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Anti-type of hesitant fuzzy sets on UP-algebras*

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Abstract. This paper aims to introduce the notions of anti-hesitant fuzzy UP-subalgebras of UP-algebras, anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals, and anti-hesitant fuzzy strongly UP-ideals, and prove some results. Furthermore, we discuss the relationships between anti-hesitant fuzzy UP-subalgebras (resp., anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals, anti-hesitant fuzzy strongly UP-ideals) and some level subsets of hesitant fuzzy sets on UP-algebras.

2010 Mathematics Subject Classifications: 03G25

Key Words and Phrases: UP-algebra, anti-hesitant fuzzy strongly UP-ideal, anti-hesitant fuzzy UP-ideal, anti-hesitant fuzzy UP-filter and anti-hesitant fuzzy UP-subalgebra

1. Introduction

The branch of the logical algebra, UP-algebras was introduced by Iampan [2] in 2017, and it is known that the class of KU-algebras [8] is a proper subclass of the class of UP-algebras. It have been examined by several researchers, for example, Somjanta et al. [14] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was introduced by Kesorn et al. [5], Kaijae et al. [4] introduced the notions of anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras, the notion of Q-fuzzy sets in UP-algebras was introduced by Tanamoon et al. [17], Sripaeng et al. [16] introduced the notion anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras, the notion of $\mathcal{N}$-fuzzy sets in UP-algebras was introduced by Songsaeng and Iampan [15], Senapati et al. [12, 13] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras, Romano [9] introduced the notion of proper UP-filters in UP-algebras, etc.

A hesitant fuzzy set on a set is a function from a reference set to a power set of the unit interval. The notion of a hesitant fuzzy set on a set was first considered by Torra [18] in 2010. The hesitant fuzzy set, which can be perfectly described in terms of the opinions

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of decision makers is a very useful tool to deal with uncertainty. The hesitant fuzzy set theories developed by Torra and others have found many applications in the domain of mathematics and elsewhere. In UP-algebras, Mosrijai et al. [6] extended the notion of fuzzy sets in UP-algebras to hesitant fuzzy sets on UP-algebras, and Satirad et al. [11] considered level subsets of a hesitant fuzzy set on UP-algebras in 2017. The notion of partial constant hesitant fuzzy sets on UP-algebras was introduced by Mosrijai et al. [7] afterwards.

In this paper, the notion of anti-hesitant fuzzy UP-subalgebras (resp., anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals and anti-hesitant fuzzy strongly UP-ideals) of UP-algebras are introduced and proved some results. Further, we discuss the relation between anti-hesitant fuzzy UP-subalgebras (resp., anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals and anti-hesitant fuzzy strongly UP-ideals) and level subsets of a hesitant fuzzy set.

2. Basic Results on UP-Algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1. *[2] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,*

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, *and*

(UP-4) $x \cdot y = 0$ *and* $y \cdot x = 0$ *imply* $x = y$.

From [2], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 1. *[10] Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A' \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the generalized power UP-algebra of type 1 with respect to Ω.*

Example 2. *[10] Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A' \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the generalized power UP-algebra of type 2 with respect to Ω.*

Example 3. *Let* $A = \{0, 1, 2, 3, 4\}$ *be a set with a binary operation* $\cdot$ *defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	3
2	0	0	0	3	3
3	0	0	0	0	3
4	0	0	0	0	0

Then $(A, \cdot, 0)$ *is a UP-algebra which is not a KU-algebra because* $(0 \cdot 2)((2 \cdot 4) \cdot (0 \cdot 4)) = 2 \cdot (3 \cdot 4) = 2 \cdot 3 = 3 \neq 0$ *(see the definition in [8]).*

In what follows, let A denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1. *[2, 3] In a UP-algebra* $A = (A, \cdot, 0)$*, the following properties hold:*

(1) $(\forall x \in A)(x \cdot x = 0)$,

(2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,

(3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,

(4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,

(5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,

(6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,

(7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,

(8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,

(9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,

(10) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0)$,

(11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,

(12) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0)$, *and*

(13) $(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0)$.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A [2] as follows: for any $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0.$$

Definition 2. *[1, 2, 14] A nonempty subset* S *of a UP-algebra* $(A, \cdot, 0)$ *is called*

(1) *a UP-subalgebra of* A *if for any* $x, y \in S, x \cdot y \in S$.

(2) *a UP-filter of A if*

(i) the constant 0 *of* A *is in* S*, and*

(ii) for any $x, y \in A, x \cdot y \in S$ *and* $x \in S$ *imply* $y \in S$.

(3) *a UP-ideal of A if*

(i) the constant 0 *of* A *is in* S*, and*

(ii) for any $x, y, z \in A, x \cdot (y \cdot z) \in S$ *and* $y \in S$ *imply* $x \cdot z \in S$.

(4) *a strongly UP-ideal of A if*

(i) the constant 0 *of* A *is in* S*, and*

(ii) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in S$ *and* $y \in S$ *imply* $x \in S$.

Guntasow et al. [1] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

3. Basic Results on Hesitant Fuzzy Sets

Definition 3. *[18] Let X be a reference set. A hesitant fuzzy set on X is defined in term of a function* $\mathrm{h_H}$ *that when applied to X return a subset of* $[0,1]$*, that is,* $\mathrm{h_H}\colon X \to \mathcal{P}([0,1])$*. A hesitant fuzzy set* $\mathrm{h_H}$ *can also be viewed as the following mathematical representation:*

$$\mathrm{H} := \{(x, \mathrm{h_H}(x)) \mid x \in X\}$$

where $\mathrm{h_H}(x)$ *is a set of some values in* $[0,1]$*, denoting the possible membership degrees of the elements* $x \in X$ *to the set* H*. We say that a hesitant fuzzy set* H *on* X *is a constant hesitant fuzzy set if its function* $\mathrm{h_H}$ *is constant.*

Definition 4. *[6] Let* H *be a hesitant fuzzy set on A. The hesitant fuzzy set* $\overline{\mathrm{H}}$ *defined by* $\mathrm{h}_{\overline{\mathrm{H}}}(x) = [0,1] - \mathrm{h_H}(x)$ *for all* $x \in A$ *is said to be the complement of* H *on* A.

Remark 1. *[6] For all hesitant fuzzy set* H *on* A*, we have* $\mathrm{H} = \overline{\overline{\mathrm{H}}}$.

Theorem 1. *A hesitant fuzzy set* H *is a constant hesitant fuzzy set on A if and only if the complement of* H *is a constant hesitant fuzzy set on* A.

Proof. Let H be a constant hesitant fuzzy set on A. Then $\mathrm{h_H}(x) = \mathrm{h_H}(0)$ for all $x \in A$. Thus $[0,1] - \mathrm{h_H}(x) = [0,1] - \mathrm{h_H}(0)$ for all $x \in A$. Therefore, $\mathrm{h}_{\overline{\mathrm{H}}}(x) = \mathrm{h}_{\overline{\mathrm{H}}}(0)$ for all $x \in A$. Hence, $\overline{\mathrm{H}}$ is a constant hesitant fuzzy set on A.

Conversely, let $\overline{\mathrm{H}}$ be a constant hesitant fuzzy set on A. Then $\mathrm{h}_{\overline{\mathrm{H}}}(x) = \mathrm{h}_{\overline{\mathrm{H}}}(0)$ for all $x \in A$. Thus $[0,1] - \mathrm{h_H}(x) = [0,1] - \mathrm{h_H}(0)$ for all $x \in A$. Therefore, $\mathrm{h_H}(x) = \mathrm{h_H}(0)$ for all $x \in A$. Hence, H is a constant hesitant fuzzy set on A.

Definition 5. *[6] A hesitant fuzzy set* H *on a* A *is called*

(1) *a hesitant fuzzy UP-subalgebra of* A *if it satisfies the following property: for any* $x, y \in A$, $\mathrm{h_H}(x \cdot y) \supseteq \mathrm{h_H}(x) \cap \mathrm{h_H}(y)$.

(2) *a hesitant fuzzy UP-filter of* A *if it satisfies the following properties: for any* $x, y \in A$,

 (1) $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x)$, *and*

 (2) $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x)$.

(3) *a hesitant fuzzy UP-ideal of* A *if it satisfies the following properties: for any* $x, y, z \in A$,

 (1) $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x)$, *and*

 (2) $\mathrm{h_H}(x \cdot z) \supseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y)$.

(4) *a hesitant fuzzy strongly UP-ideal of* A *if it satisfies the following properties: for any* $x, y, z \in A$,

 (1) $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x)$, *and*

 (2) $\mathrm{h_H}(x) \supseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h_H}(y)$.

Mosrijai et al. [6] proved that the notion of hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of hesitant fuzzy UP-filters, the notion of hesitant fuzzy UP-filters of UP-algebras is a generalization of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-ideals of UP-algebras is a generalization of hesitant fuzzy strongly UP-ideals.

Theorem 2. *[6] A hesitant fuzzy set* H *on* A *is a hesitant fuzzy strongly UP-ideal of* A *if and only if it is a constant hesitant fuzzy set on* A.

4. Anti-Type of Hesitant Fuzzy Sets

In this section, we introduce the notions of anti-hesitant fuzzy UP-subalgebras, anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals and anti-hesitant fuzzy strongly UP-ideals of UP-algebras, provide the necessary examples and prove its generalizations.

Definition 6. *A hesitant fuzzy set* H *on a* A *is called an anti-hesitant fuzzy UP-subalgebra of* A *if it satisfies the following property: for any* $x, y \in A$,

$$\mathrm{h_H}(x \cdot y) \subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y).$$

By Proposition 1 (1), we have $\mathrm{h_H}(0) = \mathrm{h_H}(x \cdot x) \subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(x) = \mathrm{h_H}(x)$ for all $x \in A$.

Example 4. *Let* $A = \{0, 1, 2, 3\}$ *be a set with a binary operation* $\cdot$ *defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $(A, \cdot, 0)$ *is a UP-algebra. We define a hesitant fuzzy set* H *on* A *as follows:*

$$\mathrm{h_H}(0) = \emptyset, \mathrm{h_H}(1) = \{0.5\}, \mathrm{h_H}(2) = \{0.6\}, \text{ and } \mathrm{h_H}(3) = [0.5, 0.6].$$

Using this data, we can show that H *is an anti-hesitant fuzzy UP-subalgebra of* A.

Definition 7. *A hesitant fuzzy set* H *on a* A *is called an anti-hesitant fuzzy UP-filter of* A *if it satisfies the following properties: for any* $x, y \in A$,

(1) $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$, *and*

(2) $\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x)$.

Example 5. *Let* $A = \{0, 1, 2, 3, 4\}$ *be a set with a binary operation* $\cdot$ *defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then $(A, \cdot, 0)$ *is a UP-algebra. We define a hesitant fuzzy set* H *on* A *as follows:*

$$\mathrm{h_H}(0) = \{0.8\}, \mathrm{h_H}(1) = [0.8, 0.9), \mathrm{h_H}(2) = [0.8, 0.9], \mathrm{h_H}(3) = [0.6, 0.9], \text{ and}$$
$$\mathrm{h_H}(4) = [0.6, 0.9].$$

Using this data, we can show that H *is an anti-hesitant fuzzy UP-filter of* A.

Definition 8. *A hesitant fuzzy set* H *on a* A *is called an anti-hesitant fuzzy UP-ideal of* A *if it satisfies the following properties: for any* $x, y, z \in A$,

(1) $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$, *and*

(2) $\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y)$.

Example 6. *Let* $A = \{0, 1, 2, 3\}$ *be a set with a binary operation* $\cdot$ *defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ *is a UP-algebra. We define a hesitant fuzzy set* H *on* A *as follows:*

$$\mathrm{h_H}(0) = \{1\}, \mathrm{h_H}(1) = \{1\}, \mathrm{h_H}(2) = \{0, 1\}, \text{ and } \mathrm{h_H}(3) = [0, 1].$$

Using this data, we can show that H *is an anti-hesitant fuzzy UP-ideal of A.*

Definition 9. *A hesitant fuzzy set* H *on a A is called an anti-hesitant fuzzy strongly UP-ideal of A if it satisfies the following properties: for any* $x, y, z \in A$,

(1) $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$, *and*

(2) $\mathrm{h_H}(x) \subseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h_H}(y)$.

Example 7. *Let* $A = \{0, 1, 2, 3\}$ *be a set with a binary operation* $\cdot$ *defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

Then $(A, \cdot, 0)$ *is a UP-algebra. We define a hesitant fuzzy set* H *on A as follows:*

$$\mathrm{h_H}(0) = \{0, 0.2\}, \mathrm{h_H}(1) = \{0, 0.2\}, \mathrm{h_H}(2) = \{0, 0.2\}, \text{ and } \mathrm{h_H}(3) = \{0, 0.2\}.$$

Using this data, we can show that H *is an anti-hesitant fuzzy strongly UP-ideal of A.*

Theorem 3. *A hesitant fuzzy set* H *on A is an anti-hesitant fuzzy strongly UP-ideal of A if and only if it is a constant hesitant fuzzy set on A.*

Proof. Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. Then $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$ and $\mathrm{h_H}(x) \subseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h_H}(y)$ for all $x, y, z \in A$. For any $x \in A$, we choose $z = x$ and $y = 0$. Then

$$\begin{aligned} \mathrm{h_H}(x) &\subseteq \mathrm{h_H}((x \cdot 0) \cdot (x \cdot x)) \cup \mathrm{h_H}(0) \\ &= \mathrm{h_H}(0 \cdot 0) \cup \mathrm{h_H}(0) && \text{((UP-3) and Proposition 1 (1))} \\ &= \mathrm{h_H}(0) \cup \mathrm{h_H}(0) && \text{((UP-2))} \\ &= \mathrm{h_H}(0) \\ &\subseteq \mathrm{h_H}(x), \end{aligned}$$

so $\mathrm{h_H}(0) = \mathrm{h_H}(x)$. Hence, H is a constant hesitant fuzzy set on A.

Conversely, assume that H is a constant hesitant fuzzy set on A. Then, for any $x \in A, \mathrm{h_H}(0) = \mathrm{h_H}(x)$, so $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$. For any $x, y, z \in A, \mathrm{h_H}(x) = \mathrm{h_H}((z{\cdot}y){\cdot}(z{\cdot}x)) = \mathrm{h_H}(y)$, so $\mathrm{h_H}(x) = \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h_H}(y)$. Thus $\mathrm{h_H}(x) \subseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h_H}(y)$. Hence, H is an anti-hesitant fuzzy strongly UP-ideal of A.

Corollary 1. *For UP-algebras, we can conclude that the notions of anti-hesitant fuzzy strongly UP-ideals and hesitant fuzzy strongly UP-ideals coincide.*

Proof. It is straightforward by Theorem 2 and 3.

Corollary 2. *A hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy strongly UP-ideal of* A *if and only if* $\overline{\mathrm{H}}$ *on* A *is an anti-hesitant fuzzy strongly UP-ideal of* A*.*

Proof. It is straightforward by Theorem 1 and 3.

By Using Corollary 1, we can show that a hesitant fuzzy set H on A is an anti-hesitant fuzzy strongly UP-ideal of A if and only if $\overline{\mathrm{H}}$ on A is an anti-hesitant fuzzy strongly UP-ideal of A.

Theorem 4. *Every anti-hesitant fuzzy UP-filter of* A *is an anti-hesitant fuzzy UP-subalgebra of* A*.*

Proof. Assume that H is an anti-hesitant fuzzy UP-filter of A. Then for any $x, y \in A$,

$$\begin{aligned} \mathrm{h_H}(x \cdot y) &\subseteq \mathrm{h_H}(y \cdot (x \cdot y)) \cup \mathrm{h_H}(y) && \text{(Definition 7 (2))} \\ &= \mathrm{h_H}(0) \cup \mathrm{h_H}(y) && \text{(Proposition 1 (5))} \\ &= \mathrm{h_H}(y) && \text{(Definition 7 (1))} \\ &\subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y). \end{aligned}$$

Hence, H is an anti-hesitant fuzzy UP-subalgebra of A.

The converse of Theorem 4 is not true in general. By Example 4, we obtain H is an anti-hesitant fuzzy UP-subalgebra of A. Since $\mathrm{h_H}(1) = \{0.5\} \nsubseteq \{0.6\} = \emptyset \cup \{0.6\} = \mathrm{h_H}(0) \cup \mathrm{h_H}(2) = \mathrm{h_H}(2 \cdot 1) \cup \mathrm{h_H}(2)$, we have H is not an anti-hesitant fuzzy UP-filter of A. Therefore, the notion of anti-hesitant fuzzy UP-subalgebras of UP-algebras is generalization of anti-hesitant fuzzy UP-filters.

Theorem 5. *Every anti-hesitant fuzzy UP-ideal of* A *is an anti-hesitant fuzzy UP-filter of* A*.*

Proof. Assume that H is an anti-hesitant fuzzy UP-ideal of A. Then for any $x, y \in A$, $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$ and

$$\begin{aligned} \mathrm{h_H}(y) &= \mathrm{h_H}(0 \cdot y) && \text{((UP-2))} \\ &\subseteq \mathrm{h_H}(0 \cdot (x \cdot y)) \cup \mathrm{h_H}(x) && \text{(Definition 8 (2))} \\ &= \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x). && \text{((UP-2))} \end{aligned}$$

Hence, H is an anti-hesitant fuzzy UP-filter of A.

The converse of Theorem 5 is not true in general. By Example 5, we obtain H is an anti-hesitant fuzzy UP-filter of A. Since $\mathrm{h_H}(3 \cdot 4) = \mathrm{h_H}(3) = [0.6, 0.9] \nsubseteq [0.8, 0.9) = \{0.8\} \cup [0.8, 0.9) = \mathrm{h_H}(0) \cup \mathrm{h_H}(2) = \mathrm{h_H}(3 \cdot (2 \cdot 4)) \cup \mathrm{h_H}(2)$, we have H is not an anti-hesitant fuzzy UP-ideal of A. Therefore, the notion of anti-hesitant fuzzy UP-filters of UP-algebras is generalization of anti-hesitant fuzzy UP-ideals.

Theorem 6. *Every anti-hesitant fuzzy strongly UP-ideal of A is an anti-hesitant fuzzy UP-ideal of A.*

Proof. Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. Then for any $x, y \in A$, $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$ and

$$\begin{aligned}
\mathrm{h_H}(x \cdot z) &\subseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot (x \cdot z))) \cap \mathrm{h_H}(y) && \text{(Definition 9 (2))}\\
&= \mathrm{h_H}((z \cdot y) \cdot 0) \cap \mathrm{h_H}(y) && \text{(Proposition 1 (5))}\\
&= \mathrm{h_H}(0) \cap \mathrm{h_H}(y) && \text{((UP-3))}\\
&= \mathrm{h_H}(y) && \text{(Definition 9 (1))}\\
&= \mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y).
\end{aligned}$$

Hence, H is an anti-hesitant fuzzy UP-ideal of A.

The converse of Theorem 6 is not true in general. By Theorem 3, we obtain an anti-hesitant fuzzy strongly UP-ideal is a constant hesitant fuzzy set. But anti-hesitant fuzzy UP-ideal is not a constant hesitant fuzzy set in general. Therefore, the notion of anti-hesitant fuzzy UP-ideals of UP-algebras is generalization of anti-hesitant fuzzy strongly UP-ideals.

Proposition 2. *Let* H *be an anti-hesitant fuzzy UP-filter (and also anti-hesitant fuzzy UP-ideal, anti-hesitant fuzzy strongly UP-ideal) of A. Then for any $x, y \in A$,*

$$x \leq y \textit{ implies } \mathrm{h_H}(x) \supseteq \mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y).$$

Proof. Let $x, y \in A$ be such that $x \leq y$. Then $x \cdot y = 0$. Since H is an anti-hesitant fuzzy UP-filter (resp., anti-hesitant fuzzy UP-ideal, anti-hesitant fuzzy strongly UP-ideal) of A, we have

$$\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x) = \mathrm{h_H}(0) \cup \mathrm{h_H}(x) = \mathrm{h_H}(x).$$

By Proposition 1 (5), we obtain $y \leq x \cdot y$ and thus $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y)$.

5. Level Subsets of a Hesitant Fuzzy Set

Definition 10. *[11] Let* H *be a hesitant fuzzy set on A. For any $\varepsilon \in \mathcal{P}([0, 1])$, the sets*

$$U(\mathrm{H}; \varepsilon) = \{x \in A \mid \mathrm{h_H}(x) \supseteq \varepsilon\} \textit{ and } U^{+}(\mathrm{H}; \varepsilon) = \{x \in A \mid \mathrm{h_H}(x) \supset \varepsilon\}$$

are called an upper ε-level subset and an upper ε-strong level subset of H*, respectively. The sets*

$$L(\mathrm{H}; \varepsilon) = \{x \in A \mid \mathrm{h_H}(x) \subseteq \varepsilon\} \textit{ and } L^{-}(\mathrm{H}; \varepsilon) = \{x \in A \mid \mathrm{h_H}(x) \subset \varepsilon\}$$

are called a lower ε-level subset and a lower ε-strong level subset of H*, respectively. The set*

$$E(\mathrm{H};\varepsilon) = \{x \in A \mid \mathrm{h_H}(x) = \varepsilon\}$$

is called an equal ε*-level subset of* H*. Then*

$$U(\mathrm{H};\varepsilon) = U^+(\mathrm{H};\varepsilon) \cup E(\mathrm{H};\varepsilon) \text{ and } L(\mathrm{H};\varepsilon) = L^-(\mathrm{H};\varepsilon) \cup E(\mathrm{H};\varepsilon).$$

Proposition 3. *Let* H *be a hesitant fuzzy set on* A *and let* $\varepsilon \in \mathcal{P}([0,1])$*. Then the following statements hold*:

(1) $U(\mathrm{H};\varepsilon) = L(\overline{\mathrm{H}};[0,1]-\varepsilon)$,

(2) $U^+(\mathrm{H};\varepsilon) = L^-(\overline{\mathrm{H}};[0,1]-\varepsilon)$,

(3) $L(\mathrm{H};\varepsilon) = U(\overline{\mathrm{H}};[0,1]-\varepsilon)$, *and*

(4) $L^-(\mathrm{H};\varepsilon) = U^+(\overline{\mathrm{H}};[0,1]-\varepsilon)$.

Proof. (1) Let $x \in A$ and let $\varepsilon \in \mathcal{P}([0,1])$. Then $x \in U(\mathrm{H};\varepsilon)$ if and only if $\mathrm{h_H}(x) \supseteq \varepsilon$ if and only if $[0,1] - \mathrm{h_H}(x) \subseteq [0,1] - \varepsilon$ if and only if $\mathrm{h}_{\overline{\mathrm{H}}}(x) \subseteq [0,1] - \varepsilon$ if and only if $x \in L(\overline{\mathrm{H}};[0,1]-\varepsilon)$. Therefore, $U(\mathrm{H};\varepsilon) = L(\overline{\mathrm{H}};[0,1]-\varepsilon)$.

(2) Let $x \in A$ and let $\varepsilon \in \mathcal{P}([0,1])$. Then $x \in U^+(\mathrm{H};\varepsilon)$ if and only if $\mathrm{h_H}(x) \supset \varepsilon$ if and only if $[0,1] - \mathrm{h_H}(x) \subset [0,1] - \varepsilon$ if and only if $\mathrm{h}_{\overline{\mathrm{H}}}(x) \subset [0,1] - \varepsilon$ if and only if $x \in L^-(\overline{\mathrm{H}};[0,1]-\varepsilon)$. Therefore, $U^+(\mathrm{H};\varepsilon) = L^-(\overline{\mathrm{H}};[0,1]-\varepsilon)$.

(3) Let $x \in A$ and let $\varepsilon \in \mathcal{P}([0,1])$. Then $x \in L(\mathrm{H};\varepsilon)$ if and only if $\mathrm{h_H}(x) \subseteq \varepsilon$ if and only if $[0,1] - \mathrm{h_H}(x) \supseteq [0,1] - \varepsilon$ if and only if $\mathrm{h}_{\overline{\mathrm{H}}}(x) \supseteq [0,1] - \varepsilon$ if and only if $x \in U(\overline{\mathrm{H}};[0,1]-\varepsilon)$. Therefore, $L(\mathrm{H};\varepsilon) = U(\overline{\mathrm{H}};[0,1]-\varepsilon)$.

(4) Let $x \in A$ and let $\varepsilon \in \mathcal{P}([0,1])$. Then $x \in L^-(\mathrm{H};\varepsilon)$ if and only if $\mathrm{h_H}(x) \subset \varepsilon$ if and only if $[0,1] - \mathrm{h_H}(x) \supset [0,1] - \varepsilon$ if and only if $\mathrm{h}_{\overline{\mathrm{H}}}(x) \supset [0,1] - \varepsilon$ if and only if $x \in U^+(\overline{\mathrm{H}};[0,1]-\varepsilon)$. Therefore, $L^-(\mathrm{H};\varepsilon) = U^+(\overline{\mathrm{H}};[0,1]-\varepsilon)$.

Lemma 1. *[11] Let* H *be a hesitant fuzzy set on* A*. Then the following statements hold: for any* $x, y \in A$,

(1) $[0,1] - (\mathrm{h_H}(x) \cup \mathrm{h_H}(y)) = ([0,1] - \mathrm{h_H}(x)) \cap ([0,1] - \mathrm{h_H}(y))$, *and*

(2) $[0,1] - (\mathrm{h_H}(x) \cap \mathrm{h_H}(y)) = ([0,1] - \mathrm{h_H}(x)) \cup ([0,1] - \mathrm{h_H}(y))$.

5.1. Lower ε-Level Subsets

Theorem 7. *A hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy UP-subalgebra of* A *if and only if for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L(\mathrm{H};\varepsilon)$ *of* A *is a UP-subalgebra of* A.

Proof. Assume that H is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x, y \in A$ be such that $x \in L(\mathrm{H};\varepsilon)$ and $y \in L(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$ and $\mathrm{h_H}(y) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-subalgebra of A, we have $\mathrm{h_H}(x \cdot y) \subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y) \subseteq \varepsilon$ and thus $x \cdot y \in L(\mathrm{H};\varepsilon)$. Hence, $L(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A. Let $x, y \in A$. Then $\mathrm{h_H}(x), \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x) \cup \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$ and $\mathrm{h_H}(y) \subseteq \varepsilon$. Thus $x, y \in L(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, $L(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A and thus $x \cdot y \in L(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(x \cdot y) \subseteq \varepsilon = \mathrm{h_H}(x) \cup \mathrm{h_H}(y)$. Hence, H is an anti-hesitant fuzzy UP-subalgebra of A.

Theorem 8. *A hesitant fuzzy set* H *on A is an anti-hesitant fuzzy UP-filter of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-filter of A.*

Proof. Assume that H is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{H};\varepsilon) \neq \emptyset$ and let $x \in A$ be such that $x \in L(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x) \subseteq \varepsilon$ and thus $0 \in L(\mathrm{H};\varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in L(\mathrm{H};\varepsilon)$ and $x \in L(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot y) \subseteq \varepsilon$ and $\mathrm{h_H}(x) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x) \subseteq \varepsilon$ and thus $y \in L(\mathrm{H};\varepsilon)$. Hence, $L(\mathrm{H};\varepsilon)$ is a UP-filter of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-filter of A. Let $x \in A$. Then $\mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$. Thus $x \in L(\mathrm{H};\varepsilon)$. By assumption, we have $L(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $0 \in L(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(0) \subseteq \varepsilon = \mathrm{h_H}(x)$.

Next, let $x, y \in A$. Then $\mathrm{h_H}(x \cdot y), \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot y) \subseteq \varepsilon$ and $\mathrm{h_H}(x) \subseteq \varepsilon$. Thus $x \cdot y, x \in L(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $L(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $y \in L(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(y) \subseteq \varepsilon = \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x)$. Hence, H is an anti-hesitant fuzzy UP-filter of A.

Theorem 9. *A hesitant fuzzy set* H *on A is an anti-hesitant fuzzy UP-ideal of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A.*

Proof. Assume that H is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{H};\varepsilon) \neq \emptyset$ and let $x \in A$ be such that $x \in L(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x) \subseteq \varepsilon$ and thus $0 \in L(\mathrm{H};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L(\mathrm{H};\varepsilon)$ and $y \in L(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \subseteq \varepsilon$ and $\mathrm{h_H}(y) \subseteq \varepsilon$. Since H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y) \subseteq \varepsilon$ and thus $x \cdot z \in L(\mathrm{H};\varepsilon)$. Hence, $L(\mathrm{H};\varepsilon)$ is a UP-ideal of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A. Let $x \in A$. Then $\mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \subseteq \varepsilon$. Thus $x \in L(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $L(\mathrm{H};\varepsilon)$ is a UP-ideal of A and so $0 \in L(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(0) \subseteq \varepsilon = \mathrm{h_H}(x)$.

Next, let $x, y, z \in A$. Then $\mathrm{h_H}(x \cdot (y \cdot z)), \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Choose $\varepsilon = \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \subseteq \varepsilon$ and $\mathrm{h_H}(y) \subseteq \varepsilon$. Thus $x \cdot (y \cdot z), y \in L(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $L(\mathrm{H};\varepsilon)$ is a UP-ideal of A and so $x \cdot z \in L(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(x \cdot z) \subseteq \varepsilon = \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y)$. Hence, H is an anti-hesitant fuzzy UP-ideal of A.

Theorem 10. *Let* H *be a hesitant fuzzy set on* A*. Then the following statements are equivalent*:

(1) H *is an anti-hesitant fuzzy strongly UP-ideal of* A*,*

(2) *a nonempty subset* $L(\mathrm{H};\varepsilon)$ *of* A *is a strongly UP-ideal of* A *for all* $\varepsilon \in \mathcal{P}([0,1])$*, and*

(3) *a nonempty subset* $U(\mathrm{H};\varepsilon)$ *of* A *is a strongly UP-ideal of* A *for all* $\varepsilon \in \mathcal{P}([0,1])$*.*

Proof. (1)⇒(2) Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 3, we obtain H is a constant hesitant fuzzy set on A and so $\mathrm{h_H}(x) = \mathrm{h_H}(y)$ for all $x, y \in A$. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L(\mathrm{H};\varepsilon) \neq \emptyset$. There exists $a \in L(\mathrm{H};\varepsilon)$ be such that $\mathrm{h_H}(a) \subseteq \varepsilon$. Thus $\mathrm{h_H}(x) = \mathrm{h_H}(a) \subseteq \varepsilon$ for all $x \in A$ and so $x \in L(\mathrm{H};\varepsilon)$ for all $x \in A$. Therefore, $L(\mathrm{H};\varepsilon) = A$. Hence, $L(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A.

(2)⇒(3) Assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{H};\varepsilon) \neq \emptyset$. If $U(\mathrm{H};\varepsilon) \neq A$, then there exist $x \in U(\mathrm{H};\varepsilon)$ and $y \notin U(\mathrm{H};\varepsilon)$. So $\mathrm{h_H}(x) \supseteq \varepsilon$ and $\mathrm{h_H}(y) \not\supseteq \varepsilon$. Consider, $\varepsilon_y = \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Then $y \in L(\mathrm{H};\varepsilon_y)$ and $\varepsilon_y \not\supseteq \varepsilon$. By assumption, we have $L(\mathrm{H};\varepsilon_y)$ is a strongly UP-ideal of A and so $L(\mathrm{H};\varepsilon_y) = A$. Thus $\mathrm{h_H}(x) \subseteq \varepsilon_y$. Since $\mathrm{h_H}(x) \supseteq \varepsilon$, we have $\varepsilon_y \supseteq \varepsilon$, a contradiction. Therefore, $U(\mathrm{H};\varepsilon) = A$. Hence, $U(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A.

(3)⇒(1) Assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A. Assume that H is not a constant hesitant fuzzy set on A. There exist $x, y \in A$ be such that $\mathrm{h_H}(x) \neq \mathrm{h_H}(y)$. Now, $x \in U(\mathrm{H};\mathrm{h_H}(x)) \neq \emptyset$ and $y \in U(\mathrm{H};\mathrm{h_H}(y)) \neq \emptyset$. By assumption, we have $U(\mathrm{H};\mathrm{h_H}(x))$ and $U(\mathrm{H};\mathrm{h_H}(y))$ are strongly UP-ideals of A and thus $U(\mathrm{H};\mathrm{h_H}(x)) = A = U(\mathrm{H};\mathrm{h_H}(y))$. Then $x \in U(\mathrm{H};\mathrm{h_H}(y))$ and $y \in U(\mathrm{H};\mathrm{h_H}(x))$. Thus $\mathrm{h_H}(x) \supseteq \mathrm{h_H}(y)$ and $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x)$. So $\mathrm{h_H}(x) = \mathrm{h_H}(y)$, a contradiction. Therefore, H is a constant hesitant fuzzy set on A. By Theorem 3, we obtain H is an anti-hesitant fuzzy strongly UP-ideal of A.

5.2. Lower ε-Strong Level Subsets

Theorem 11. *Let* H *be a hesitant fuzzy set on* A*. Then the following statements hold*:

(1) *if* H *is an anti-hesitant fuzzy UP-subalgebra of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $L^-(\mathrm{H};\varepsilon)$ *is a UP-subalgebra of* A *if* $L^-(\mathrm{H};\varepsilon)$ *is nonempty, and*

(2) *if* $\mathrm{Im}(\mathrm{H})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $L^-(\mathrm{H};\varepsilon)$ *of* A *is a UP-subalgebra of* A*, then* H *is an anti-hesitant fuzzy UP-subalgebra of* A*.*

Proof. (1) Assume that H is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L^-(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x, y \in A$ be such that $x \in L^-(\mathrm{H};\varepsilon)$ and $y \in L^-(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \subset \varepsilon$ and $\mathrm{h_H}(y) \subset \varepsilon$. Since H is an anti-hesitant fuzzy UP-subalgebra of A, we have $\mathrm{h_H}(x \cdot y) \subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y) \subset \varepsilon$ and thus $x \cdot y \in L^-(\mathrm{H};\varepsilon)$. Hence, $L^-(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A.

(2) Assume that $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A. Assume that there exist $x, y \in A$ such that $\mathrm{h_H}(x \cdot y) \not\subseteq$

$h_H(x) \cup h_H(y)$. Since Im(H) is a chain, we have $h_H(x \cdot y) \supset h_H(x) \cup h_H(y)$. Choose $\varepsilon = h_H(x \cdot y) \in \mathcal{P}([0,1])$. Then $h_H(x) \subset \varepsilon$ and $h_H(y) \subset \varepsilon$. Thus $x, y \in L^-(H; \varepsilon) \neq \emptyset$. By assumption, we have $L^-(H; \varepsilon)$ is a UP-subalgebra of A and so $x \cdot y \in L^-(H; \varepsilon)$. Thus $h_H(x \cdot y) \subset \varepsilon = h_H(x \cdot y)$, a contradiction. Therefore, $h_H(x \cdot y) \subseteq h_H(x) \cup h_H(y)$ for all $x, y \in A$. Hence, H is an anti-hesitant fuzzy UP-subalgebra of A.

Example 8. *Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	2	0	0	0
3	0	2	2	0	0
4	0	2	2	4	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H *on A as follows:*

$$h_H(0) = (0,1), h_H(1) = [0,1), h_H(2) = (0,1], h_H(3) = [0,1), \textit{ and } h_H(4) = [0,1].$$

Then Im(H) *is not a chain. If $\varepsilon \subseteq (0,1)$, then $L^-(H; \varepsilon) = \emptyset$. If $\varepsilon = [0,1)$ or $\varepsilon = (0,1]$, then $L^-(H; \varepsilon) = \{0\}$. If $\varepsilon = [0,1]$, then. $L^-(H; \varepsilon) = \{0, 1, 2, 3\}$. Using this data, we can show that all nonempty subset $L^-(H; \varepsilon)$ of A is a UP-subalgebra of A. Since $h_H(3 \cdot 1) = h_H(2) = (0,1] \nsubseteq [0,1) = h_H(3) \cup h_H(1)$, we have* H *is not an anti-hesitant fuzzy UP-subalgebra of A.*

Theorem 12. *Let* H *be a hesitant fuzzy set on A. Then the following statements hold*:

(1) *if* H *is an anti-hesitant fuzzy UP-filter of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $L^-(H; \varepsilon)$ is a UP-filter of A if $L^-(H; \varepsilon)$ is nonempty, and*

(2) *if* Im(H) *is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(H; \varepsilon)$ of A is a UP-filter of A, then* H *is an anti-hesitant fuzzy UP-filter of A.*

Proof. (1) Assume that H is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L^-(H; \varepsilon) \neq \emptyset$ and let $x \in A$ be such that $x \in L^-(H; \varepsilon)$. Then $h_H(x) \subset \varepsilon$. Since H is an anti-hesitant fuzzy UP-filter of A, we have $h_H(0) \subseteq h_H(x) \subset \varepsilon$ and thus $0 \in L^-(H; \varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in L^-(H; \varepsilon)$ and $x \in L^-(H; \varepsilon)$. Then $h_H(x \cdot y) \subset \varepsilon$ and $h_H(x) \subset \varepsilon$. Since H is an anti-hesitant fuzzy UP-filter of A, we have $h_H(y) \subseteq h_H(x \cdot y) \cup h_H(x) \subset \varepsilon$ and thus $y \in L^-(H; \varepsilon)$. Hence, $L^-(H; \varepsilon)$ is a UP-filter of A.

(2) Assume that Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(H; \varepsilon)$ of A is a UP-filter of A. Assume that there exists $x \in A$ such that $h_H(0) \nsubseteq h_H(x)$. Since Im(H) is a chain, we have $h_H(0) \supset h_H(x)$. Choose $\varepsilon = h_H(0) \in \mathcal{P}([0,1])$. Then $h_H(x) \subset h_H(0) = \varepsilon$. Thus $x \in L^-(H; \varepsilon) \neq \emptyset$. By assumption, we have $L^-(H; \varepsilon)$ is a UP-filter of A and so $0 \in L^-(H; \varepsilon)$. Therefore, $h_H(0) \subset \varepsilon = h_H(0)$, a contradiction. Hence, $h_H(0) \subseteq h_H(x)$ for all $x \in A$.

Next, assume that there exist $x, y \in A$ such that $\mathrm{h_H}(y) \not\subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x)$. Since Im(H) is a chain, we have $\mathrm{h_H}(y) \supset \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x)$. Choose $\varepsilon = \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot y) \subset \varepsilon$ and $\mathrm{h_H}(x) \subset \varepsilon$. Thus $x \cdot y, x \in L^-(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $L^-(\mathrm{H}; \varepsilon)$ is a UP-filter of A and so $y \in L^-(\mathrm{H}; \varepsilon)$. Thus $\mathrm{h_H}(y) \subset \varepsilon = \mathrm{h_H}(y)$, a contradiction. Therefore, $\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x)$ for all $x, y \in A$. Hence, H is an anti-hesitant fuzzy UP-filter of A.

Example 9. *Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	3	4
2	0	0	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H *on A as follows:*

$$\mathrm{h_H}(0) = (0,1), \mathrm{h_H}(1) = [0,1), \mathrm{h_H}(2) = (0,1], \mathrm{h_H}(3) = [0,1], \textit{ and } \mathrm{h_H}(4) = [0,1].$$

Then Im(H) *is not a chain. If $\varepsilon \subseteq (0,1)$, then $L^-(\mathrm{H}; \varepsilon) = \emptyset$. If $\varepsilon = [0,1)$ or $\varepsilon = (0,1]$, then $L^-(\mathrm{H}; \varepsilon) = \{0\}$. If $\varepsilon = [0,1]$, then $L^-(\mathrm{H}; \varepsilon) = \{0,1,2\}$. Using this data, we can show that all nonempty subset $L^-(\mathrm{H}; \varepsilon)$ of A is a UP-filter of A. Since $\mathrm{h_H}(2) = (0,1] \not\subseteq [0,1) = \mathrm{h_H}(1) \cup \mathrm{h_H}(1) = \mathrm{h_H}(1 \cdot 2) \cup \mathrm{h_H}(1)$, we have* H *is not an anti-hesitant fuzzy UP-filter of A.*

Theorem 13. *Let* H *be a hesitant fuzzy set on A. Then the following statements hold*:

(1) *if* H *is an anti-hesitant fuzzy UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $L^-(\mathrm{H}; \varepsilon)$ is a UP-ideal of A if $L^-(\mathrm{H}; \varepsilon)$ is nonempty, and*

(2) *if* Im(H) *is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{H}; \varepsilon)$ of A is a UP-ideal of A, then* H *is an anti-hesitant fuzzy UP-ideal of A.*

Proof. (1) Assume that H is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L^-(\mathrm{H}; \varepsilon) \neq \emptyset$ and let $x \in A$ be such that $x \in L^-(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x) \subset \varepsilon$. Since H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x) \subset \varepsilon$ and thus $0 \in L^-(\mathrm{H}; \varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L^-(\mathrm{H}; \varepsilon)$ and $y \in L^-(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \subset \varepsilon$ and $\mathrm{h_H}(y) \subset \varepsilon$. Since H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y) \subset \varepsilon$ and thus $x \cdot z \in L^-(\mathrm{H}; \varepsilon)$. Hence, $L^-(\mathrm{H}; \varepsilon)$ is a UP-ideal of A.

(2) Assume that Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{H}; \varepsilon)$ of A is a UP-ideal of A. Assume that there exists $x \in A$ such that $\mathrm{h_H}(0) \not\subseteq \mathrm{h_H}(x)$. Since Im(H) is a chain, we have $\mathrm{h_H}(0) \supset \mathrm{h_H}(x)$. Choose $\varepsilon = \mathrm{h_H}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \subset \mathrm{h_H}(0) = \varepsilon$. Thus $x \in L^-(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $L^-(\mathrm{H}; \varepsilon)$ is a UP-ideal of A and so $0 \in L^-(\mathrm{H}; \varepsilon)$. Therefore, $\mathrm{h_H}(0) \subset \varepsilon = \mathrm{h_H}(0)$, a contradiction. Hence, $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x)$ for all $x \in A$.

Next, assume that there exist $x, y, z \in A$ such that $\mathrm{h_H}(x \cdot z) \nsubseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y)$. Since $\mathrm{Im(H)}$ is a chain, we have $\mathrm{h_H}(x \cdot z) \supset \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y)$. Choose $\varepsilon = \mathrm{h_H}(x \cdot z) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \subset \varepsilon$ and $\mathrm{h_H}(y) \subset \varepsilon$. Thus $x \cdot (y \cdot z), y \in L^-(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $L^-(\mathrm{H}; \varepsilon)$ is a UP-ideal of A and so $x \cdot z \in L^-(\mathrm{H}; \varepsilon)$. Thus $\mathrm{h_H}(x \cdot z) \subset \varepsilon = \mathrm{h_H}(x \cdot z)$, a contradiction. Therefore, $\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y)$ for all $x, y, z \in A$. Hence, H is an anti-hesitant fuzzy UP-ideal of A.

Example 10. *Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	2	0	4
4	0	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H *on A as follows:*

$$\mathrm{h_H}(0) = (0,1), \mathrm{h_H}(1) = [0,1), \mathrm{h_H}(2) = [0,1], \mathrm{h_H}(3) = (0,1], \textit{ and } \mathrm{h_H}(4) = [0,1].$$

Then $\mathrm{Im(H)}$ *is not a chain. If $\varepsilon \subseteq (0,1)$, then $L^-(\mathrm{H}; \varepsilon) = \emptyset$. If $\varepsilon = [0,1)$ or $\varepsilon = (0,1]$, then $L^-(\mathrm{H}; \varepsilon) = \{0\}$. If $\varepsilon = [0,1]$, then $L^-(\mathrm{H}; \varepsilon) = \{0, 1, 3\}$. Using this data, we can show that all nonempty subset $L^-(\mathrm{H}; \varepsilon)$ of A is a UP-ideal of A. Since $\mathrm{h_H}(0 \cdot 1) = \mathrm{h_H}(1) = [0,1) \nsubseteq (0,1] = \mathrm{h_H}(0) \cup \mathrm{h_H}(3) = \mathrm{h_H}(0 \cdot (3 \cdot 1)) \cup \mathrm{h_H}(3)$, we have* H *is not an anti-hesitant fuzzy UP-ideal of A.*

Theorem 14. *Let* H *be a hesitant fuzzy set on A. Then the following statements hold*:

1. *if* H *is an anti-hesitant fuzzy strongly UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, $L^-(\mathrm{H}; \varepsilon)$ is a strongly UP-ideal of A if $L^-(\mathrm{H}; \varepsilon)$ is nonempty, and*
2. *if* $\mathrm{Im(H)}$ *is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{H}; \varepsilon)$ of A is a strongly UP-ideal of A, then* H *is an anti-hesitant fuzzy strongly UP-ideal of A.*

Proof. (1) Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 3, we obtain H is a constant hesitant fuzzy set on A and so $\mathrm{h_H}(x) = \mathrm{h_H}(y)$ for all $x, y \in A$. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $L^-(\mathrm{H}; \varepsilon) \neq \emptyset$. There exists $a \in L^-(\mathrm{H}; \varepsilon)$ be such that $\mathrm{h_H}(a) \subset \varepsilon$. Thus $\mathrm{h_H}(x) = \mathrm{h_H}(a) \subset \varepsilon$ for all $x \in A$ and so $x \in L^-(\mathrm{H}; \varepsilon)$ for all $x \in A$. Therefore, $L^-(\mathrm{H}; \varepsilon) = A$. Hence, $L^-(\mathrm{H}; \varepsilon)$ is a strongly UP-ideal of A.

(2) Assume that $\mathrm{Im(H)}$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $L^-(\mathrm{H}; \varepsilon)$ of A is a strongly UP-ideal of A. Assume that H is not a constant hesitant fuzzy set on A. There exist $x, y \in A$ be such that $\mathrm{h_H}(x) \neq \mathrm{h_H}(y)$. Since $\mathrm{Im(H)}$ is a chain, we have $\mathrm{h_H}(x) \subset \mathrm{h_H}(y)$ or $\mathrm{h_H}(x) \supset \mathrm{h_H}(y)$. Without loss of generality, assume that $\mathrm{h_H}(x) \subset \mathrm{h_H}(y)$, then $x \in L^-(\mathrm{H}; \mathrm{h_H}(y)) \neq \emptyset$. By assumption, we have $L^-(\mathrm{H}; \mathrm{h_H}(y))$ is a strongly UP-ideal of A and so $L^-(\mathrm{H}; \mathrm{h_H}(y)) = A$. Thus $y \in A = L^-(\mathrm{H}; \mathrm{h_H}(y))$ and so $\mathrm{h_H}(y) \subset \mathrm{h_H}(y)$, a contradiction. Therefore, H is a constant hesitant fuzzy set on A. By Theorem 3, we obtain H is an anti-hesitant fuzzy strongly UP-ideal of A.

Example 11. *Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	0	1
0	0	1
1	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H *on A as follows:*

$$\mathrm{h_H}(0) = (0, 1], \text{ and } \mathrm{h_H}(1) = [0, 1).$$

Then Im(H) *is not a chain. If $\varepsilon \subseteq [0, 1)$ or $\varepsilon \subseteq (0, 1]$, then $L^-(\mathrm{H}; \varepsilon) = \emptyset$. If $\varepsilon = [0, 1]$, then $L^-(\mathrm{H}; \varepsilon) = A$. Thus a nonempty subset $L^-(\mathrm{H}; \varepsilon)$ of A is a strongly UP-ideal of A. By Theorem 3 and* H *is not a constant hesitant fuzzy set on A, we have* H *is not an anti-hesitant fuzzy strongly UP-ideal of A.*

5.3. Upper ε-Level Subsets

Theorem 15. *A hesitant fuzzy set $\overline{\mathrm{H}}$ on A is an anti-hesitant fuzzy UP-subalgebra of A if and only if for all $\varepsilon \in \mathcal{P}([0, 1])$, a nonempty subset $U(\mathrm{H}; \varepsilon)$ of A is a UP-subalgebra of A.*

Proof. Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0, 1])$ be such that $U(\mathrm{H}; \varepsilon) \neq \emptyset$, and let $x, y \in A$ be such that $x \in U(\mathrm{H}; \varepsilon)$ and $y \in U(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x) \supseteq \varepsilon$ and $\mathrm{h_H}(y) \supseteq \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A, we obtain $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 1 (2), we have $[0, 1] - \mathrm{h_H}(x \cdot y) \subseteq ([0, 1] - \mathrm{h_H}(x)) \cup ([0, 1] - \mathrm{h_H}(y)) = [0, 1] - (\mathrm{h_H}(x) \cap \mathrm{h_H}(y))$. Thus $\mathrm{h_H}(x \cdot y) \supseteq \mathrm{h_H}(x) \cap \mathrm{h_H}(y) \supseteq \varepsilon$. Therefore, $x \cdot y \in U(\mathrm{H}; \varepsilon)$. Hence, $U(\mathrm{H}; \varepsilon)$ is a UP-subalgebra of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0, 1])$, a nonempty subset $U(\mathrm{H}; \varepsilon)$ of A is a UP-subalgebra of A. Let $x, y \in A$. Choose $\varepsilon = \mathrm{h_H}(x) \cap \mathrm{h_H}(y) \in \mathcal{P}([0, 1])$. Then $\mathrm{h_H}(x) \supseteq \varepsilon$ and $\mathrm{h_H}(y) \supseteq \varepsilon$. Thus $x, y \in U(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H}; \varepsilon)$ is a UP-subalgebra of A and so $x \cdot y \in U(\mathrm{H}; \varepsilon)$. Therefore, $\mathrm{h_H}(x \cdot y) \supseteq \varepsilon = \mathrm{h_H}(x) \cap \mathrm{h_H}(y)$. By Lemma 1 (2), we have

$$\begin{aligned} \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) &= [0, 1] - \mathrm{h_H}(x \cdot y) \\ &\subseteq [0, 1] - (\mathrm{h_H}(x) \cap \mathrm{h_H}(y)) \\ &= ([0, 1] - \mathrm{h_H}(x)) \cup ([0, 1] - \mathrm{h_H}(y)) \\ &= \mathrm{h}_{\overline{\mathrm{H}}}(x) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y). \end{aligned}$$

Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A.

Theorem 16. *A hesitant fuzzy set $\overline{\mathrm{H}}$ on A is an anti-hesitant fuzzy UP-filter of A if and only if for all $\varepsilon \in \mathcal{P}([0, 1])$, a nonempty subset $U(\mathrm{H}; \varepsilon)$ of A is a UP-filter of A.*

Proof. Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0, 1])$ be such that $U(\mathrm{H}; \varepsilon) \neq \emptyset$, and let $x \in A$ be such that $x \in U(\mathrm{H}; \varepsilon)$. Then $\mathrm{h_H}(x) \supseteq \varepsilon$. Since

$\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Thus $[0,1] - \mathrm{h}_{\mathrm{H}}(0) \subseteq [0,1] - \mathrm{h}_{\mathrm{H}}(x)$. Therefore, $\mathrm{h}_{\mathrm{H}}(0) \supseteq \mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Hence, $0 \in U(\mathrm{h}_{\mathrm{H}}; \varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in U(\mathrm{H}; \varepsilon)$ and $x \in U(\mathrm{H}; \varepsilon)$. Then $\mathrm{h}_{\mathrm{H}}(x \cdot y) \supseteq \varepsilon$ and $\mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$. By Lemma 1 (2), we have $[0,1] - \mathrm{h}_{\mathrm{H}}(y) \subseteq ([0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot y)) \cup ([0,1] - \mathrm{h}_{\mathrm{H}}(x)) = [0,1] - (\mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x))$. Thus $\mathrm{h}_{\mathrm{H}}(y) \supseteq \mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Therefore, $y \in U(\mathrm{H}; \varepsilon)$. Hence, $U(\mathrm{H}; \varepsilon)$ is a UP-filter of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H}; \varepsilon)$ of A is a UP-filter of A. Let $x \in A$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Thus $x \in U(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H}; \varepsilon)$ is a UP-filter of A and so $0 \in U(\mathrm{H}; \varepsilon)$. Therefore, $\mathrm{h}_{\mathrm{H}}(0) \supseteq \varepsilon = \mathrm{h}_{\mathrm{H}}(x)$. Hence, $\mathrm{h}_{\overline{\mathrm{H}}}(0) = [0,1] - \mathrm{h}_{\mathrm{H}}(0) \subseteq [0,1] - \mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\overline{\mathrm{H}}}(x)$.

Next, let $x, y \in A$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x \cdot y) \supseteq \varepsilon$ and $\mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Thus $x \cdot y, x \in U(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H}; \varepsilon)$ is a UP-filter of A and so $y \in U(\mathrm{H}; \varepsilon)$. Therefore, $\mathrm{h}_{\mathrm{H}}(y) \supseteq \varepsilon = \mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x)$. By Lemma 1 (2), we have

$$\begin{aligned}
\mathrm{h}_{\overline{\mathrm{H}}}(y) &= [0,1] - \mathrm{h}_{\mathrm{H}}(y) \\
&\subseteq [0,1] - (\mathrm{h}_{\mathrm{H}}(x \cdot y) \cap \mathrm{h}_{\mathrm{H}}(x)) \\
&= ([0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot y)) \cup ([0,1] - \mathrm{h}_{\mathrm{H}}(x)) \\
&= \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x).
\end{aligned}$$

Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A.

Theorem 17. *A hesitant fuzzy set $\overline{\mathrm{H}}$ on A is an anti-hesitant fuzzy UP-ideal of A if and only if for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H}; \varepsilon)$ of A is a UP-ideal of A.*

Proof. Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U(\mathrm{H}; \varepsilon) \neq \emptyset$, and let $x \in A$ be such that $x \in U(\mathrm{H}; \varepsilon)$. Then $\mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Thus $[0,1] - \mathrm{h}_{\mathrm{H}}(0) \subseteq [0,1] - \mathrm{h}_{\mathrm{H}}(x)$. Therefore, $\mathrm{h}_{\mathrm{H}}(0) \supseteq \mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Hence, $0 \in U(\mathrm{H}; \varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(\mathrm{H}; \varepsilon)$ and $y \in U(\mathrm{H}; \varepsilon)$. Then $\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \supseteq \varepsilon$ and $\mathrm{h}_{\mathrm{H}}(y) \supseteq \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A, we obtain $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 1 (2), we have $[0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot z) \subseteq ([0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z))) \cup ([0,1] - \mathrm{h}_{\mathrm{H}}(y)) = [0,1] - (\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y))$. Thus $\mathrm{h}_{\mathrm{H}}(x \cdot z) \supseteq \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\mathrm{H}}(y) \supseteq \varepsilon$. Therefore, $x \cdot z \in U(\mathrm{H}; \varepsilon)$. Hence, $U(\mathrm{H}; \varepsilon)$ is a UP-ideal of A.

Conversely, assume that for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U(\mathrm{H}; \varepsilon)$ of A is a UP-ideal of A. Let $x \in A$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(x) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x) \supseteq \varepsilon$. Thus $x \in U(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H}; \varepsilon)$ is a UP-ideal of A and so $0 \in U(\mathrm{H}; \varepsilon)$. Therefore, $\mathrm{h}_{\mathrm{H}}(0) \supseteq \varepsilon = \mathrm{h}_{\mathrm{H}}(x)$. Hence, $\mathrm{h}_{\overline{\mathrm{H}}}(0) = [0,1] - \mathrm{h}_{\mathrm{H}}(0) \subseteq [0,1] - \mathrm{h}_{\mathrm{H}}(x) = \mathrm{h}_{\overline{\mathrm{H}}}(x)$.

Next, let $x, y, z \in A$. Choose $\varepsilon = \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \supseteq \varepsilon$ and $\mathrm{h}_{\mathrm{H}}(y) \supseteq \varepsilon$. Thus $x \cdot (y \cdot z), y \in U(\mathrm{H}; \varepsilon) \neq \emptyset$. By assumption, we have $U(\mathrm{H}; \varepsilon)$ is a UP-ideal of A and so $x \cdot z \in U(\mathrm{H}; \varepsilon)$. Therefore, $\mathrm{h}_{\mathrm{H}}(x \cdot z) \supseteq \varepsilon = \mathrm{h}_{\mathrm{H}}(x \cdot (y \cdot z)) \cap \mathrm{h}_{\mathrm{H}}(y)$. By Lemma 1 (2), we have

$$\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) = [0,1] - \mathrm{h}_{\mathrm{H}}(x \cdot z)$$

$$\subseteq [0,1] - (\mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y))$$
$$= ([0,1] - \mathrm{h_H}(x \cdot (y \cdot z))) \cup ([0,1] - \mathrm{h_H}(y))$$
$$= \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y).$$

Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A.

Theorem 18. *Let* H *be a hesitant fuzzy set on* A. *Then the following statements are equivalent*:

(1) $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy strongly UP-ideal of* A*,*

(2) *a nonempty subset* $U(\mathrm{H};\varepsilon)$ *of* A *is a strongly UP-ideal of* A *for all* $\varepsilon \in \mathcal{P}([0,1])$*, and*

(3) *a nonempty subset* $L(\mathrm{H};\varepsilon)$ *of* A *is a strongly UP-ideal of* A *for all* $\varepsilon \in \mathcal{P}([0,1])$.

Proof. It is straightforward by Theorem 10 and Corollary 2.

5.4. Upper ε-Strong Level Subsets

Theorem 19. *Let* H *be a hesitant fuzzy set on* A. *Then the following statements hold*:

(1) *if* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-subalgebra of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $U^+(\mathrm{H};\varepsilon)$ *is a UP-subalgebra of* A *if* $U^+(\mathrm{H};\varepsilon)$ *is nonempty, and*

(2) *if* $\mathrm{Im}(\mathrm{H})$ *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $U^+(\mathrm{H};\varepsilon)$ *of* A *is a UP-subalgebra of* A*, then* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-subalgebra of* A.

Proof. (1) Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^+(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x, y \in A$ be such that $x \in U^+(\mathrm{H};\varepsilon)$ and $y \in U^+(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \supset \varepsilon$ and $\mathrm{h_H}(y) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A, we obtain $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 1 (2), we have $[0,1] - \mathrm{h_H}(x \cdot y) \subseteq ([0,1]-\mathrm{h_H}(x)) \cup ([0,1]-\mathrm{h_H}(y)) = [0,1]-(\mathrm{h_H}(x) \cap \mathrm{h_H}(y))$. Thus $\mathrm{h_H}(x \cdot y) \supseteq \mathrm{h_H}(x) \cap \mathrm{h_H}(y) \supset \varepsilon$. Therefore, $x \cdot y \in U^+(\mathrm{H};\varepsilon)$. Hence, $U^+(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A.

(2) Assume that $\mathrm{Im}(\mathrm{H})$ is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{H};\varepsilon)$ of A is a UP-subalgebra of A. Assume that there exist $x, y \in A$ such that $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \nsubseteq \mathrm{h}_{\overline{\mathrm{H}}}(x) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. Since $\mathrm{Im}(\mathrm{H})$ is a chain, we have $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \supset \mathrm{h}_{\overline{\mathrm{H}}}(x) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 1 (2), we have $[0,1] - \mathrm{h_H}(x \cdot y) \supset ([0,1] - \mathrm{h_H}(x)) \cup ([0,1] - \mathrm{h_H}(y)) = [0,1] - (\mathrm{h_H}(x) \cap \mathrm{h_H}(y))$. Thus $\mathrm{h_H}(x \cdot y) \subset \mathrm{h_H}(x) \cap \mathrm{h_H}(y)$. Choose $\varepsilon = \mathrm{h_H}(x \cdot y) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \supset \varepsilon$ and $\mathrm{h_H}(y) \supset \varepsilon$. Thus $x, y \in U^+(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U^+(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A and so $x \cdot y \in U^+(\mathrm{H};\varepsilon)$. Thus $\mathrm{h_H}(x \cdot y) \supset \varepsilon = \mathrm{h_H}(x \cdot y)$, a contradiction. Therefore, $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$ for all $x, y \in A$. Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-subalgebra of A.

Example 12. *Let* $A = \{0,1,2,3,4\}$ *be a set with a binary operation* $\cdot$ *defined by the Cayley table from Example 8. Then* $(A, \cdot, 0)$ *is a UP-algebra. We define a hesitant fuzzy set* H *on* A *as follows:*

$$\mathrm{h_H}(0) = \{0,1\}, \mathrm{h_H}(1) = \{1\}, \mathrm{h_H}(2) = \{0\}, \mathrm{h_H}(3) = \{1\}, \textit{ and } \mathrm{h_H}(4) = \emptyset.$$

Then Im(H) *is not a chain. If* $\varepsilon = \{1\}$ *or* $\varepsilon = \{0\}$*, then* $U^+(\mathrm{H};\varepsilon) = \{0\}$*. If* $\varepsilon = \emptyset$*, then* $U^+(\mathrm{H};\varepsilon) = \{0,1,3\}$*. Otherwise,* $U^+(\mathrm{H};\varepsilon) = \emptyset$*. Using this data, we can show that all nonempty subset* $U^+(\mathrm{H};\varepsilon)$ *of* A *is a UP-subalgebra of* A*. By Definition 4, we have*

$$\mathrm{h}_{\overline{\mathrm{H}}}(0) = (0,1), \mathrm{h}_{\overline{\mathrm{H}}}(1) = [0,1), \mathrm{h}_{\overline{\mathrm{H}}}(2) = (0,1], \mathrm{h}_{\overline{\mathrm{H}}}(3) = [0,1), \textit{ and } \mathrm{h}_{\overline{\mathrm{H}}}(3) = [0,1].$$

Since $\mathrm{h}_{\overline{\mathrm{H}}}(3\cdot 1) = \mathrm{h}_{\overline{\mathrm{H}}}(2) = (0,1] \nsubseteq [0,1) = \mathrm{h}_{\overline{\mathrm{H}}}(3) \cup \mathrm{h}_{\overline{\mathrm{H}}}(1)$*, we have* H *is not an anti-hesitant fuzzy UP-subalgebra of* A*.*

Theorem 20. *Let* H *be a hesitant fuzzy set on* A*. Then the following statements hold*:

(1) *if* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-filter of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $U^+(\mathrm{H};\varepsilon)$ *is a UP-filter of* A *if* $U^+(\mathrm{H};\varepsilon)$ *is nonempty, and*

(2) *if* Im(H) *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $U^+(\mathrm{H};\varepsilon)$ *of* A *is a UP-filter of* A*, then* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-filter of* A*.*

Proof. (1) Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^+(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x \in A$ be such that $x \in U^+(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Thus $[0,1] - \mathrm{h_H}(0) \subseteq [0,1] - \mathrm{h_H}(x)$. Therefore, $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x) \supset \varepsilon$. Hence, $0 \in U^+(\mathrm{h_H};\varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in U^+(\mathrm{H};\varepsilon)$ and $x \in U^+(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x\cdot y) \supset \varepsilon$ and $\mathrm{h_H}(x) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$. By Lemma 1 (2), we have $[0,1]-\mathrm{h_H}(y) \subseteq ([0,1]-\mathrm{h_H}(x\cdot y)) \cup ([0,1]-\mathrm{h_H}(x)) = [0,1] - (\mathrm{h_H}(x\cdot y) \cap \mathrm{h_H}(x))$. Thus $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x\cdot y) \cap \mathrm{h_H}(x) \supset \varepsilon$. Therefore, $y \in U^+(\mathrm{H};\varepsilon)$. Hence, $U^+(\mathrm{H};\varepsilon)$ is a UP-filter of A.

(2) Assume that Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{H};\varepsilon)$ of A is a UP-filter of A. Assume that there exists $x \in A$ such that $\mathrm{h}_{\overline{\mathrm{H}}}(0) \nsubseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Since Im(H) is a chain, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \supset \mathrm{h}_{\overline{\mathrm{H}}}(x)$. and thus $[0,1] - \mathrm{h_H}(0) \supset [0,1] - \mathrm{h_H}(x)$. So $\mathrm{h_H}(0) \subset \mathrm{h_H}(x)$. Choose $\varepsilon = \mathrm{h_H}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \supset \varepsilon$. Thus $x \in U^+(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U^+(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $0 \in L^-(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(0) \supset \varepsilon = \mathrm{h_H}(0)$, a contradiction. Hence, $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$ for all $x \in A$.

Next, assume that there exist $x, y \in A$ such that $\mathrm{h}_{\overline{\mathrm{H}}}(y) \nsubseteq \mathrm{h}_{\overline{\mathrm{H}}}(x\cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Since Im(H) is a chain, we have $\mathrm{h}_{\overline{\mathrm{H}}}(y) \supset \mathrm{h}_{\overline{\mathrm{H}}}(x\cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$. By Lemma 1 (2), we have $[0,1]-\mathrm{h_H}(y) \supset ([0,1]-\mathrm{h_H}(x\cdot y)) \cup ([0,1]-\mathrm{h_H}(x)) = [0,1]-(\mathrm{h_H}(x\cdot y) \cap \mathrm{h_H}(x))$. Thus $\mathrm{h_H}(y) \subset \mathrm{h_H}(x\cdot y) \cap \mathrm{h_H}(x)$. Choose $\varepsilon = \mathrm{h_H}(y) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x\cdot y) \supset \varepsilon$ and $\mathrm{h_H}(x) \supset \varepsilon$. Thus $x\cdot y, x \in U^+(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U^+(\mathrm{H};\varepsilon)$ is a UP-filter of A and so $y \in U^+(\mathrm{H};\varepsilon)$. Thus $\mathrm{h_H}(y) \supset \varepsilon = \mathrm{h_H}(y)$, a contradiction. Therefore, $\mathrm{h}_{\overline{\mathrm{H}}}(y) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x\cdot y) \cup \mathrm{h}_{\overline{\mathrm{H}}}(x)$ for all $x, y \in A$. Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-filter of A.

Example 13. *Let* $A = \{0,1,2,3,4\}$ *be a set with a binary operation* $\cdot$ *defined by the Cayley table from Example 9. Then* $(A,\cdot,0)$ *is a UP-algebra. We define a hesitant fuzzy set* H *on* A *as follows:*

$$\mathrm{h_H}(0) = \{0,1\}, \mathrm{h_H}(1) = \{1\}, \mathrm{h_H}(2) = \{0\}, \mathrm{h_H}(3) = \emptyset, \textit{ and } \mathrm{h_H}(4) = \emptyset.$$

Then Im(H) *is not a chain. If* $\varepsilon = \{1\}$ *or* $\varepsilon = \{0\}$*, then* $U^+(\mathrm{H};\varepsilon) = \{0\}$*. If* $\varepsilon = \emptyset$*, then* $U^+(\mathrm{H};\varepsilon) = \{0,1,2\}$*. Otherwise,* $U^+(\mathrm{H};\varepsilon) = \emptyset$*. Using this data, we can show that all nonempty subset* $U^+(\mathrm{H};\varepsilon)$ *of* A *is a UP-filter of* A*. By Definition 4, we have*

$$\mathrm{h}_{\overline{\mathrm{H}}}(0) = (0,1), \mathrm{h}_{\overline{\mathrm{H}}}(1) = [0,1), \mathrm{h}_{\overline{\mathrm{H}}}(2) = (0,1], \mathrm{h}_{\overline{\mathrm{H}}}(3) = [0,1], \textit{ and } \mathrm{h}_{\overline{\mathrm{H}}}(4) = [0,1].$$

Since $\mathrm{h}_{\overline{\mathrm{H}}}(2) = (0,1] \nsubseteq [0,1) = \mathrm{h}_{\overline{\mathrm{H}}}(1) \cup \mathrm{h}_{\overline{\mathrm{H}}}(1) = \mathrm{h}_{\overline{\mathrm{H}}}(1 \cdot 2) \cup \mathrm{h}_{\overline{\mathrm{H}}}(1)$*, we have* $\overline{\mathrm{H}}$ *is not an anti-hesitant fuzzy UP-filter of* A*.*

Theorem 21. *Let* H *be a hesitant fuzzy set on* A*. Then the following statements hold:*

(1) *if* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-ideal of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $U^+(\mathrm{H};\varepsilon)$ *is a UP-ideal of* A *if* $U^+(\mathrm{H};\varepsilon)$ *is nonempty, and*

(2) *if* Im(H) *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $U^+(\mathrm{H};\varepsilon)$ *of* A *is a UP-ideal of* A*, then* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy UP-ideal of* A*.*

Proof. (1) Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^+(\mathrm{H};\varepsilon) \neq \emptyset$, and let $x \in A$ be such that $x \in U(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Thus $[0,1] - \mathrm{h_H}(0) \subseteq [0,1] - \mathrm{h_H}(x)$. Therefore, $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x) \supset \varepsilon$. Hence, $0 \in U^+(\mathrm{H};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U^+(\mathrm{H};\varepsilon)$ and $y \in U^+(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \supset \varepsilon$ and $\mathrm{h_H}(y) \supset \varepsilon$. Since $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A, we obtain $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 1 (2), we have $[0,1] - \mathrm{h_H}(x \cdot z) \subseteq ([0,1] - \mathrm{h_H}(x \cdot (y \cdot z))) \cup ([0,1] - \mathrm{h_H}(y)) = [0,1] - (\mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y))$. Thus $\mathrm{h_H}(x \cdot z) \supseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y) \supset \varepsilon$. Therefore, $x \cdot z \in U^+(\mathrm{H};\varepsilon)$. Hence, $U^+(\mathrm{H};\varepsilon)$ is a UP-ideal of A.

(2) Assume that Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A. Assume that there exists $x \in A$ such that $\mathrm{h}_{\overline{\mathrm{H}}}(0) \nsubseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Since Im(H) is a chain, we have $\mathrm{h}_{\overline{\mathrm{H}}}(0) \supset \mathrm{h}_{\overline{\mathrm{H}}}(x)$. Then $[0,1] - \mathrm{h_H}(0) \supset [0,1] - \mathrm{h_H}(x)$. Thus $\mathrm{h_H}(0) \subset \mathrm{h_H}(x)$. Choose $\varepsilon = \mathrm{h_H}(0) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x) \supset \varepsilon$. Thus $x \in U^+(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U^+(\mathrm{H};\varepsilon)$ is a UP-ideal of A and so $0 \in U^+(\mathrm{H};\varepsilon)$. Therefore, $\mathrm{h_H}(0) \supset \varepsilon = \mathrm{h_H}(0)$, a contradiction. Hence, $\mathrm{h}_{\overline{\mathrm{H}}}(0) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x)$ for any $x \in A$.

Next, assume that there exist $x, y, z \in A$ such that $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \nsubseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. Since Im(H) is a chain, we have $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \supset \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$. By Lemma 1 (2), we have $[0,1] - \mathrm{h_H}(x \cdot z) \supset ([0,1] - \mathrm{h_H}(x \cdot (y \cdot z))) \cup ([0,1] - \mathrm{h_H}(y)) = [0,1] - (\mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y))$. Thus $\mathrm{h_H}(x \cdot z) \subset \mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y)$. Choose $\varepsilon = \mathrm{h_H}(x \cdot z) \in \mathcal{P}([0,1])$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) \supset \varepsilon$ and $\mathrm{h_H}(y) \supset \varepsilon$. Thus $x \cdot (y \cdot z), y \in U^+(\mathrm{H};\varepsilon) \neq \emptyset$. By assumption, we have $U^+(\mathrm{H};\varepsilon)$ is a UP-ideal of A and so $x \cdot z \in L^-(\mathrm{H};\varepsilon)$. Thus, $\mathrm{h_H}(x \cdot z) \supset \varepsilon = \mathrm{h_H}(x \cdot z)$, a contradiction. Therefore, $\mathrm{h}_{\overline{\mathrm{H}}}(x \cdot z) \subseteq \mathrm{h}_{\overline{\mathrm{H}}}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(y)$ for all $x, y, z \in A$. Hence, $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy UP-ideal of A.

Example 14. *Let* $A = \{0,1,2,3,4\}$ *be a set with a binary operation* $\cdot$ *defined by the Cayley table from Example 10. Then* $(A, \cdot, 0)$ *is a UP-algebra. We define a hesitant fuzzy set* H *on* A *as follows:*

$$\mathrm{h_H}(0) = \{0,1\}, \mathrm{h_H}(1) = \{1\}, \mathrm{h_H}(2) = \emptyset, \mathrm{h_H}(3) = \{0\}, \text{ and } \mathrm{h_H}(4) = \emptyset.$$

Then Im(H) *is not a chain. If* $\varepsilon = \{1\}$ *or* $\varepsilon = \{0\}$*, then* $U^+(\mathrm{H};\varepsilon) = \{0\}$*. If* $\varepsilon = \emptyset$*, then* $U^+(\mathrm{H};\varepsilon) = \{0,1,3\}$*. Otherwise,* $U^+(\mathrm{H};\varepsilon) = \emptyset$*. Using this data, we can show that all nonempty subset* $U^+(\mathrm{H};\varepsilon)$ *of* A *is a UP-ideal of* A*. By Definition 4, we have*

$$\mathrm{h}_{\overline{\mathrm{H}}}(0) = (0,1), \mathrm{h}_{\overline{\mathrm{H}}}(1) = [0,1), \mathrm{h}_{\overline{\mathrm{H}}}(2) = [0,1], \mathrm{h}_{\overline{\mathrm{H}}}(3) = (0,1], \text{ and } \mathrm{h}_{\overline{\mathrm{H}}}(4) = [0,1].$$

Since $\mathrm{h}_{\overline{\mathrm{H}}}(0 \cdot 1) = \mathrm{h}_{\overline{\mathrm{H}}}(1) = [0,1) \nsubseteq (0,1] = \mathrm{h}_{\overline{\mathrm{H}}}(0) \cup \mathrm{h}_{\overline{\mathrm{H}}}(3) = \mathrm{h}_{\overline{\mathrm{H}}}(0 \cdot (3 \cdot 1)) \cup \mathrm{h}_{\overline{\mathrm{H}}}(3)$*, we have* $\overline{\mathrm{H}}$ *is not an anti-hesitant fuzzy UP-ideal of* A*.*

Theorem 22. *Let* H *be a hesitant fuzzy set on* A*. Then the following statements hold*:

(1) *if* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy strongly UP-ideal of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*,* $U^+(\mathrm{H};\varepsilon)$ *is a strongly UP-ideal of* A *if* $U^+(\mathrm{H};\varepsilon)$ *is nonempty, and*

(2) *if* Im(H) *is a chain and for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $U^+(\mathrm{H};\varepsilon)$ *of* A *is a strongly UP-ideal of* A*, then* $\overline{\mathrm{H}}$ *is an anti-hesitant fuzzy strongly UP-ideal of* A*.*

Proof. (1) Assume that $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 3, we obtain $\overline{\mathrm{H}}$ is a constant hesitant fuzzy set on A. By Corollary 2, we have H is a constant hesitant fuzzy set on A and so $\mathrm{h_H}(x) = \mathrm{h_H}(y)$ for all $x, y \in A$. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $U^+(\mathrm{H};\varepsilon) \neq \emptyset$. There exists $a \in U^+(\mathrm{H};\varepsilon)$ be such that $\mathrm{h_H}(a) \supset \varepsilon$. Thus $\mathrm{h_H}(x) = \mathrm{h_H}(a) \supset \varepsilon$ for all $x \in A$ and so $x \in U^+(\mathrm{H};\varepsilon)$ for all $x \in A$. Therefore, $U^+(\mathrm{H};\varepsilon) = A$. Hence, $U^+(\mathrm{H};\varepsilon)$ is a strongly UP-ideal of A.

(2) Assume that Im(H) is a chain and for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $U^+(\mathrm{H};\varepsilon)$ of A is a strongly UP-ideal of A. Assume that $\overline{\mathrm{H}}$ is not a constant hesitant fuzzy set on A. By Corollary 2, we have H is not a constant hesitant fuzzy set on A. There exist $x, y \in A$ be such that $\mathrm{h_H}(x) \neq \mathrm{h_H}(y)$. Since Im(H) is a chain, we have $\mathrm{h_H}(x) \subset \mathrm{h_H}(y)$ or $\mathrm{h_H}(x) \supset \mathrm{h_H}(y)$. Without loss of generality, assume that $\mathrm{h_H}(x) \subset \mathrm{h_H}(y)$, then $y \in U^+(\mathrm{H};\mathrm{h_H}(x)) \neq \emptyset$. By assumption, we have $U^+(\mathrm{H};\mathrm{h_H}(x))$ is a strongly UP-ideal of A and so $U^+(\mathrm{H};\mathrm{h_H}(x)) = A$. Thus $x \in A = U^+(\mathrm{H};\mathrm{h_H}(x))$ and so $\mathrm{h_H}(x) \subset \mathrm{h_H}(x)$, a contradiction. Therefore, $\overline{\mathrm{H}}$ is a constant hesitant fuzzy set on A. By Theorem 3, we obtain $\overline{\mathrm{H}}$ is an anti-hesitant fuzzy strongly UP-ideal of A.

Example 15. *Let* $A = \{0,1\}$ *be a set with a binary operation* $\cdot$ *defined by the Cayley table from Example 11. Then* $(A, \cdot, 0)$ *is a UP-algebra. We define a hesitant fuzzy set* H *on* A *as follows:*

$$\mathrm{h_H}(0) = \{0\}, \text{ and } \mathrm{h_H}(1) = \{1\}.$$

Then Im(H) *is not a chain. If* $\varepsilon = \emptyset$*, then* $U^+(\mathrm{H};\varepsilon) = A$*. Otherwise,* $U^+(\mathrm{H};\varepsilon) = \emptyset$*. Thus a nonempty subset* $U^+(\mathrm{H};\varepsilon)$ *of* A *is a strongly UP-ideal of* A*. By Definition 4, we have*

$$\mathrm{h}_{\overline{\mathrm{H}}}(0) = (0,1], \text{ and } \mathrm{h}_{\overline{\mathrm{H}}}(1) = [0,1).$$

By Theorem 3 and because $\overline{\mathrm{H}}$ *is not a constant hesitant fuzzy set on* A*, we have* $\overline{\mathrm{H}}$ *is not an anti-hesitant fuzzy strongly UP-ideal of* A*.*

5.5. Equal ε-Level Subsets

Theorem 23. *If a hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy UP-subalgebra of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $E(\mathrm{H};\varepsilon)$ *of* A *is a UP-subalgebra of* A *where* $L^-(\mathrm{H};\varepsilon)$ *is empty.*

Proof. Assume that H is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $E(\mathrm{H};\varepsilon) \neq \emptyset$ but $L^-(\mathrm{H};\varepsilon) = \emptyset$, and let $x, y \in A$ be such that $x \in E(\mathrm{H};\varepsilon)$ and $y \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$ and $\mathrm{h_H}(y) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-subalgebra of A, we have $\mathrm{h_H}(x \cdot y) \subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y) = \varepsilon$. Thus $x \cdot y \in L(\mathrm{H};\varepsilon)$. Since $L^-(\mathrm{H};\varepsilon)$ is empty, we obtain $L(\mathrm{H};\varepsilon) = L^-(\mathrm{H};\varepsilon) \cup E(\mathrm{H};\varepsilon) = \emptyset \cup E(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$. Therefore, $x \cdot y \in E(\mathrm{H};\varepsilon)$. Hence, $E(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A.

The following example show that the converse of Theorem 23 is not true in general.

Example 16. *Let* $A = \{0, 1, 2, 3\}$ *be a set with a binary operation* $\cdot$ *defined by the following Cayley table:*

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	1	0

Then $(A, \cdot, 0)$ *is a UP-algebra. We define a hesitant fuzzy set* H *on* A *as follows:*

$$\mathrm{h_H}(0) = \emptyset, \mathrm{h_H}(1) = [0, 0.6], \mathrm{h_H}(2) = [0, 0.3], \text{ and } \mathrm{h_H}(3) = [0, 0.3].$$

If $\varepsilon \neq \emptyset$*, then* $L^-(\mathrm{H};\varepsilon) \neq \emptyset$*. If* $\varepsilon = \emptyset$*, then* $L^-(\mathrm{H};\varepsilon) = \emptyset$ *and* $E(\mathrm{H};\varepsilon) = \{0\}$*. Thus* $E(\mathrm{H};\varepsilon)$ *is clearly a UP-subalgebra of* A*. Since* $\mathrm{h_H}(3 \cdot 2) = \mathrm{h_H}(1) = [0, 0.6] \not\subseteq [0, 0.3] = \mathrm{h_H}(3) \cup \mathrm{h_H}(2)$*, we have* H *is not an anti-hesitant fuzzy UP-subalgebra of* A*.*

Theorem 24. *If a hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy UP-filter of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $E(\mathrm{H};\varepsilon)$ *of* A *is a UP-filter of* A *where* $L^-(\mathrm{H};\varepsilon)$ *is empty.*

Proof. Assume that H is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $E(\mathrm{H};\varepsilon) \neq \emptyset$ but $L^-(\mathrm{H};\varepsilon) = \emptyset$, and let $x \in A$ be such that $x \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-filter of A, we obtain $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x) = \varepsilon$ and thus $0 \in L(\mathrm{H};\varepsilon)$. Since $L^-(\mathrm{H};\varepsilon)$ is empty, we have $0 \in L(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in E(\mathrm{H};\varepsilon)$ and $x \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot y) = \varepsilon$ and $\mathrm{h_H}(x) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x) = \varepsilon$. Thus $y \in L(\mathrm{H};\varepsilon)$. Since $L^-(\mathrm{H};\varepsilon)$ is empty, we obtain $L(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$. Therefore, $y \in E(\mathrm{H};\varepsilon)$. Hence, $E(\mathrm{H};\varepsilon)$ is a UP-filter of A.

The converse of Theorem 24 is not true in general. By Example 16, we still have $E(\mathrm{H};\varepsilon) = \{0\}$ is a UP-filter of A. Since $\mathrm{h_H}(1) = [0, 0.6] \not\subseteq [0, 0.3] = \mathrm{h_H}(0) \cup \mathrm{h_H}(2) = \mathrm{h_H}(2 \cdot 1) \cup \mathrm{h_H}(2)$, we have H is not an anti-hesitant fuzzy UP-filter of A.

Theorem 25. *If a hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy UP-ideal of* A*, then* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $E(\mathrm{H};\varepsilon)$ *of* A *is a UP-ideal of* A *where* $L^{-}(\mathrm{H};\varepsilon)$ *is empty.*

Proof. Assume that H is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $E(\mathrm{H};\varepsilon) \neq \emptyset$ but $L^{-}(\mathrm{H};\varepsilon) = \emptyset$, and let $x \in A$ be such that $x \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-ideal of A, we obtain $\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x) = \varepsilon$ and thus $0 \in L(\mathrm{H};\varepsilon)$. Since $L^{-}(\mathrm{H};\varepsilon)$ is empty, we have $0 \in L(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in E(\mathrm{H};\varepsilon)$ and $y \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) = \varepsilon$ and $\mathrm{h_H}(y) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y) = \varepsilon$. Thus $x \cdot z \in L(\mathrm{H};\varepsilon)$. Since $L^{-}(\mathrm{H};\varepsilon)$ is empty, we obtain $L(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$. Therefore, $x \cdot z \in E(\mathrm{H};\varepsilon)$. Hence, $E(\mathrm{H};\varepsilon)$ is a UP-ideal of A.

The converse of Theorem 25 is not true in general. By Example 16, we still have $E(\mathrm{H};\varepsilon) = \{0\}$ is a UP-ideal of A. Since $\mathrm{h_H}(0 \cdot 1) = \mathrm{h_H}(1) = [0, 0.6] \nsubseteq [0, 0.3] = \mathrm{h_H}(0) \cup \mathrm{h_H}(2) = \mathrm{h_H}(0 \cdot (2 \cdot 1)) \cup \mathrm{h_H}(2)$, we have H is not an anti-hesitant fuzzy UP-ideal of A.

Theorem 26. *A hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy strongly UP-ideal of* A *if and only if* $E(\mathrm{H};\mathrm{h_H}(0))$ *is a strongly UP-ideal of* A.

Proof. Assume that H is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 3, we obtain H is a constant hesitant fuzzy set on A and so $\mathrm{h_H}(x) = \mathrm{h_H}(0)$ for all $x \in A$. Then $E(\mathrm{H};\mathrm{h_H}(0)) = A$. Hence, $E(\mathrm{H};\mathrm{h_H}(0))$ is a strongly UP-ideal of A.

Conversely, assume that $E(\mathrm{H};\mathrm{h_H}(0))$ is a strongly UP-ideal of A. Then $E(\mathrm{H};\mathrm{h_H}(0)) = A$ and so $\mathrm{h_H}(x) = \mathrm{h_H}(0)$ for all $x \in A$. Therefore, H is a constant hesitant fuzzy set on A. By Theorem 3, H is an anti-hesitant fuzzy strongly UP-ideal of A.

Moreover, we still obtain theorems of equal ε-level subsets with a hesitant fuzzy UP-subalgebra. (resp., hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal)

Theorem 27. *If a hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy UP-subalgebra of* A*, then for all* $\varepsilon \in \mathcal{P}([0,1])$*, a nonempty subset* $E(\mathrm{H};\varepsilon)$ *of* A *is a UP-subalgebra of* A *where* $U^{+}(\mathrm{H};\varepsilon)$ *is empty.*

Proof. Assume that H is an anti-hesitant fuzzy UP-subalgebra of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $E(\mathrm{H};\varepsilon) \neq \emptyset$ but $U^{+}(\mathrm{H};\varepsilon) = \emptyset$, and let $x, y \in A$ be such that $x \in E(\mathrm{H};\varepsilon)$ and $y \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$ and $\mathrm{h_H}(y) = \varepsilon$. Since H is an anti-hesitant fuzzy UP-subalgebra of A, we have $\mathrm{h_H}(x \cdot y) \supseteq \mathrm{h_H}(x) \cap \mathrm{h_H}(y) = \varepsilon$. Thus $x \cdot y \in U(\mathrm{H};\varepsilon)$. Since $U^{+}(\mathrm{H};\varepsilon)$ is empty, we obtain $U(\mathrm{H};\varepsilon) = U^{+}(\mathrm{H};\varepsilon) \cup E(\mathrm{H};\varepsilon) = \emptyset \cup E(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$. Therefore, $x \cdot y \in E(\mathrm{H};\varepsilon)$. Hence, $E(\mathrm{H};\varepsilon)$ is a UP-subalgebra of A.

The following example show that the converse of Theorem 27 is not true in general.

Example 17. *Let* $A = \{0, 1, 2, 3\}$ *be a set with a binary operation* $\cdot$ *defined by the following*

Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	0
3	0	0	1	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a hesitant fuzzy set H *on A as follows:*

$$\mathrm{h_H}(0) = [0,1], \mathrm{h_H}(1) = \{0\}, \mathrm{h_H}(2) = [0,0.1], \textit{ and } \mathrm{h_H}(3) = [0,0.1].$$

If $\varepsilon \neq [0,1]$, then $U^{+}(\mathrm{H};\varepsilon) \neq \emptyset$. If $\varepsilon = [0,1]$, then $U^{+}(\mathrm{H};\varepsilon) = \emptyset$ and $E(\mathrm{H};\varepsilon) = \{0\}$. Thus $E(\mathrm{H};\varepsilon)$ is clearly a UP-subalgebra of A. Since $\mathrm{h_H}(3 \cdot 2) = \mathrm{h_H}(1) = \{0\} \not\supseteq [0,0.1] = \mathrm{h_H}(3) \cap \mathrm{h_H}(2)$, we have H *is not an anti-hesitant fuzzy UP-subalgebra of A.*

Theorem 28. *If a hesitant fuzzy set* H *on A is an anti-hesitant fuzzy UP-filter of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $E(\mathrm{H};\varepsilon)$ of A is a UP-filter of A where $U^{+}(\mathrm{H};\varepsilon)$ is empty.*

Proof. Assume that H is an anti-hesitant fuzzy UP-filter of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $E(\mathrm{H};\varepsilon) \neq \emptyset$ but $U^{+}(\mathrm{H};\varepsilon) = \emptyset$, and let $x \in A$ be such that $x \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-filter of A, we obtain $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x) = \varepsilon$ and thus $0 \in U(\mathrm{H};\varepsilon)$. Since $U^{+}(\mathrm{H};\varepsilon)$ is empty, we have $0 \in U(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$.

Next, let $x, y \in A$ be such that $x \cdot y \in E(\mathrm{H};\varepsilon)$ and $x \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot y) = \varepsilon$ and $\mathrm{h_H}(x) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-filter of A, we have $\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x) = \varepsilon$. Thus $y \in L(\mathrm{H};\varepsilon)$. Since $U^{+}(\mathrm{H};\varepsilon)$ is empty, we obtain $U(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$. Therefore, $y \in E(\mathrm{H};\varepsilon)$. Hence, $E(\mathrm{H};\varepsilon)$ is a UP-filter of A.

The converse of Theorem 28 is not true in general. By Example 17, we still have $E(\mathrm{H};\varepsilon) = \{0\}$ is a UP-filter of A. Since $\mathrm{h_H}(1) = \{0\} \not\supseteq [0,0.1] = \mathrm{h_H}(0) \cap \mathrm{h_H}(3) = \mathrm{h_H}(3 \cdot 1) \cap \mathrm{h_H}(3)$, we have H is not an anti-hesitant fuzzy UP-filter of A.

Theorem 29. *If a hesitant fuzzy set* H *on A is an anti-hesitant fuzzy UP-ideal of A, then for all $\varepsilon \in \mathcal{P}([0,1])$, a nonempty subset $E(\mathrm{H};\varepsilon)$ of A is a UP-ideal of A where $U^{+}(\mathrm{H};\varepsilon)$ is empty.*

Proof. Assume that H is an anti-hesitant fuzzy UP-ideal of A. Let $\varepsilon \in \mathcal{P}([0,1])$ be such that $E(\mathrm{H};\varepsilon) \neq \emptyset$ but $U^{+}(\mathrm{H};\varepsilon) = \emptyset$, and let $x \in A$ be such that $x \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x) = \varepsilon$. Because H is an anti-hesitant fuzzy UP-filter of A, we obtain $\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x) = \varepsilon$ and thus $0 \in U(\mathrm{H};\varepsilon)$. Since $U^{+}(\mathrm{H};\varepsilon)$ is empty, we have $0 \in U(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$.

Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in E(\mathrm{H};\varepsilon)$ and $y \in E(\mathrm{H};\varepsilon)$. Then $\mathrm{h_H}(x \cdot (y \cdot z)) = \varepsilon$ and $\mathrm{h_H}(y) = \varepsilon$. Since H is an anti-hesitant fuzzy UP-ideal of A, we have $\mathrm{h_H}(x \cdot z) \supseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y) = \varepsilon$. Thus $x \cdot z \in U(\mathrm{H};\varepsilon)$. Since $L^{-}(\mathrm{H};\varepsilon)$ is empty, we obtain $U(\mathrm{H};\varepsilon) = E(\mathrm{H};\varepsilon)$. Therefore, $x \cdot z \in E(\mathrm{H};\varepsilon)$. Hence, $E(\mathrm{H};\varepsilon)$ is a UP-ideal of A.

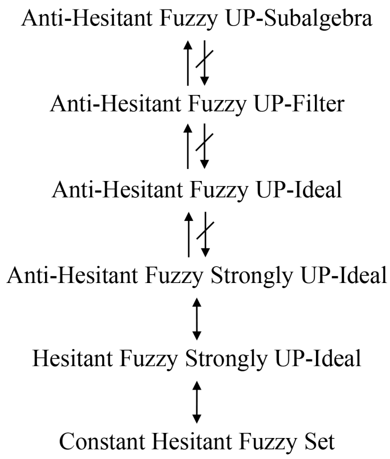
The converse of Theorem 29 is not true in general. By Example 17, we still have $E(\mathrm{H};\varepsilon) = \{0\}$ is a UP-ideal of A. Since $\mathrm{h_H}(3 \cdot 2) = \mathrm{h_H}(1) = \{0\} \not\supseteq [0,0.1] = \mathrm{h_H}(0) \cap \mathrm{h_H}(2) = \mathrm{h_H}(3 \cdot (2 \cdot 2)) \cap \mathrm{h_H}(2)$, we have H is not an anti-hesitant fuzzy UP-ideal of A.

Theorem 30. *A hesitant fuzzy set* H *on* A *is an anti-hesitant fuzzy strongly UP-ideal of* A *if and only if* $E(\mathrm{H}; \mathrm{h}_{\mathrm{H}}(0))$ *is a strongly UP-ideal of* A*.*

Proof. It is straightforward by Theorem 26 and 3.

6. Conclusions and Future Work

In this paper, we have introduced the notion of anti-hesitant fuzzy UP-subalgebras (resp., anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals and anti-hesitant fuzzy strongly UP-ideals) of UP-algebras and investigated some of its important properties. Then we have the diagram of anti-type of hesitant fuzzy sets on UP-algebras below.



In our future study of UP-algebras, may be the following topics should be considered:

- To get more results in anti-hesitant fuzzy UP-subalgebras, anti-hesitant fuzzy UP-filters, anti-hesitant fuzzy UP-ideals, and anti-hesitant fuzzy strongly UP-ideals of UP-algebras.
- To define anti-hesitant fuzzy soft UP-subalgebras, anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals, and anti-hesitant fuzzy soft strongly UP-ideals over UP-algebras.
- To define operations of hesitant fuzzy soft sets over UP-algebras.

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