t_1} \otimes v_1 \rangle$, $a_2 = \langle \delta_{t_2} \otimes v_2 \rangle$ and $a = a_1 \cdot a_2$.

Proof. From Proposition 2.12, we know that



For $\alpha, \alpha' \in C^*$, we follow the above figure from top to bottom and calculate the following: Put $\mathrm{coev}(1) = \beta \otimes \gamma$ for some $\beta \in C$ and $\gamma \in C^*$ with $\triangle_C(\beta) = \beta_1 \otimes \beta_2$, $\alpha \otimes \alpha' = \gamma$, $\mathrm{ev}(\alpha \otimes \beta_2) = 1$ and $\mathrm{ev}\left(\alpha' \otimes \beta_1\right) = 1$ that imply $\langle \beta \rangle \cdot \langle \gamma \rangle = e$, $\langle \alpha \rangle \cdot \langle \beta_2 \rangle = e$, $\left\langle \alpha' \right\rangle \cdot \langle \beta_1 \rangle = e$, $\langle \beta \rangle = \langle \beta_1 \rangle \cdot \langle \beta_2 \rangle$ and $\langle \alpha \rangle . \langle \alpha' \rangle = \langle \gamma \rangle$.

We start with

$$(\alpha \otimes \alpha') \otimes \mathrm{coev}(1) = (\alpha \otimes \alpha') \otimes (\beta \otimes \gamma). \tag{6}$$

Applying the associator Φ on the right hand side of (6) and then the comultiplication on β give

$$\begin{aligned} \alpha \bar{\triangleleft} \tau(\langle \alpha' \rangle, \langle \beta \rangle \cdot \langle \gamma \rangle) \otimes \big(\alpha' \otimes (\beta \otimes \gamma)\big) &= \alpha \otimes \big(\alpha' \otimes (\beta \otimes \gamma)\big) \\ &= \alpha \otimes \big(\alpha' \otimes \big((\beta_1 \otimes \beta_2) \otimes \gamma\big)\big), \end{aligned} \tag{7}$$

since $\alpha \bar{\triangleleft} \tau((\langle \alpha' \rangle, \langle \beta \rangle \cdot \langle \gamma \rangle) = \alpha \bar{\triangleleft} \tau(\langle \alpha' \rangle, e) = \alpha \bar{\triangleleft} e = \alpha$ and $\triangle_C(\beta) = \beta_1 \otimes \beta_2$.

Now, Applying the associator Φ and then the associator inverse Φ^{-1} on the right hand side of (7) give

$$\alpha \otimes \big(\alpha' \otimes \big(\beta_1 \bar{\triangleleft} \tau(\langle \beta_2 \rangle, \langle \gamma \rangle) \otimes (\beta_2 \otimes \gamma)\big),$$

$$\alpha \otimes \big((\alpha^{'} \bar{\triangleleft} \tau(\langle \beta^{'} \rangle, \langle \beta_2 \rangle . \langle \gamma \rangle)^{-1} \otimes \beta^{'} \big) \otimes (\beta_2 \otimes \gamma) \big), \tag{8}$$

where

$$\beta^{'} = \beta_1 \bar{\triangleleft} \tau(\langle \beta_2 \rangle, \langle \gamma \rangle).$$

Next, we apply the evaluation map on $\big((\alpha^{'} \bar{\triangleleft} \tau(\langle \beta^{'} \rangle, \langle \beta_2 \rangle . \langle \gamma \rangle)^{-1} \otimes \beta^{'} \big)$ of (8) to get

$$\alpha^{'} \bar{\triangleleft} \tau(\langle \beta^{'} \rangle, \langle \beta_2 \rangle \cdot \langle \gamma \rangle)^{-1} (\beta^{'}) \tag{9}$$

$$= \alpha^{'} (\beta^{'} \bar{\triangleleft} \tau(\langle \beta^{'} \rangle^{l}, \langle \beta^{'} \rangle)^{-1} (\langle \beta^{'} \rangle^{l} \triangleright \tau(\langle \beta^{'} \rangle, \langle \beta_2 \rangle \cdot \langle \gamma \rangle)) \tau(\langle \beta^{'} \rangle^{L} \triangleleft \tau(\langle \beta^{'} \rangle, \langle \beta_2 \rangle \cdot \langle \gamma \rangle), (\langle \beta^{'} \rangle^{L} \triangleleft \tau(\langle \beta^{'} \rangle, \langle \beta_2 \rangle \cdot \langle \gamma \rangle))^{R})).$$

To make this equation simpler we need to do following calculations:
We first show that $\langle \beta^{'} \rangle = (\langle \beta_2 \rangle \cdot \langle \gamma \rangle)^{L}$ as follows:

$$\langle \beta^{'} \rangle \cdot (\langle \beta_2 \rangle \cdot \langle \gamma \rangle) = \big(\langle \beta_1 \rangle \triangleleft \tau(\langle \beta_2 \rangle, \langle \gamma \rangle) \big) \cdot (\langle \beta_2 \rangle \cdot \langle \gamma \rangle) = (\langle \beta_1 \rangle \cdot \langle \beta_2 \rangle) \cdot \langle \gamma \rangle = \langle \beta \rangle \cdot \langle \gamma \rangle = e.$$

But we know $\langle \alpha \rangle \cdot \langle \alpha^{'} \rangle = \langle \gamma \rangle$ and $\langle \alpha \rangle \cdot \langle \beta_2 \rangle = e$, that imply $\langle \beta_2 \rangle \cdot \langle \gamma \rangle = \langle \alpha^{'} \rangle$. Hence,

$$\langle \beta^{'} \rangle = (\langle \beta_2 \rangle . \langle \gamma \rangle)^{L} = \langle \alpha' \rangle^{L}.$$

Substituting in (9) gives

$$\alpha^{'} \bar{\triangleleft} \tau(\langle \alpha^{'} \rangle^{L}, \langle \alpha^{'} \rangle)^{-1} (\beta^{'}) = \tag{10}$$

$$\alpha^{'} (\beta^{'} \bar{\triangleleft} \tau(\langle \beta^{'} \rangle^{l}, \langle \beta^{'} \rangle)^{-1} (\langle \beta^{'} \rangle^{l} \triangleright \tau(\langle \alpha^{'} \rangle^{l}, \langle \alpha^{'} \rangle)) \tau(\langle \beta^{'} \rangle^{l} \triangleleft \tau(\langle \alpha^{'} \rangle^{l}, \langle \alpha^{'} \rangle), (\langle \beta^{'} \rangle^{l} \triangleleft \tau(\langle \alpha^{'} \rangle^{l}, \langle \alpha^{'} \rangle))^{R})).$$

Next, we need to do the following calculations:

$$\big(\langle \alpha^{'} \rangle^{LL} \triangleleft \tau(\langle \alpha^{'} \rangle^{L}, \langle \alpha^{'} \rangle) \big) \cdot (\langle \alpha^{'} \rangle^{L} \cdot \langle \alpha^{'} \rangle) = (\langle \alpha^{'} \rangle^{LL} \cdot \langle \alpha^{'} \rangle^{L}) \cdot \langle \alpha^{'} \rangle,$$

which implies that

$$\langle \alpha^{'} \rangle^{LL} \triangleleft \tau(\langle \alpha^{'} \rangle^{L}, \langle \alpha^{'} \rangle) = \langle \alpha^{'} \rangle.$$

Thus, we can consider the following

$$\langle \alpha^{'} \rangle^{LL} \langle \alpha^{'} \rangle^{L} \langle \alpha^{'} \rangle = \langle \alpha^{'} \rangle^{LL} \tau(\langle \alpha^{'} \rangle^{L}, \langle \alpha^{'} \rangle) = \big(\langle \alpha^{'} \rangle)^{LL} \triangleright \tau(\langle \alpha^{'} \rangle^{L}, \langle \alpha^{'} \rangle) \big) \langle \alpha^{'} \rangle,$$

which implies that

$$\langle \alpha^{'} \rangle^{LL} \langle \alpha^{'} \rangle^{L} = \tau(\langle \alpha^{'} \rangle^{LL}, \langle \alpha^{'} \rangle^{L}) = \langle \alpha^{'} \rangle^{LL} \triangleright \tau(\langle^{'} \alpha \rangle^{L}, \langle \alpha^{'} \rangle).$$

Now, substituting in equation (10) gives

$$\alpha^{'} \bar{\triangleleft} \tau(\langle \alpha^{'} \rangle^{L}, \langle \alpha^{'} \rangle)^{-1} (\beta^{'}) = \alpha^{'} (\beta^{'} \bar{\triangleleft} \tau(\langle \alpha^{'} \rangle, \langle \alpha^{'} \rangle^{R})).$$

After applying the evaluation map and since $\tau(\langle \alpha' \rangle, \langle \alpha' \rangle^{R}) = e$, (8) becomes

$$\alpha \otimes \big((\alpha^{'} (\beta^{'} \bar{\triangleleft} \tau(\langle \alpha^{'} \rangle, \langle \alpha^{'} \rangle^{R})) \otimes (\beta_2 \otimes \gamma) \big) = \alpha \otimes \big(\alpha^{'} (\beta^{'}) \otimes (\beta_2 \otimes \gamma) \big). \tag{11}$$

We now apply the associator inverse Φ^{-1} on (11) to get

$$\alpha \otimes \big(\alpha'(\beta^{'})\bar{\triangleleft}\tau(\langle\beta_2\rangle, \langle\gamma\rangle)^{-1} \otimes \beta_2\big) \otimes \gamma.$$

Applying the associator inverse again gives

$$\big(\alpha\bar{\triangleleft}\tau(\langle\beta''\rangle, \langle\gamma\rangle)^{-1} \otimes \beta''\big) \otimes \gamma, \tag{12}$$

where

$$\beta'' = \alpha^{'}(\beta^{'})\bar{\triangleleft}\tau(\langle\beta_2\rangle, \langle\gamma\rangle)^{-1} \otimes \beta_2 = \alpha'\big(\beta_1 \triangleleft \tau(\langle\beta_2\rangle, \langle\gamma\rangle)\big)\bar{\triangleleft}\tau(\langle\beta_2\rangle, \langle\gamma\rangle)^{-1} \otimes \beta_2 = \alpha'(\beta_1) \otimes \beta_2,$$

which implies that

$$\langle\beta''\rangle = (\langle\alpha'\rangle \cdot \langle\beta_1\rangle) \cdot \langle\beta_2\rangle = e \cdot \langle\beta_2\rangle = \langle\beta_2\rangle.$$

Now, we apply the evaluation map on (12) to get

$$\begin{aligned} &\big(\big(\alpha\bar{\triangleleft}\tau(\langle\beta_2\rangle, \langle\gamma\rangle)^{-1}\big)(\beta'')\big)(\gamma) \\ &= \alpha\big(\beta''\bar{\triangleleft}\tau(\langle\beta_2\rangle^L, \langle\beta_2\rangle)^{-1}\big(\langle\beta_2\rangle^L \triangleright \tau(\langle\beta_2\rangle, \langle\gamma\rangle)\big)\tau\big(\langle\beta_2\rangle^L \triangleleft \tau(\langle\beta_2\rangle, \langle\gamma\rangle), (\langle\beta_2\rangle^L \triangleleft \tau(\langle\beta_2\rangle, \langle\gamma\rangle))^R\big)\big)(\gamma) \\ &= \alpha\big(\beta''\bar{\triangleleft}(\langle\beta_2\rangle^L \triangleright \tau(\langle\beta_2\rangle, \langle\gamma\rangle))\big)(\gamma). \end{aligned} \tag{13}$$

Considering the equality of the diagram, we should have

$$\mu_{C^*}(\alpha \otimes \alpha') = \alpha\big(\beta''\bar{\triangleleft}\big(\langle\beta_2\rangle^L \triangleright \tau(\langle\beta_2\rangle, \langle\gamma\rangle)\big)\big)(\gamma), \tag{14}$$

where $\beta'' = \alpha'(\beta_1) \otimes \beta_2$.

But, from the definition of the coevaluation map, we know that

$$\mathrm{coev}(1) = \sum_{\xi \in \text{ basis of } V} \xi \,\bar{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1} \otimes \hat{\xi}, .$$

So we can put

$$\beta = \xi \,\bar{\triangleleft}\, \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1} \qquad \text{and} \qquad \gamma = \hat{\xi},$$

that imply that

$$\langle\beta\rangle = \langle\xi\rangle \triangleleft \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1} \qquad \text{and} \qquad \langle\gamma\rangle = \langle\hat{\xi}\rangle = \langle\xi\rangle^L.$$

Thus, if we apply the coproduct on β, we get

$$\Delta_C(\beta) = \Delta_C(\xi \triangleleft \tau(\,\langle\xi\rangle^L, \langle\xi\rangle\,)^{-1}) = \xi_1\bar{\triangleleft}\tau(\langle\xi_1\rangle^L, \langle\xi_1\rangle)^{-1} \otimes \xi_2\bar{\triangleleft}\tau(\xi_2^L, \xi_2)^{-1}.$$

Consequently, we can write

$$\beta_1 = \xi_1 \triangleleft \tau(\,\langle\xi_1\rangle^L, \langle\xi_1\rangle\,)^{-1} \qquad \text{and} \qquad \beta_2 = \xi_2\bar{\triangleleft}\tau(\langle\xi_2\rangle^L, \langle\xi_2\rangle)^{-1},$$

with

$$\langle\beta_1\rangle = \langle\xi_1\rangle \triangleleft \tau(\langle\xi_1\rangle^L, \langle\xi_1\rangle)^{-1} \qquad \text{and} \qquad \langle\beta_2\rangle = \langle\xi_2\rangle \triangleleft \tau(\langle\xi_2\rangle^L, \langle\xi_2\rangle)^{-1}.$$

Now, let $\xi = \delta_t \otimes v$, $\gamma = t \otimes \delta_v$, $\xi_1 = \delta_{t_1} \otimes v_1$ and $\xi_2 = \delta_{t_2} \otimes v_2$, with $a = \langle\xi\rangle = \langle\delta_t \otimes v\rangle$, $a^L = \langle\gamma\rangle = \langle t \otimes \delta_v\rangle$, $a_1 = \langle\xi_1\rangle = \langle\delta_{t_1} \otimes v_1\rangle$ and $a_2 = \langle\xi_2\rangle = \langle\delta_{t_2} \otimes v_2\rangle$.

As $\tau(\langle\xi_1\rangle^L,\langle\xi_1\rangle)^{-1}=e^{-1}=e$ and $\tau(\langle\xi_2\rangle^L,\langle\xi_2\rangle)^{-1}=e^{-1}=e$, it follows that $\beta_1=\xi_1\bar{\triangleleft}e=\xi_1$ and $\beta_2=\xi_2\bar{\triangleleft}e=\xi_2$, which means that

$$\beta''=\alpha'(\delta_{t_1}\otimes v_1)\otimes(\delta_{t_2}\otimes v_2).$$

If we put $\alpha'=t_1\otimes\delta_{v_1}$ in the right hand side of the above equation, then it can be rewritten as

$$\beta''=\mathrm{ev}\big((t_1\otimes\delta_{v_1})\otimes(\delta_{t_1}\otimes v_1)\big)\otimes(\delta_{t_2}\otimes v_2)=\delta_{t_1,t_1}\delta_{v_1,v_1}(\delta_{t_2}\otimes v_2)=(\delta_{t_2}\otimes v_2).$$

Also, if we put $q=\langle\beta_2\rangle^L\triangleright\tau(\langle\beta_2\rangle,\langle\gamma\rangle)=a_2^L\triangleright\tau(a_2,a^L)$, then

$$\beta''\bar{\triangleleft}q=(\delta_{t_2}\otimes v_2)\bar{\triangleleft}q=(\delta_{t_2\triangleleft(a_2\triangleright q)}\otimes(a_2\triangleright q)^{-1}v_2q).$$

Now we substitute these simplified parts in equation (14) to get

$$\mu_{C*}(\alpha\otimes\alpha')=\alpha\big((\delta_{t_2\triangleleft(a_2\triangleright q)}\otimes(a_2\triangleright q)^{-1}v_2q)\big)(\gamma).$$

If we put $\alpha=t_2\otimes\delta_{v_2}$, the above equation becomes

$$\begin{aligned}\mu_{C*}(\alpha\otimes\alpha')=&\ \mathrm{ev}\big((t_2\otimes\delta_{v_2})\otimes(\delta_{t_2\triangleleft(a_2\triangleright q)}\otimes(a_2\triangleright q)^{-1}v_2q)\big)(\gamma)\\ &=\delta_{t_2,t_2\triangleleft(a_2\triangleright q)}\delta_{v_2,(a_2\triangleright q)^{-1}v_2q)}(\gamma)\end{aligned}\tag{15}$$

which implies that

$$t_2=t_2\triangleleft(a_2\triangleright q)=t_2\triangleleft\big(a_2\triangleright\big(a_2^L\triangleright\tau(a_2,a^L)\big)\big),$$

$$v_2=(a_2\triangleright q)^{-1}v_2q=\big(a_2\triangleright(a_2^L\triangleright\tau(a_2,a^L)\big)\big)^{-1}v_2\big(a_2^L\triangleright\tau(a_2,a^L)\big).$$

To have these equations satisfied we should have $\tau(a_2,a^L)=e$. Hence, $a_2^L\triangleright\tau(a_2,a^L)=e$ and $a_2\triangleright(a_2^L\triangleright\tau(a_2,a^L))=e$.

On the other hand, we know that

$$\delta_t\otimes v=(\delta_{t_1}\otimes v_1)\otimes(\delta_{t_2}\otimes v_2)=\delta_{t_2,t_1\triangleleft v_1}\delta_{t_1\triangleleft\tau(a_1,a_2)}\otimes\tau(a_1,a_2)^{-1}v_1v_2$$

Thus,

$$v=\tau(a_1,a_2)^{-1}v_1v_2\quad\text{and}\quad t=t_1\triangleleft\tau(a_1,a_2).$$

Therefore,

$$\mu_{C^*}(\alpha\otimes\alpha')=\delta_{t_1\triangleleft v_1,t_2}\ \big(t_1\triangleleft\tau(a_1,a_2)\ \otimes\ \delta_{\tau(a_1,a_2)^{-1}v_1v_2}\big).$$

To confirm our calculation we show $t\triangleleft v=t\cdot a$ knowing that $t_1\triangleleft v_1=t_1\cdot a_1$, $t_2\triangleleft v_2=t_2\cdot a_2$, $t_1\triangleleft v_1=t_2$, and $a=a_1\cdot a_2$. We start with the right hand side as follows:

$$t\cdot a=t_1\triangleleft\tau(a_1,a_2)\cdot(a_1\cdot a_2)=(t_1\cdot a_1)\cdot a_2=(t_1\triangleleft v_1)\cdot a_2=t_2\cdot a_2=t_2\triangleleft v_2.$$

On the other hand,

$$t\triangleleft v=t_1\triangleleft\tau(a_1,a_2)\triangleleft\tau(a_1,a_2)^{-1}\triangleleft v_1v_2=t_1\triangleleft v_1v_2=t_1\triangleleft v_1\triangleleft v_2=t_2\triangleleft v_2.\quad\blacksquare$$

Proposition 3.2. *Let A be an algebra in the category $\mathcal{C}$. Then the counit ϵ_{A^*} on A^* for any element $\alpha = (s \otimes \delta_u) \in A^*$ is given by*

$$\epsilon_{A^*}(s \otimes \delta_u) = \delta_{u,e},$$

for $u \in G$ and $s \in M$.

Proof. From Proposition (2.11), we know that

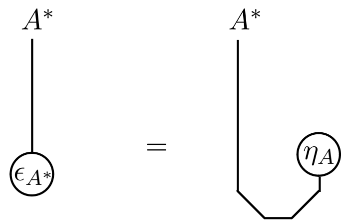


Figure 5: Definition of counit on A^*.

We follow figure 5 from top to bottom and start with the following for $\alpha \in A^*$ and $k \in K$:

$$\alpha = \alpha \otimes k. \tag{16}$$

Knowing that $\eta_A : K \longrightarrow A$, by definition (2.1), we apply the map $(I_{A^*} \otimes \eta_A)$ on equation (16) to get

$$(I_{A^*} \otimes \eta_A)(\alpha \otimes k) = I_{A^*}(\alpha) \otimes \eta_A(k) = \alpha \otimes \beta, \tag{17}$$

where $\beta = (\delta_s \otimes e) \in A$.

Now, we put $\alpha = (s \otimes \delta_u)$ and apply the evaluation map on the right hand side of equation (17) to have

$$\mathrm{ev}(\alpha \otimes \beta) = \mathrm{ev}((s \otimes \delta_u) \otimes (\delta_s \otimes e)) = \delta_{u,e}\delta_{s,s} = \delta_{u,e}.$$

Finally, considering the left hand side of the equality in figure 5 gives

$$\epsilon_{A^*}(s \otimes \delta_u) = \delta_{u,e}. \quad \blacksquare$$

Proposition 3.3. *Let C be a coalgebra in the category $\mathcal{C}$. Then the unit η_{C^*} on C^* can be given by*

$$\eta_{C^*}(1_K) = \sum_{v \in G} e \otimes \delta_v,$$

where 1_K is the unity of K.

Proof. From Proposition (2.12), we know that

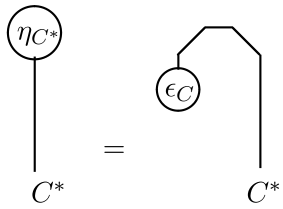


Figure 6: Definition of unit on C^*.

We follow figure 6 from top to bottom and start by considering the following:

$$\operatorname{coev}(1) = \beta \otimes \gamma, \tag{18}$$

for $\beta \in C$ and $\gamma \in C^*$, which implies $\langle \beta \rangle \cdot \langle \gamma \rangle = e$.

But, from the definition of the coevaluation map, we know

$$\operatorname{coev}(1) = \sum_{\xi \in \text{ basis of } V} \xi \,\bar{\triangleleft}\, \tau(\,\langle \xi \rangle^L, \langle \xi \rangle\,)^{-1} \otimes \hat{\xi}.$$

We let $\beta = \xi \,\bar{\triangleleft}\, \tau(\,\langle \xi \rangle^L, \langle \xi \rangle\,)^{-1}$, $\gamma = \hat{\xi}$ and $w = \tau(\langle \xi \rangle^L, \langle \xi \rangle)^{-1}$. If $\xi = \delta_t \otimes v$, $\gamma = t \otimes \delta_v$, then $a = \langle \xi \rangle = \langle \delta_t \otimes v \rangle$, and $a^L = \langle \gamma \rangle = \langle t \otimes \delta_v \rangle$. Hence,

$$\beta = (\delta_t \otimes v) \bar{\triangleleft} w = \delta_{t \triangleleft (a \triangleright w)} \otimes (a \triangleright w)^{-1} v w.$$

Now, applying the map $(\epsilon_C \otimes I_{C^*})$ on equation (18) gives

$$\begin{aligned} \epsilon_C(\beta) \otimes I_{C^*}(\gamma) &= \sum_{v \in G} \delta_{t \triangleleft (a \triangleright w), e} \otimes (t \otimes \delta_v) \\ &= \sum_{v \in G} \delta_{t \triangleleft (a \triangleright w), e} \, (t \otimes \delta_v). \end{aligned} \tag{19}$$

To get a nonzero solution we should have $t \triangleleft (a \triangleright w) = e \Rightarrow t \triangleleft (a \triangleright w) \triangleleft (a \triangleright w)^{-1} = e \triangleleft (a \triangleright w)^{-1} \Rightarrow t = e$ which leads to $a = \langle \delta_t \otimes v \rangle = \langle \delta_e \otimes v \rangle = e$. Thus, equation (19) can be rewritten as

$$\epsilon_C(\beta) \otimes I_{C^*}(\gamma) = \sum_{v \in G} e \otimes \delta_v.$$

Finally, considering the left hand side of the equality in figure 6 gives

$$\eta_{C^*}(1_k) = \sum_{v \in G} e \otimes \delta_v. \quad \blacksquare$$

In the next propositions we will check the unit property and the counit property for η_{C^*} and ϵ_{A^*} respectively.

Proposition 3.4. *Let A be an algebra in the category $\mathcal{C}$. Then the counit property for the counit on A^* is satisfied, i.e.*

$$(\epsilon_{A^*} \otimes I_{A^*})\Delta_{A^*}(t \otimes \delta_v) = (I_{A^*} \otimes \epsilon_{A^*})\Delta_{A^*}(t \otimes \delta_v)$$

for any element $\gamma = (t \otimes \delta_v) \in A^*$ with $v \in G, t \in M$.

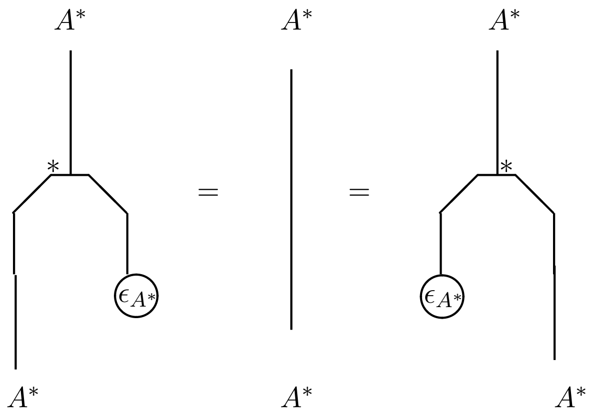


Figure 7: Counit property on A^*.

Proof. As A is an algebra in the category $\mathcal{C}$, it has a unit map $\eta_A : K \longrightarrow A$ satisfying

$$\mu_A(I_A \otimes \eta_A)(\beta \otimes k) = k\beta = \mu_A(\eta_A \otimes I_A)(k \otimes \beta).$$

We consider the dual map $\eta_A^* : A^* \longrightarrow K^* = K$ and let $\epsilon_{A^*} : A^* \longrightarrow K$ denote the restriction of η_A^* to A^*.

Now, for $\gamma = (t \otimes \delta_v) \in A^*$, $k \in K$, we have

$$\epsilon_{A^*}(\gamma)(k) = \gamma\big(\eta_A(k)\big) = \gamma\big(\eta_A(1_K)k\big) = \gamma(1_A)(k). \tag{20}$$

Hence, $\epsilon_{A^*}(\gamma) = \gamma(1_A)$.

Next, let $\mu_A^* : A^* \longrightarrow (A \otimes A)^*$ be the transpose of the multiplication map μ_A defined as

$$\mu_A^*(\gamma)(\beta_1 \otimes \beta_2) = \gamma(\mu_A(\beta_1 \otimes \beta_2)) = \gamma(\beta_1.\beta_2). \tag{21}$$

It is known that $\mu_A^*(A^*) \subseteq A^* \otimes A^*$ [11]. Let Δ_{A^*} denote the restriction of μ_A^* to A^*. Then $\Delta_{A^*} : A^* \longrightarrow A^* \otimes A^*$ is a K-linear map defined as

$$\Delta_{A^*}(\gamma) = \mu_A^*(\gamma), \quad \text{for } \gamma \in A^*. \tag{22}$$

Thus, for $\gamma = (t \otimes \delta_v) \in A^*$, $\beta = (\delta_s \otimes u) \in A$ and $k \in K$, we have

$$\begin{aligned}
(\epsilon_{A^*} \otimes I_{A^*})\Delta_{A^*}(\gamma)(k \otimes \beta) &= (\eta_A^* \otimes I_A^*)\mu_A^*(\gamma)(k \otimes \beta)\\
&= \mu_A^*(\gamma)(\eta_A \otimes I_A)(k \otimes \beta)\\
&= \gamma\big(\mu_A(\eta_A(k) \otimes \beta)\big)\\
&= \gamma\big(\mu_A\big((\delta_t \otimes e) \otimes (\delta_s \otimes u)\big)\big)\\
&= \gamma\big(\delta_{s,t\triangleleft e}\delta_{t\triangleleft\tau(a,b)} \otimes \tau(a,b)^{-1}eu\big)\\
&= \gamma\big(\delta_{s,t}\delta_{t\triangleleft\tau(e,b)} \otimes \tau(e,b)^{-1}eu\big)\\
&= \gamma(\delta_s \otimes u)\\
&= \gamma(\beta)\\
&= \gamma\big(\mu_A(\beta \otimes \eta_A(k))\big)\\
&= \mu_A^*(\gamma)(I_A \otimes \eta_A)(\beta \otimes k)\\
&= (I_A^* \otimes \eta_A^*)\mu_A^*(\gamma)(\beta \otimes k)\\
&= (I_{A^*} \otimes \epsilon_{A^*})\Delta_{A^*}(\gamma)(\beta \otimes k),
\end{aligned}$$

where $a = \langle \delta_t \otimes e\rangle = e$ and $b = \langle \delta_s \otimes u\rangle$. We have used the following calculations: $\tau(a,b) = \tau(e,b) = e$, $\tau(a,b)^{-1} = \tau(e,b)^{-1} = e^{-1} = e$, $t \triangleleft \tau(a,b) = t \triangleleft e = t$, $\tau(a,b)^{-1}eu = u$ and $t = s$. Therefore, ϵ_{A^*} satisfies the counit property as claimed. ■

Proposition 3.5. *let C be a coalgebra in category $\mathcal{C}$. Then the unit property on C^* is satisfied, i.e.*

$$\mu_{C^*}(I_{C^*} \otimes \eta_{C^*})\big(\gamma \otimes k\big) = \mu_{C^*}(\eta_{C^*} \otimes I_{C^*})\big(k \otimes \gamma\big)$$

for any element $\gamma = (t \otimes \delta_v) \in C^*$ with $v \in G, t \in M$, and $k \in K$.

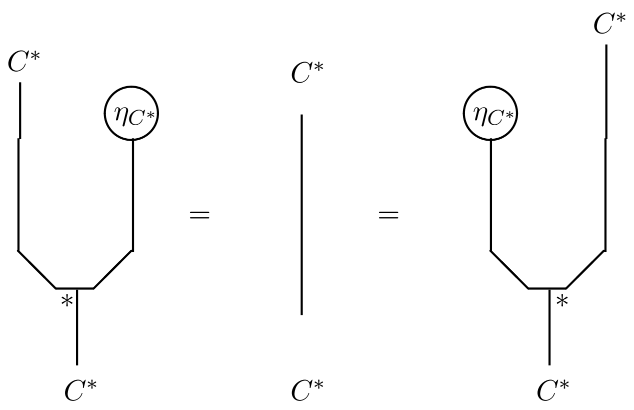


Figure 8: Unit property on C^*.

Proof. As C is a coalgebra in the category $\mathcal{C}$, it has a counit map $\epsilon_C : C \longrightarrow K$ satisfying the counit property, i.e

$$\sum_{(\beta)} \epsilon_C(\beta_{(1)})\beta_{(2)} = \beta = \sum_{(\beta)} \epsilon_C(\beta_{(2)})\beta_{(1)}.$$

The transpose of the counit map of C is $\epsilon_C^* : K^* \longrightarrow C^*$ which is defined by

$$\epsilon_C^*(\gamma)(\beta) = \gamma\big(\epsilon_C(\beta)\big),$$

for $\gamma \in K^*$, $\beta \in C$. If we identify K with K^*, we get $\epsilon_C^* : K \longrightarrow C^*$ defined as

$$\epsilon_C^*(k)(\beta) = k\big(\epsilon_C(\beta)\big) = k\epsilon_C(\beta), \tag{23}$$

for $k \in K$, $\beta \in A$.

Now, we use the same techniques as in the proof of the previous proposition to have $\eta_{C^*} = \epsilon_C^*$ and $\mu_{C^*} = \Delta_C^*$. We define the following maps: $I_{C^*} \otimes \eta_{C^*} : C^* \otimes K \longrightarrow C^* \otimes C^*$ by

$$\gamma \otimes k \longmapsto \gamma \otimes \eta_{C^*}(k),$$

and $\eta_{C^*} \otimes I_{C^*} : K \otimes C^* \longrightarrow C^* \otimes C^*$ by

$$k \otimes \gamma \longmapsto \eta_{C^*}(k) \otimes \gamma.$$

Next, it is known that the transpose of Δ_C is a K-linear map $\Delta_C^* : (C \otimes C)^* \longrightarrow C^*$, defined by

$$\Delta_C^*(\psi)(\beta) = \psi(\Delta_C(\beta)), \tag{24}$$

for $\psi \in (C \otimes C)^*$, $\beta \in C$. Also, by Corollary 2.4, we have $C^* \otimes C^* \subseteq (C \otimes C)^*$. Hence, Δ_C^* leads to a K-linear map $\mu_{C^*} : C^* \otimes C^* \longrightarrow C^*$, defined by

$$\mu_{C^*}(\gamma_1 \otimes \gamma_2)(\beta) = \Delta_C^*(\gamma_1 \otimes \gamma_2)(\beta) = (\gamma_1 \otimes \gamma_2)(\Delta_C(\beta)) = \sum_{(\beta)} \gamma_1(\beta_1) \otimes \gamma_2(\beta_2). \tag{25}$$

Thus, if we put $\gamma = (t \otimes \delta_v)$, $\beta = (\delta_s \otimes u)$, $\beta_1 = (\delta_{s_1} \otimes u_1)$ and $\beta_2 = (\delta_{s_2} \otimes u_2)$ with $\beta =$

$\beta_1 \otimes \beta_2$ and $\langle\beta\rangle = \langle\beta_1\rangle \cdot \langle\beta_2\rangle$, for $k \in K$, $\beta, \beta_1, \beta_2 \in C$ and $\gamma \in C^*$, we get

$$\begin{aligned}
\mu_{C^*}(I_{C^*} \otimes \eta_{C^*})(\gamma \otimes k)(\beta) &= \Delta_C^*\big(\gamma \otimes \epsilon_C^*(k)\big)(\beta) \\
&= \big(\gamma \otimes \epsilon_C^*(k)\big)\big(\Delta_C(\beta)\big) = \big(\gamma \otimes \epsilon_C^*(k)\big)(\beta_1 \otimes \beta_2) \\
&= \sum_{(\beta)} \gamma(\beta_1) \otimes \epsilon_C^*(k)(\beta_2) = \sum_{(\beta)} \gamma(\beta_1) \otimes k\big(\epsilon_C(\beta_2)\big) \\
&= \sum_{(\beta)} \gamma(\beta_1) k\big(\epsilon_C(\beta_2)\big) = k \sum_{(\beta)} \gamma(\beta_1)\epsilon_C(\beta_2) \\
&= k \sum_{(\beta)} \gamma(\delta_{s_1} \otimes u_1)\delta_{s_2,e} = k \sum_{(\beta)} \delta_{s_2,e}\gamma(\delta_{s_1} \otimes u_1) \\
&= k \sum_{(\beta)} \epsilon_C(\beta_2)\gamma(\beta_1) = k \sum_{(\beta)} \gamma\big(\epsilon_C(\beta_2)\beta_1\big) \\
&= k\gamma\Big(\sum_{(\beta)} \epsilon_C(\beta_2)\beta_1\Big) = k\gamma(\beta) = k\gamma\Big(\sum_{(\beta)} \beta_2\epsilon_C(\beta_1)\Big) \\
&= k \sum_{(\beta)} \gamma\big(\beta_2\epsilon_C(\beta_1)\big) = k \sum_{(\beta)} \gamma(\delta_{s_2} \otimes u_2)\delta_{s_1,e} \\
&= k \sum_{(\beta)} \delta_{s_1,e}\gamma(\delta_{s_2} \otimes u_2) = k \sum_{(\beta)} \epsilon_C(\beta_1)\gamma(\beta_2) \\
&= \sum_{(\beta)} k\big(\epsilon_C(\beta_1)\big)\gamma(\beta_2) = \sum_{(\beta)} \epsilon_C^*(k)(\beta_1)\gamma(\beta_2) \\
&= \sum_{(\beta)} k\big(\epsilon_C(\beta_1)\big) \otimes \gamma(\beta_2) = \sum_{(\beta)} \epsilon_C^*(k)(\beta_1) \otimes \gamma(\beta_2) \\
&= \sum_{(\beta)} (\epsilon_C^*(k) \otimes \gamma)(\beta_1 \otimes \beta_2) = \big(\epsilon_C^*(k) \otimes \gamma\big)\Delta_C(\beta) \\
&= \Delta_{C^*}\big(\epsilon_C^*(k) \otimes \gamma\big)(\beta) = \Delta_{C^*}\big(\epsilon_C^* \otimes I_{C^*}\big)(k \otimes \gamma)(\beta) \\
&= \mu_{C^*}\big(\eta_{C^*} \otimes I_{C^*}\big)(k \otimes \gamma)(\beta).
\end{aligned}$$

In the above calculations we have used equations (23), (24) and (25), the facts that $k\big(\epsilon_C(\beta_2)\big) \in K$ and $C \otimes K \cong C$, Proposition 2.9 and Definition 2.2. Therefore, η_{C^*} satisfies the unit property as required. ■

Example 1. *Let X be the dihedral group $D_6 = \big\langle x, y : x^6 = y^2 = 1, xy = yx^5 \big\rangle$ and let G be the non-normal subgroup $\{1, x^3, y, x^3y\}$. If we choose $M = \{1, x, x^5\}$ to be the set of left coset representatives, then the $\cdot$, τ, the action $\triangleright$ and the coaction $\triangleleft$, are given by the following tables:*

$\cdot$	1	x	x^5
1	1	x	x^5
x	x	x^5	1
x^5	x^5	1	x

τ	1	x	x^5
1	1	1	1
x	1	x^3	1
x^5	1	1	x^3

$s \triangleright u$	1	x^3	y	x^3y
1	1	x^3	y	x^3y
x	1	x^3	y	x^3y
x^5	1	x^3	y	x^3y

$s \triangleleft u$	1	x^3	y	x^3y
1	1	1	1	1
x	x	x	x^5	x^5
x^5	x^5	x^5	x	x

We take our field to be the binary field $F = \{0, 1\}$.

We check multiplication μ_{C^*} *in Proposition 3.1. For two elements* $\alpha' = (t_1 \otimes \delta_{v_1})$ *and* $\alpha = (t_2 \otimes \delta_{v_2})$ *in* C^* *with* $v_1, v_2 \in G$, $t_1, t_2 \in M$, *if we put* $t_1 = x$, $t_2 = x^5$ *in* M, $v_1 = y$, $v_2 = x^3$ *in* G, *then* $\alpha = \big(x^5 \otimes \delta_{x^3}\big)$, $\alpha' = \big(x \otimes \delta_y\big)$, $a_2 = \langle \delta_{t_2} \otimes v_2 \rangle$, $a_1 = \langle \delta_{t_1} \otimes v_1 \rangle$, *and* $a = a_1 \cdot a_2$.

We start by calculating the following:

$$t_2 \cdot a_2 = t_2 \triangleleft v_2,$$

$$x^5 \cdot a_2 = x^5 \triangleleft x^3 \Rightarrow x^5 \cdot a_2 = x^5 \Rightarrow a_2 = 1,$$

and

$$t_1 \cdot a_1 = t_1 \triangleleft v_1,$$

$$x \cdot a_1 = x \triangleleft y \Rightarrow x \cdot a_1 = x^5 \Rightarrow a_1 = x.$$

Also,

$$a = a_1 \cdot a_2 \Rightarrow a = x \cdot 1 = x \;\Rightarrow\; a^L = x^5.$$

The following calculations are needed as well:

$$t_1 \triangleleft \tau(a_1, a_2) = x \triangleleft \tau(x, 1) = x \triangleleft 1 = x,$$

$$\tau(a_1, a_2)^{-1}\, v_1\, v_2 \;= \tau(x, 1)^{-1}\, v_1\, v_2 \;= 1\, y\, x^3 \;= x^3 y,$$

and

$$t_1 \triangleleft v_1 = x \triangleleft y = x^5, \quad \text{and} \quad t_2 = x^5.$$

Now, we substitute in the formula of μ_{C^*} *as follows:*

$$\mu_{C^*}(\alpha \otimes \alpha') = \delta_{t_1 \triangleleft v_1, t_2} \;\big(t_1 \triangleleft \tau(a_1, a_2) \;\otimes\; \delta_{\tau(a_1, a_2)^{-1} v_1 v_2}\big),$$

$$\mu_{C^*}\big((x^5 \otimes \delta_{x^3}) \otimes (x \otimes \delta_y)\big) = \delta_{x^5,x^5}\big(x \otimes \delta_{x^3 y}\big) = x \otimes \delta_{x^3 y} \in C^*.$$

Next, we check the counit ϵ_{A^*} in Proposition 3.2, for any element $\alpha = (s \otimes \delta_u) \in A^*$ with $s \in M$ and $u \in G$ as follows: Choose $s = x$. If $u = e = 1$, then

$$\epsilon_{A^*}(x \otimes \delta_1) = \delta_{1,1} = 1 \in F.$$

If $u \neq e$, for example $u = y$, then

$$\epsilon_{A^*}(x \otimes \delta_y) = \delta_{y,1} = 0 \in F.$$

Finally, we check the unit η_{C^*} in Proposition 3.3, for $1 \in F$. If we let $t = 1 \in M$, $v = y \in G$, then

$$\eta_{C^*}(1) = 1 \otimes \delta_y.$$

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