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Characterizations of non-associative rings by their intuitionistic fuzzy bi-ideals

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Abstract. The purpose of this paper is to initiate and study on the generalization of the fuzzification of ideals in a class of non-associative and non-commutative algebraic structures (LA-ring). We characterize different classes of LA-ring in terms of intuitionistic fuzzy left (resp. right, bi-, generalized bi-, $(1,2)$-) ideals.

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In 1972, a generalization of abelian semigroups initiated by Kazim et al [11]. In ternary commutative (abelian) law: $abc = cba$, they introduced braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup), if it satisfies the left invertive law: $(ab)c = (cb)a$. An LA-semigroup is a midway structure between an abelian semigroup and a groupoid. Ideals in LA-semigroup have been investigated by [16].

In [9] (resp. [4]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [11], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial by Protic et al [16] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Kamran [10], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup S is said to be a left almost group, if there exists left identity $e \in S$ such that $ea = a$ for all $a \in S$ and for every $a \in S$, there exists $b \in S$ such that $ba = e$.

Shah et al [20], initiated the concept of left almost ring (abbreviated as LA-ring) of finitely nonzero functions, which is a generalization of a commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by

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defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Despite the fact that the structure is non-associative and non-commutative, however it possesses properties which usually come across in associative and commutative algebraic structures.

A non-empty subset A of an LA-ring R is called an LA-subring of R if $a - b$ and $ab \in A$ for all $a, b \in A$. A is called a left (resp. right) ideal of R if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R.

An LA-subring A of R is called a bi-ideal of R if $(AR)A \subseteq A$. A non-empty subset A of R is called a generalized bi-ideal of R if $(A, +)$ is an LA-group and $(AR)A \subseteq A$. Every bi-ideal of R is a generalized bi-ideal of R. An LA-subring A of R is called $(1, 2)$-ideal of R if $(AR)A^2 \subseteq A$.

We will initiate the concept of regular (resp. left regular, right regular, $(2, 2)$-regular, left weakly regular, right weakly regular, intra-regular) LA-rings. We will also define the concept of intuitionistic fuzzy left (resp. right, bi-,generalized bi-, $(1, 2)$-) ideals.

We will describe a study of regular (resp. left regular, right regular, $(2, 2)$-regular, left weakly regular, right weakly regular, intra-regular) LA-rings by the properties of intuitionistic fuzzy left (right, bi-, generalized bi-) ideals. In this regard, we will prove that in regular (resp. left weakly regular) LA-rings, the concept of intuitionistic fuzzy (right, two-sided) ideals coincides. We will also show that in right regular (resp. $(2, 2)$-regular, right weakly regular, intra-regular) LA-rings, the concept of intuitionistic fuzzy (left, right, two-sided) ideals coincides. Also in left regular LA-rings with left identity, the concept of intuitionistic fuzzy (left, right, two-sided) ideals coincides. We will also characterize left weakly regular LA-rings in terms of intuitionistic fuzzy right (two-sided, bi-, generalize bi-) ideals.

1. Basic Definitions and Preliminary Results

After the introduction of fuzzy set by Zadeh [22], several researchers explored on the generalization of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1, 2], as a generalization of the notion of fuzzy set.

Liu [13], introduced the concept of fuzzy subrings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings (for example [6, 12, 14, 15, 21]). Gupta et al [6], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [12], characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (right, quasi, bi-) ideals.

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$ [1, 2].

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ in X can be identified to be an ordered pair (μ_A, γ_A) in $I^X \times I^X$, where I^X is the set of all functions from X to $[0, 1]$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

Banerjee et al [3] and Hur et al [7], initiated the notion of intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring. Subsequently many authors studied the intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring by describing the different properties (see [8]). Shah et al [18], have initiated the concept of intuitionistic fuzzy normal LA-subrings of an LA-ring.

We initiate the notion of intuitionistic fuzzy left (resp. right, bi-, generalized bi-,$(1,2)$-) ideals of an LA-ring R.

[18] An intuitionistic fuzzy set (IFS) $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy LA-subring of R if

(1) $\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$,
(2) $\gamma_A(x-y) \leq max\{\gamma_A(x), \gamma_A(y)\}$,
(3) $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$,
(4) $\gamma_A(xy) \leq max\{\gamma_A(x), \gamma_A(y)\}$ for all $x, y \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy left ideal of R if

(1) $\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$,
(2) $\gamma_A(x-y) \leq max\{\gamma_A(x), \gamma_A(y)\}$,
(3) $\mu_A(xy) \geq \mu_A(y)$,
(4) $\gamma_A(xy) \leq \gamma_A(y)$ for all $x, y \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy right ideal of R if

(1) $\mu_A(x-y) \geq min\{\mu_A(x), \mu_A(y)\}$,
(2) $\gamma_A(x-y) \leq max\{\gamma_A(x), \gamma_A(y)\}$,
(3) $\mu_A(xy) \geq \mu_A(x)$,
(4) $\gamma_A(xy) \leq \gamma_A(x)$ for all $x, y \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy ideal of R if it is both an intuitionistic fuzzy left ideal and an intuitionistic fuzzy right ideal of R.

Example 1. *Let* $R = \{a, b, c, d\}$. *Define* $+$ *and* $\cdot$ *in* R *as follows* :

$+$	a	b	c	d
a	a	b	c	d
b	d	a	b	c
c	c	d	a	b
d	b	c	d	a

and

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

Then R *is an LA-ring and* $A = (\mu_A, \gamma_A)$ *be an IFS of* R. *We define* $\mu_A(a) = \mu_A(c) = 0.7$, $\mu_A(b) = \mu_A(d) = 0$ *and* $\gamma_A(a) = \gamma_A(c) = 0$, $\gamma_A(b) = \gamma_A(d) = 0.7$. *Then* $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy ideal of* R.

Every intuitionistic fuzzy left (resp. right, two-sided) ideal of an LA-ring R is an intuitionistic fuzzy LA-subring of R, but the converse is not true.

Example 2. $R = \{0,1,2,3,4,5,6,7\}$ *is an LA-ring.*

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	0	3	1	6	4	7	5
2	1	3	0	2	5	7	4	6
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	6	4	7	5	2	0	3	1
6	5	7	4	6	1	3	0	2
7	7	6	5	4	3	2	1	0

and

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	4	4	0	0	4	4	0
2	0	4	4	0	0	4	4	0
3	0	0	0	0	0	0	0	0
4	0	3	3	0	0	3	3	0
5	0	7	7	0	0	7	7	0
6	0	7	7	0	0	7	7	0
7	0	3	3	0	0	3	3	0

Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R. We define $\mu_A(0) = \mu_A(4) = 0.7$, $\mu_A(1) = \mu_A(2) = \mu_A(3) = \mu_A(5) = \mu_A(6) = \mu_A(7) = 0$ and $\gamma_A(0) = \gamma_A(4) = 0$, $\gamma_A(1) = \gamma_A(2) = \gamma_A(3) = \gamma_A(5) = \gamma_A(6) = \gamma_A(7) = 0.7$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy LA-subring of R, but not an intuitionistic fuzzy right ideal of R, because

$$\begin{aligned} \mu_A(41) &= \mu_A(3) = 0. \\ \mu_A(4) &= 0.7. \\ &\Rightarrow \mu_A(41) \ngeq \mu_A(4). \\ \text{and } \gamma_A(41) &= \gamma_A(3) = 0.7. \\ \gamma_A(4) &= 0. \\ &\Rightarrow \gamma_A(41) \nleq \gamma_A(4). \end{aligned}$$

An Intuitionistic fuzzy LA-subring $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy bi-ideal of R if

(1) $\mu_A((xy)z) \geq min\{\mu_A(x), \mu_A(z)\}$,

(2) $\gamma_A((xy)z) \leq max\{\gamma_A(x), \gamma_A(y)\}$ for all $x, y, z \in R$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy generalized bi-ideal of R if

(1) $\mu_A(x - y) \geq min\{\mu_A(x), \mu_A(y)\}$,

(2) $\gamma_A(x - y) \leq max\{\gamma_A(x), \gamma_A(y)\}$,

(3) $\mu_A((xy)z) \geq min\{\mu_A(x), \mu_A(z)\}$,

(4) $\gamma_A((xy)z) \leq max\{\gamma_A(x), \gamma_A(z)\}$ for all $x, y, z \in R$.

An intuitionistic fuzzy LA-subring $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy $(1,2)$-ideal of R if

(1) $\mu_A((xw)(yz)) \geq min\{\mu_A(x), \mu_A(y), \mu_A(z)\}$,

(2) $\gamma_A((xw)(yz)) \leq max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\}$ for all $x, y, z, w \in R$.

We note that an LA-ring R can be considered an intuitionistic fuzzy set of itself and we write $R = I_R$, i.e., $R = (\mu_R, \gamma_R) = (1, 0)$ for all $x \in R$.

Let A and B be two intuitionistic fuzzy sets of an LA-ring R. Then

(1) $A \subseteq B \Leftrightarrow \mu_A \subseteq \mu_B$ and $\gamma_A \supseteq \gamma_B$,

(2) $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$,

(3) $A^c = (\gamma_A, \mu_A)$,
(4) $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B) = (\mu_{A\wedge B}, \gamma_{A\vee B})$,
(5) $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B) = (\mu_{A\vee B}, \gamma_{A\wedge B})$,
(6) $0 \sim= (0,1), 1 \sim= (1,0)$.

[18] Let A be a non-empty subset of an LA-ring R. Then the intuitionistic characteristic of A is denoted by $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ and defined by

$$\mu_{\chi_A}(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases} \text{ and } \gamma_{\chi_A}(x) = \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \notin A \end{cases}$$

The product of $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ is denoted by $A \circ B = (\mu_A \circ \mu_B, \gamma_A \circ \gamma_B)$ and defined by:

$$(\mu_A \circ \mu_B)(x) = \begin{cases} \vee_{x=\sum\limits_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n}\{\mu_A(a_i) \wedge \mu_B(b_i)\}\} & \text{if } x = \sum\limits_{i=1}^{n} a_i b_i,\ a_i, b_i \in R \\ 0 & \text{if } x \neq \sum\limits_{i=1}^{n} a_i b_i \end{cases}$$

$$\text{and } (\gamma_A \circ \gamma_B)(x) = \begin{cases} \wedge_{x=\sum\limits_{i=1}^{n} a_i b_i} \{\vee_{i=1}^{n}\{\gamma_A(a_i) \vee \gamma_B(b_i)\}\} & \text{if } x = \sum\limits_{i=1}^{n} a_i b_i,\ a_i, b_i \in R \\ 1 & \text{if } x \neq \sum\limits_{i=1}^{n} a_i b_i \end{cases}$$

Now we are giving the some fundamental properties, which will be very helpful for next section.

Theorem 1. *Let A and B be two non-empty subsets of an LA-ring R. Then the following conditions hold.*
(1) *If $A \subseteq B$ then $\chi_A \subseteq \chi_B$.*
(2) $\chi_A \circ \chi_B = \chi_{AB}$.
(3) $\chi_A \cup \chi_B = \chi_{A\cup B}$.
(4) $\chi_A \cap \chi_B = \chi_{A\cap B}$.

Proof. Straight forward.

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy sets of an LA-ring R. The sum of A and B is denoted by $A + B = (\mu_A + \mu_B, \gamma_A + \gamma_B)$ and defined by

$$\begin{aligned} (\mu_A + \mu_B)(x) &= \vee_{x=y+z}(\mu_A(y) \wedge \mu_B(z)) \\ \text{and } (\gamma_A + \gamma_B)(x) &= \wedge_{x=y+z}(\gamma_A(y) \vee \gamma_B(z)), \text{ for all } x \in R. \end{aligned}$$

Lemma 1. *Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy sets of an LA-ring R. Then $A + B$ is also an intuitionistic fuzzy set of R.*

Proof. It is sufficient to show that $0 \leq (\mu_A + \mu_B)(x) + (\gamma_A + \gamma_B)(x) \leq 1$ for all $x \in R$. Now

$$(\mu_A + \mu_B)(x) = \vee_{x=y+z}(\mu_A(y) \wedge \mu_B(z)) \leq \vee_{x=y+z}((1 - \gamma_A(y)) \wedge (1 - \gamma_B(z)))$$

$$= 1 - \wedge_{x=y+z}(\gamma_A(y) \vee \gamma_B(z)) = 1 - (\gamma_A + \gamma_B)(x).$$

Since $\mu_A(y) \leq 1 - \gamma_A(y)$ and $\mu_A(z) \leq 1 - \gamma_A(z)$ for all $y, z \in R$. Hence $A + B$ is an intuitionistic fuzzy set of R.

Lemma 2. *Every intuitionistic fuzzy left (resp. right, two-sided) ideal of an LA-ring R is an intuitionistic fuzzy bi-ideal of R.*

Proof. Straight forward.

Lemma 3. *Every intuitionistic fuzzy bi-ideal of an LA-ring R is an intuitionistic fuzzy $(1, 2)$-ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy bi-ideal of R and $a, x, y, z \in R$. Thus

$$\begin{aligned} \mu_A((xa)(yz)) &\geq min\{\mu_A(x), \mu_A(yz)\} \geq min\{\mu_A(x), \mu_A(y), \mu_A(z)\} \\ \text{and } \gamma_A((xa)(yz)) &\leq max\{\gamma_A(x), \gamma_A(yz)\} \leq max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\}. \end{aligned}$$

Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $(1, 2)$-ideal of R.

Remark 1. *Every intuitionistic fuzzy left (resp. right, two-sided) ideal of an LA-ring R is an intuitionistic fuzzy $(1, 2)$-ideal of R.*

Proposition 1. *Let R be an LA-ring having the property $a = a^2$ for every $a \in R$. Then every intuitionistic fuzzy $(1, 2)$-ideal of R is an intuitionistic fuzzy bi-ideal of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $(1, 2)$-ideal of R and $a, x, y \in R$. Thus

$$\begin{aligned} \mu_A((xa)y) &= \mu_A((xa)(yy)) \geq min\{\mu_A(x), \mu_A(y), \mu_A(y)\} \\ &= min\{\mu_A(x), \mu_A(y)\} \\ \text{and } \gamma_A((xa)y) &= \gamma_A((xa)(yy)) \leq max\{\gamma_A(x), \gamma_A(y), \gamma_A(y)\} \\ &= max\{\gamma_A(x), \gamma_A(y)\}. \end{aligned}$$

Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of R.

Theorem 2. *If $\{A_i\}_{i \in I}$ is a family of intuitionistic fuzzy $(1, 2)$-ideals of an LA-ring R, then $\cap A_i$ is also an intuitionistic fuzzy $(1, 2)$-ideal of R, where $\cap A_i = (\wedge \mu_{A_i}, \vee \gamma_{A_i})$ and*

$$\begin{aligned} \wedge \mu_{A_i}(x) &= inf\,\{\mu_{A_i}(x) \mid i \in I,\ x \in R\} \\ \text{and } \vee \gamma_{A_i}(x) &= sup\,\{\gamma_{A_i}(x) \mid i \in I,\ x \in R\}. \end{aligned}$$

Proof. Straight forward.

Remark 2. *Intersection of a family of intuitionistic fuzzy bi-ideals of an LA-ring R, is also an intuitionistic fuzzy bi-ideal of R.*

Lemma 4. *[18] Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is an LA-subring of R if and only if the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy LA-subring of R.*

Proposition 2. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a left (resp. right) ideal of R if and only if the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy left (resp. right) ideal of R.*

Proof. Straight forward.

Theorem 3. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a $(1,2)$-ideal of R if and only if the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy $(1,2)$-ideal of R.*

Proof. Let A be a $(1,2)$-ideal of R, this implies that A is an LA-subring of R. Then χ_A is an intuitionistic fuzzy LA-subring of R by the Lemma 4. Let $a, x, y, z \in R$. If $x, y, z \in A$, then by definition of intuitionistic characteristic function $\mu_{\chi_A}(x) = 1 = \mu_{\chi_A}(y) = \mu_{\chi_A}(z)$ and $\gamma_{\chi_A}(x) = 0 = \gamma_{\chi_A}(y) = \mu_{\chi_A}(z)$. Since $(xa)(yz) \in A$, A being a $(1,2)$-ideal of R, so $\mu_{\chi_A}((xa)(yz)) = 1$ and $\gamma_{\chi_A}((xa)(yz)) = 0$. Thus

$$\begin{aligned} \mu_{\chi_A}((xa)(yz)) &\geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \mu_{\chi_A}(z)\} \\ \text{and } \gamma_{\chi_A}((xa)(yz)) &\leq max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), \gamma_{\chi_A}(z)\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mu_{\chi_A}((xa)(yz)) &\geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \mu_{\chi_A}(z)\} \\ \text{and } \gamma_{\chi_A}((xa)(yz)) &\leq max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), \gamma_{\chi_A}(z)\}, \end{aligned}$$

when $x, y, z \notin A$. Hence the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy $(1,2)$-ideal of R.

Conversely, suppose that the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy $(1,2)$-ideal of R, this means that χ_A is an intuitionistic fuzzy LA-subring of R. Then A is an LA-subring of R by the Lemma 4. Let $t \in (AR)A^2$, this implies that $t = (xa)(yz)$, where $x, y, z \in A$ and $a \in R$. Then by definition $\mu_{\chi_A}(x) = 1 = \mu_{\chi_A}(y) = \mu_{\chi_A}(z)$ and $\gamma_{\chi_A}(x) = 0 = \gamma_{\chi_A}(y) = \gamma_{\chi_A}(z)$. Now

$$\begin{aligned} \mu_{\chi_A}((xa)(yz)) &\geq \mu_{\chi_A}(x) \wedge \mu_{\chi_A}(y) \wedge \mu_{\chi_A}(z) = 1 \\ \text{and } \gamma_{\chi_A}((xa)(yz)) &\leq \gamma_{\chi_A}(x) \vee \gamma_{\chi_A}(y) \vee \gamma_{\chi_A}(z) = 0, \end{aligned}$$

χ_A being an intuitionistic fuzzy $(1,2)$-ideal of R. Thus $\mu_{\chi_A}((xa)(yz)) = 1$ and $\gamma_{\chi_A}((xa)(yz)) = 0$, i.e., $(xa)(yz) \in A$. Hence A is a $(1,2)$-ideal of R.

Remark 3. *Let R be an LA-ring and $\emptyset \neq A \subseteq R$. Then A is a bi-ideal of R if and only if the intuitionistic characteristic function $\chi_A = \langle \mu_{\chi_A}, \gamma_{\chi_A} \rangle$ of A is an intuitionistic fuzzy bi-ideal of R.*

Zadeh [22], introduced the concept of level set. Das [5], studied the fuzzy groups, level subgroups and gave the proper definition of a level set such that: let μ be a fuzzy subset of a non-empty set S, for $t \in [0,1]$, the set $\mu_t = \{x \in S \mid \mu(x) \geq t\}$, is called a level subset of the fuzzy subset μ. Now we give the definition of strong level set.

Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set of an LA-ring R, then for all $r, t \in (0,1]$, we define a set $A^{(r,t)} = \{x \in R \mid \mu_A(x) \geq r \text{ and } \gamma_A(x) \leq t\}$, which is called the (r,t)-strong level set of A. It is clear that $A^{(r,t)} = U(\mu_A; r) \cap L(\gamma_A; t)$ for all $r, t \in (0,1]$.

Lemma 5. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R. Then A is an intuitionistic fuzzy LA-subring of R if and only if $A^{(r,t)}$ is an LA-subring of R for all $r, t \in (0,1]$.*

Proof. Straight forward.

Proposition 3. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R. Then A is an intuitionistic fuzzy left (resp. right) ideal of R if and only if $A^{(r,t)}$ is a left (resp. right) ideal of R for all $r, t \in (0,1]$.*

Proof. Straight forward.

Theorem 4. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R. Then A is an intuitionistic fuzzy $(1,2)$-ideal of R if and only if $A^{(r,t)}$ is a $(1,2)$-ideal of R for all $r, t \in (0,1]$.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy $(1,2)$-ideal of R, this implies that A is an intuitionistic fuzzy LA-subring of R. Then $A^{(r,t)}$ is an LA-subring of R by the Lemma 5. Let $x, y, z \in A^{(r,t)}$ and $a \in R$, so $\mu_A(y), \mu_A(y), \mu_A(z) \geq r$ and $\gamma_A(y), \gamma_A(y), \gamma_A(z) \leq t$. By our assumption

$$\begin{aligned} \mu_A((xy)(az)) &\geq \mu_A(y) \wedge \mu_A(y) \wedge \mu_A(z) \geq r \\ \text{and } \gamma_A((xy)(az)) &\leq \mu_A(x) \vee \mu_A(y) \vee \mu_A(z) \leq t. \end{aligned}$$

Thus $\mu_A((xy)(az)) \geq r$ and $\gamma_A((xy)(az)) \leq t$, i.e., $(xy)(az) \in A^{(r,t)}$. So $A^{(r,t)}$ is a $(1,2)$-ideal of R.

Conversely, suppose that $A^{(r,t)}$ is a $(1,2)$-ideal of R, this means that $A^{(r,t)}$ is an LA-subring of R. Then A is an intuitionistic fuzzy LA-subring of R by the Lemma 5. Let $x, y, z, a \in R$. We have to show that

$$\begin{aligned} \mu_A((xy)(az)) &\geq \mu_A(x) \wedge \mu_A(y) \wedge \mu_A(y) \\ \text{and } \gamma_A((xy)(az)) &\leq \gamma_A(x) \vee \gamma_A(y) \vee \gamma_A(y). \end{aligned}$$

We assume a contradiction

$$\begin{aligned} \mu_A((xy)(az)) &\leq \mu_A(x) \vee \mu_A(y) \vee \mu_A(y) \\ \text{and } \gamma_A((xy)(az)) &\geq \gamma_A(x) \wedge \gamma_A(y) \wedge \gamma_A(y). \end{aligned}$$

Let $\mu_A(x) = r = \mu_A(y) = \mu_A(z)$ and $\gamma_A(x) = t = \gamma_A(y) = \gamma_A(z)$, this implies that $\mu_A(x), \mu_A(y), \mu_A(z) \geq r$ and $\gamma_A(x), \gamma_A(y), \gamma_A(z) \leq t$, i.e., $x, y, z \in A^{(r,t)}$. But $\mu_A((xy)(az)) \leq r$ and $\gamma_A((xy)(az)) \geq t$, i.e., $(xy)(az) \notin A^{(r,t)}$, which is a contradiction. So

$$\begin{aligned} \mu_A((xy)(az)) &\geq \mu_A(x) \wedge \mu_A(y) \wedge \mu_A(y) \\ \text{and } \gamma_A((xy)(az)) &\leq \gamma_A(x) \vee \gamma_A(y) \vee \gamma_A(y). \end{aligned}$$

Remark 4. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R. Then A is an intuitionistic fuzzy bi-ideal of R if and only if $A^{(r,t)}$ is a bi-ideal of R for all $r, t \in (0, 1]$.*

2. Characterizations of LA-rings

In this section, we characterize different classes of LA-ring in terms of intuitionistic fuzzy left (right, bi-, generalized bi-) ideals. An LA-ring R is called regular, if for every element $x \in R$, there exists an element $a \in R$ such that $x = (xa)x$. An LA-ring R is called intra-regular, if for every element $x \in R$, there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$.

An LA-ring R is called left (resp. right) regular, if for every element $x \in R$, there exists an element $a \in R$ such that $x = ax^2$ (resp. $x^2 a$). An LA-ring R is called completely regular, if it is regular, left regular and right regular. An LA-ring R is called $(2, 2)$-regular, if for every element $x \in R$, there exists an element $a \in R$ such that $x = (x^2 a)x^2$. An LA-ring R is called locally associative LA-ring if $(a.a).a = a.(a.a)$ for all $a \in R$.

A ring R is called left (resp. right) weakly regular if $I^2 = I$, for every left (resp. right) ideal I of R, equivalently $x \in RxRx (x \in xRxR)$ for every $x \in R$. An LA-ring R is called weakly regular if it is both left weakly regular and right weakly regular [17]. Now we define this notion in a class of non-associative and non-commutative rings (LA-ring).

An LA-ring R is called left (resp. right) weakly regular, if for every element $x \in R$, there exist elements $a, b \in R$ such that $x = (ax)(bx)$ (resp. $x = (xa)(xb)$) . An LA-ring R is called weakly regular if it is both left weakly regular and right weakly regular.

Lemma 6. *Every intuitionistic fuzzy right ideal of an LA-ring R with left identity e, is an intuitionistic fuzzy ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of R and $x, y \in R$. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A((ex)y) = \mu_A((yx)e) \geq \mu_A(yx) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A((ex)y) = \gamma_A((yx)e) \leq \gamma_A(yx) \leq \gamma_A(y). \end{aligned}$$

Hence A is an intuitionistic fuzzy ideal of R.

Lemma 7. *Every intuitionistic fuzzy right ideal of a regular LA-ring R, is an intuitionistic fuzzy ideal of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R. Let $x, y \in R$, this implies that there exists $a \in R$, such that $x = (xa)x$. Thus $\mu_A(xy) = \mu_A(((xa)x)y) = \mu_A((yx)(xa)) \geq \mu_A(yx) \geq \mu_A(y)$ and $\gamma_A(xy) = \gamma_A(((xa)x)y) = \gamma_A((yx)(xa)) \leq \gamma_A(yx) \leq \gamma_A(y)$. Therefore A is an intuitionistic fuzzy ideal of R.

Proposition 4. *Let R be a regular LA-ring having the property $a = a^2$ for every $a \in R$, with left identity e. Then every intuitionistic fuzzy generalized bi-ideal of R is an intuitionistic fuzzy bi-ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = (xa)x$. We have to show that A is an intuitionistic fuzzy LA-subring of R. Thus

$$\begin{aligned}
\mu_A(xy) &= \mu_A(((xa)x)y) = \mu_A(((xa)x^2)y) = \mu_A(((xa)(xx))y) \\
&= \mu_A((x((xa)x))y) \geq min\{\mu_A(x), \mu_A(y)\} \\
\text{and } \gamma_A(xy) &= \gamma_A(((xa)x)y) = \gamma_A(((xa)x^2)y) = \gamma_A(((xa)(xx))y) \\
&= \gamma_A((x((xa)x))y) \leq max\{\gamma_A(x), \gamma_A(y)\}.
\end{aligned}$$

Hence A is an intuitionistic fuzzy LA-subring of R.

Lemma 8. *Let R be an LA-ring with left identity e. Then Ra is the smallest left ideal of R containing a.*

Proof. Let $x, y \in Ra$ and $r \in R$. This implies that $x = r_1a$ and $y = r_2a$, where $r_1, r_2 \in R$. Now

$$\begin{aligned}
x - y &= r_1a - r_2a = (r_1 - r_2)a \in Ra \\
\text{and } rx &= r(r_1a) = (er)(r_1a) = ((r_1a)r)e = ((r_1a)(er))e \\
&= ((r_1e)(ar))e = (e(ar))(r_1e) = (ar)(r_1e) \\
&= ((r_1e)r)a \in Ra.
\end{aligned}$$

Since $a = ea \in Ra$. Thus Ra is a left ideal of R containing a. Let I be another left ideal of R containing a. So $ra \in I$, where $ra \in Ra$, i.e., $Ra \subseteq I$. Hence Ra is the smallest left ideal of R containing a.

Lemma 9. *Let R be an LA-ring with left identity e. Then aR is a left ideal of R.*

Proof. Straight forward.

Proposition 5. *Let R be an LA-ring with left identity e. Then $aR \cup Ra$ is the smallest right ideal of R containing a.*

Proof. Let $x, y \in aR \cup Ra$, this means that $x, y \in aR$ or Ra. Since aR and Ra both are left ideals of R, so $x - y \in aR$ and Ra, i.e., $x - y \in aR \cup Ra$. We have to show that $(aR \cup Ra)R \subseteq (aR \cup Ra)$. Now

$$\begin{aligned} (aR \cup Ra)R &= (aR)R \cup (Ra)R = (RR)a \cup (Ra)(eR) \\ &\subseteq Ra \cup (Re)(aR) = Ra \cup R(aR) \\ &= Ra \cup a(RR) \subseteq Ra \cup aR = aR \cup Ra. \\ &\Rightarrow (aR \cup Ra)R \subseteq aR \cup Ra. \end{aligned}$$

As $a \in Ra$, i.e., $a \in aR \cup Ra$. Let I be another right ideal of R containing a. Since $aR \in IR \subseteq I$ and $Ra = (RR)a = (aR)R \in (IR)R \subseteq IR \subseteq I$, i.e., $aR \cup Ra \subseteq I$. Therefore $aR \cup Ra$ is the smallest right ideal of R containing a.

Lemma 10. *Let R be an LA-ring. Then $A \circ B \subseteq A \cap B$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy left ideal B of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right and $B = (\mu_B, \gamma_B)$ be an intuitionistic fuzzy left ideal of R and $x \in R$. If x cannot be expressible as $x = \sum_{i=1}^{n} a_i b_i$, where $a_i, b_i \in R$ and n is any positive integer, then obvious $A \circ B \subseteq A \cap B$, otherwise we have

$$\begin{aligned} (\mu_A \circ \mu_B)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu_A(a_i) \wedge \mu_B(b_i)\}\} \\ &\leq \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu_A(a_i b_i) \wedge \mu_B(a_i b_i)\}\} \\ &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} (\mu_A \wedge \mu_B)(a_i b_i)\} \\ &= (\mu_A \wedge \mu_B)(x) = (\mu_A \cap \mu_B)(x) \\ &\Rightarrow \mu_A \circ \mu_B \subseteq \mu_A \cap \mu_B \end{aligned}$$

Similarly, we have $\gamma_A \circ \gamma_B \supseteq \gamma_A \cup \gamma_B$. Hence $A \circ B \subseteq A \cap B$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy left ideal B of R.

Theorem 5. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) R is a regular.

(2) $A \cap B = A \circ B$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy left ideal B of R.

Proof. Suppose that (1) holds. Since $A \circ B \subseteq A \cap B$, for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy left ideal B of R by the Lemma 10. Let $x \in R$, this implies that there exists an element $a \in R$ such that $x = (xa)x$. Thus

$$\begin{aligned} (\mu_A \circ \mu_B)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{\mu_A(a_i) \wedge \mu_B(b_i)\}\} \\ &\geq min\{\mu_A(xa), \mu_B(x)\} \geq min\{\mu_A(x), \mu_B(x)\} \\ &= (\mu_A \wedge \mu_B)(x) = (\mu_A \cap \mu_B)(x). \end{aligned}$$

$$\Rightarrow \quad \mu_A \cap \mu_B \subseteq \mu_A \circ \mu_B.$$

Similarly, we have $\gamma_A \cup \gamma_B \supseteq \gamma_A \circ \gamma_B$. Hence $A \cap B = A \circ B$, i.e., $(1) \Rightarrow (2)$. Assume that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 8 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 5. So χ_{Ra} is an intuitionistic fuzzy left ideal and $\chi_{aR\cup Ra}$ is an intuitionistic fuzzy right ideal of R, by the Lemma 2. By our assumption $\chi_{aR\cup Ra} \cap \chi_{Ra} = \chi_{aR\cup Ra} \circ \chi_{Ra}$, i.e., $\chi_{(aR\cup Ra)\cap Ra} = \chi_{(aR\cup Ra)Ra}$. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. Now $(Ra)(Ra) = ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra)$. This implies that

$$(aR)(Ra) \cup (Ra)(Ra) = (aR)(Ra) \cup (aR)(Ra) = (aR)(Ra).$$

Thus $a \in (aR)(Ra)$. Then

$$\begin{aligned} a &= (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a \\ &= ((xe)(ay))a = (a((xe)y))a \in (aR)a, \text{ for any } x, y \in R. \end{aligned}$$

This means that $a \in (aR)a$, i.e., a is regular. Hence R is a regular, i.e., $(2) \Rightarrow (1)$.

Theorem 6. *Let R be a regular locally associative LA-ring having the property $a = a^2$ for every $a \in R$. Then for every intuitionistic fuzzy bi-ideal $A = (\mu_A, \gamma_A)$ of R, $A(a^n) = A(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof. For $n = 1$. Let $a \in R$, this implies that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a = (a^2x)a^2$, because $a = a^2$. Thus

$$\begin{aligned} \mu_A(a) &= \mu_A\left((a^2x)a^2\right) \geq min\{\mu_A\left(a^2\right), \mu_A\left(a^2\right)\} = \mu_A\left(a^2\right) \\ &= \mu_A(aa) \geq min\{\mu_A(a), \mu_A(a)\} = \mu_A(a). \end{aligned}$$

Similarly, $\gamma_A(a) = \gamma_A\left(a^2\right)$, therefore $A(a) = A\left(a^2\right)$.

Now $a^2 = aa = ((a^2x)a^2)((a^2x)a^2) = (a^4x^2)a^4$, then the result is true for $n = 2$. Suppose that the result is true for $n = k$, i.e., $A(a^k) = A\left(a^{2k}\right)$. Now $a^{k+1} = a^k a = ((a^{2k}x^k)a^{2k})((a^2x)a^2) = (a^{2(k+1)}x^{k+1})a^{2(k+1)}$. Thus

$$\begin{aligned} \mu_A\left(a^{k+1}\right) &= \mu_A\left((a^{2(k+1)}x^{k+1})a^{2(k+1)}\right) \geq min\{\mu_A(a^{2(k+1)}), \mu_A\left(a^{2(k+1)}\right)\} \\ &= \mu_A\left(a^{2(k+1)}\right) = \mu_A\left(a^{k+1}a^{k+1)}\right) \\ &\geq min\{\mu_A\left(a^{k+1)}\right), \mu_A\left(a^{k+1)}\right)\} = \mu_A(a^{k+1)}). \end{aligned}$$

Similarly, $\gamma_A(a^{k+1}) = \gamma_A\left(a^{2(k+1)}\right)$, therefore $A(a^{k+1}) = A(a^{2(k+1)})$. Hence by induction method, the result is true for all positive integers.

Lemma 11. *Every intuitionistic fuzzy left (right) ideal of $(2,2)$-regular LA-ring R, is an intuitionistic fuzzy ideal of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R and $x, y \in R$, this means that there exists $a \in R$ such that $x = (x^2a)x^2$. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((x^2a)x^2)y) = \mu_A((yx^2)(x^2a)) \geq \mu_A(yx^2) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A(((x^2a)x^2)y) = \gamma_A((yx^2)(x^2a)) \leq \gamma_A(yx^2) \leq \gamma_A(y). \end{aligned}$$

Therefore A is an intuitionistic fuzzy ideal of R. Similarly, for left ideal.

Remark 5. *The concept of intuitionistic fuzzy (left, right, two-sided) ideals coincides in $(2,2)$-regular LA-rings.*

Proposition 6. *Every intuitionistic fuzzy generalized bi-ideal of $(2,2)$-regular LA-ring R with left identity e, is an intuitionistic fuzzy bi-ideal of R.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy generalized bi-ideal of R and $x, y \in R$, then there exists an element $a \in R$ such that $x = (x^2a)x^2$. We have to show that A is an intuitionistic fuzzy LA-subring of R. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((x^2a)x^2)y) = \mu_A(((x^2a)(xx))y) \\ &= \mu_A((x((x^2a)x))y) \geq min\{\mu_A(x), \mu_A(y)\} \\ \text{and } \gamma_A(xy) &= \gamma_A(((x^2a)x^2)y) = \gamma_A(((x^2a)(xx))y) \\ &= \gamma_A((x((x^2a)x))y) \leq max\{\gamma_A(x), \gamma_A(y)\}. \end{aligned}$$

So A is an intuitionistic fuzzy LA-subring of R.

Theorem 7. *Let R be a $(2,2)$-regular locally associative LA-ring. Then for every intuitionistic fuzzy bi-ideal $A = (\mu_A, \gamma_A)$ of R, $A(a^n) = A(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof. Same as Theorem 6.

Lemma 12. *Let R be a right regular LA-ring. Then every intuitionistic fuzzy left (right) ideal of R is an intuitionistic fuzzy ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = x^2a$. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A((x^2a)y) = \mu_A(((xx)a)y) = \mu_A(((ax)x)y) \\ &= \mu_A((yx)(ax)) \geq \mu_A(yx) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A((x^2a)y) = \gamma_A(((xx)a)y) = \gamma_A(((ax)x)y) \\ &= \gamma_A((yx)(ax)) \leq \gamma_A(yx) \leq \gamma_A(y). \end{aligned}$$

Hence A is an intuitionistic fuzzy ideal of R. Similarly, for left ideal.

Remark 6. *The concept of intuitionistic fuzzy (left, right, two-sided) ideals coincides in right regular LA-rings.*

Proposition 7. *Let R be a right regular LA-ring with left identity e. Then every intuitionistic fuzzy generalized bi-ideal of R is an intuitionistic fuzzy bi-ideal of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy generalized bi-ideal of R and $x, y \in R$, this means that there exists $a \in R$ such that $x = x^2a$. We have to show that A is an intuitionistic fuzzy LA-subring of R. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A((x^2a)y) = \mu_A(((xx)(ea))y) = \mu_A(((ae)(xx))y) \\ &= \mu_A((x((ae)x))y) \geq min\{\mu_A(x), \mu_A(y)\} \\ \text{and } \gamma_A(xy) &= \gamma_A((x^2a)y) = \gamma_A(((xx)(ea))y) = \gamma_A(((ae)(xx))y) \\ &= \gamma_A((x((ae)x))y) \leq max\{\gamma_A(x), \gamma_A(y)\}. \end{aligned}$$

Therefore A is an intuitionistic fuzzy LA-subring of R.

Lemma 13. *Let R be a left regular LA-ring with left identity e. Then every intuitionistic fuzzy left (right) ideal of R is an intuitionistic fuzzy ideal of R.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R and $x, y \in R$, then there exists an element $a \in R$ such that $x = ax^2$. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A((ax^2)y) = \mu_A((a(xx))y) = \mu_A((x(ax))y) \\ &= \mu_A((y(ax))x) \geq \mu_A(y(ax)) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A((ax^2)y) = \gamma_A((a(xx))y) = \gamma_A((x(ax))y) \\ &= \gamma_A((y(ax))x) \leq \gamma_A(y(ax)) \leq \gamma_A(y). \end{aligned}$$

So A is an intuitionistic fuzzy ideal of R. Similarly, for left ideal.

Remark 7. *The concept of intuitionistic fuzzy (left, right, two-sided) ideals coincides in left regular LA-rings with left identity.*

Proposition 8. *Every intuitionistic fuzzy generalized bi-ideal of a left regular LA-ring R with left identity e, is an intuitionistic fuzzy bi-ideal of R.*

Proof. Let A be an intuitionistic fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exists $a \in R$ such that $x = ax^2$. We have to show that A is an intuitionistic fuzzy LA-subring of R. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A((ax^2)y) = \mu_A((a(xx))y) = \mu_A((x(ax))y) \geq min\{\mu_A(x), \mu_A(y)\} \\ \text{and } \gamma_A(xy) &= \gamma_A((ax^2)y) = \gamma_A((a(xx))y) = \gamma_A((x(ax))y) \leq max\{\gamma_A(x), \gamma_A(y)\}. \end{aligned}$$

Hence A is an intuitionistic fuzzy LA-subring of R.

Theorem 8. *Let R be a regular and right regular locally associative LA-ring. Then for every intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ of R, $A(a^n) = A(a^{3n})$ for all $a \in R$, where n is any positive integer.*

Proof. For $n = 1$. Let $a \in R$, this means that there exists an element $x \in R$ such that $a = (ax)a$ and $a = a^2x$. Now $a = (ax)a = (ax)(a^2x) = a^3x^2$. Thus

$$\begin{aligned} \mu_A(a) &= \mu_A(a^3x^2) \geq \mu_A(a^3) = \mu_A(aa^2) \geq min\{\mu_A(a), \mu_A(a^2)\} \\ &\geq min\{\mu_A(a), \mu_A(a), \mu_A(a)\} = \mu_A(a). \end{aligned}$$

Similarly, $\gamma_A(a) = \gamma_A(a^3)$, so $A(a) = A(a^3)$. Here $a^2 = aa = (a^3x^2)(a^3x^2) = a^6x^4$, then the result is true for $n = 2$. Assume that the result is true for $n = k$, i.e., $A(a^k) = A(a^{3k})$. Now $a^{k+1} = a^ka = (a^{3k}x^{2k})(a^3x^2) = a^{3(k+1)}x^{2(k+1)}$. Thus

$$\begin{aligned} \mu_A(a^{k+1}) &= \mu_A(a^{3(k+1)}x^{2(k+1)}) \geq \mu_A(a^{3(k+1)}) = \mu_A(a^{3k+3}) \\ &= \mu_A(a^{k+1}a^{2k+2}) \geq min\{\mu_A\left(a^{k+1}\right), \mu_A\left(a^{2k+2}\right)\} \\ &\geq min\{\mu_A\left(a^{k+1}\right), \mu_A\left(a^{k+1}\right), \mu_A\left(a^{k+1}\right)\} = \mu_A\left(a^{k+1}\right). \end{aligned}$$

Similarly, $\gamma_A\left(a^{k+1}\right) = \gamma_A\left(a^{3(k+1)}\right)$, so $A(a^{k+1}) = A(a^{3(k+1)})$. Hence by induction method, the result is true for all positive integers.

Theorem 9. *Let R be a right regular locally associative LA-ring. Then for every intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ of R, $A(a^n) = A(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Proof. For $n = 1$. Let $a \in R$, then there exists an element $x \in R$ such that $a = a^2x$. Thus

$$\begin{aligned} \mu_A(a) &= \mu_A(a^2x) \geq \mu_A(a^2) = \mu_A(aa) \\ &\geq min\{\mu_A(a), \mu_A(a)\} = \mu_A(a). \end{aligned}$$

Similarly, $\gamma_A(a) = \gamma_A(a^2)$, therefore $A(a) = A(a^2)$. Now $a^2 = aa = (a^2x)(a^2x) = a^4x^2$, then the result is true for $n = 2$. Suppose that the result is true for $n = k$, i.e., $A(a^k) = A(a^{2k})$. Now $a^{k+1} = a^ka = (a^{2k}x^k)(a^2x) = a^{2(k+1)}x^{(k+1)}$. Thus

$$\begin{aligned} \mu_A(a^{k+1}) &= \mu_A(a^{2(k+1)}x^{(k+1)}) \geq \mu_A(a^{2(k+1)}) \\ &= \mu_A(a^{2k+2}) = \mu_A(a^{k+1}a^{k+1}) \\ &\geq min\{\mu_A\left(a^{k+1}\right), \mu_A\left(a^{k+1}\right)\} = \mu_A\left(a^{k+1}\right). \end{aligned}$$

Similarly, $\gamma_A(a^{k+1}) = \gamma_A(a^{2(k+1)})$, therefore $A(a^{k+1}) = A(a^{2(k+1)})$. Hence by induction method, the result is true for all positive integers.

Lemma 14. *Let R be a right regular locally associative LA-ring with left identity e. Then for every intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ of R, $A(ab) = A(ba)$ for all $a, b \in R$.*

Proof. Let $a, b \in R$. By using Theorem 9 (for $n = 1$) . Now

$$\begin{aligned} \mu_A(ab) &= \mu_A((ab)^2) = \mu_A((ab)(ab)) \\ &= \mu_A((ba)(ba)) = \mu_A((ba)^2) = \mu_A(ba) \\ \text{and } \gamma_A(ab) &= \gamma_A((ab)^2) = \gamma_A((ab)(ab)) \\ &= \gamma_A((ba)(ba)) = \gamma_A((ba)^2) = \gamma_A(ba). \end{aligned}$$

Thus $A(ab) = A(ba)$.

Remark 8. *It is easy to see that, if R is a left regular locally associative LA-ring with left identity e. Then for every intuitionistic fuzzy left ideal $A = (\mu_A, \gamma_A)$ of R, $A(a^n) = A(a^{2n})$ for all $a \in R$, where n is any positive integer. And also for every intuitionistic fuzzy left ideal $A = (\mu_A, \gamma_A)$ of R, $A(ab) = A(ba)$ for all $a, b \in R$.*

Lemma 15. *Let R be a right weakly regular LA-ring. Then every intuitionistic fuzzy left (right) ideal of R is an intuitionistic fuzzy ideal of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R and $x, y \in R$, this means that there exist $a, b \in R$ such that $x = (xa)(xb)$. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((xa)(xb))y) = \mu_A((((xb)a)x)y) \\ &= \mu_A((((ab)x)x)y) = \mu_A((yx)((ab)x)) \\ &= \mu_A((yx)(nx)) \text{ say } ab = n \\ &\geq \mu_A(yx) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A(((xa)(xb))y) = \gamma_A((((xb)a)x)y) \\ &= \gamma_A((((ab)x)x)y) = \gamma_A((yx)((ab)x)) \\ &= \gamma_A((yx)(nx)) \leq \gamma_A(yx) \leq \gamma(y). \end{aligned}$$

Therefore A is an intuitionistic fuzzy ideal of R. Similarly, for left ideal.

Remark 9. *The concept of intuitionistic fuzzy (left, right, two-sided) ideals coincides in right weakly regular LA-rings.*

Proposition 9. *Every intuitionistic fuzzy generalized bi-ideal of a right weakly regular LA-ring R with left identity e, is an intuitionistic fuzzy bi-ideal of R.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy generalized bi-ideal of R and $x, y \in R$, then there exist elements $a, b \in R$ such that $x = (xa)(xb)$. We have to show that A is an intuitionistic fuzzy LA-subring of R. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((xa)(xb))y) = \mu_A((x((xa)b))y) \geq min\{\mu_A(x), \mu_A(y)\} \\ \text{and } \gamma_A(xy) &= \gamma_A(((xa)(xb))y) = \gamma_A((x((xa)b))y) \leq max\{\gamma_A(x), \gamma_A(y)\}. \end{aligned}$$

So A is an intuitionistic fuzzy LA-subring of R.

Lemma 16. *Let R be a left weakly regular LA-ring. Then every intuitionistic fuzzy right ideal of R is an intuitionistic fuzzy ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of R and $x, y \in R$, this implies that there exist $a, b \in R$ such that $x = (ax)(bx)$. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((ax)(bx))y) = \mu_A(y(bx))(ax) \geq \mu_A(y(bx)) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A(((ax)(bx))y) = \gamma_A(y(bx))(ax) \leq \gamma_A(y(bx)) \leq \gamma_A(y). \end{aligned}$$

Hence A is an intuitionistic fuzzy ideal of R.

Remark 10. *The concept of intuitionistic fuzzy (right, two-sided) ideals coincides in left weakly regular LA-rings.*

Lemma 17. *Let R be a left weakly regular LA-ring with left identity e. Then every intuitionistic fuzzy left ideal of R is an intuitionistic fuzzy ideal of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of R and $x, y \in R$, this means that there exist $a, b \in R$ such that $x = (ax)(bx)$. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((ax)(bx))y) = \mu_A(((ab)(xx))y) \\ &= \mu_A((x((ab)x))y) = \mu_A((y((ab)x))x) \geq \mu_A(x) \\ \text{and } \gamma_A(xy) &= \gamma_A(((ax)(bx))y) = \gamma_A(((ab)(xx))y) \\ &= \gamma_A((x((ab)x))y) = \gamma_A((y((ab)x))x) \leq \gamma_A(x). \end{aligned}$$

Therefore A is an intuitionistic fuzzy ideal of R.

Remark 11. *The concept of intuitionistic fuzzy (left, two-sided) ideals coincides in left weakly regular LA-rings with left identity e.*

Proposition 10. *Every intuitionistic fuzzy generalized bi-ideal of a left weakly regular LA-ring R with left identity e, is an intuitionistic fuzzy bi-ideal of R.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy generalized bi-ideal of R and $x, y \in R$, then there exist elements $a, b \in R$ such that $x = (ax)(bx)$. We have to show that A is an intuitionistic fuzzy LA-subring of R. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((ax)(bx))y) = \mu_A(((ab)(xx))y) \\ &= \mu_A((x((ab)x))y) \geq min\{\mu_A(x), \mu_A(y)\} \\ \text{and } \gamma_A(xy) &= \gamma_A(((ax)(bx))y) = \gamma_A(((ab)(xx))y) \\ &= \gamma_A((x((ab)x))y) \leq max\{\gamma_A(x), \gamma_A(y)\}. \end{aligned}$$

So A is an intuitionistic fuzzy LA-subring of R.

Remark 12. *It is easy to see that, if R is a left (right) weakly regular locally associative LA-ring. Then for every intuitionistic fuzzy left (right) ideal $A = (\mu_A, \gamma_A)$ of R, $A(a^n) = A(a^{2n})$ for all $a \in R$, where n is any positive integer.*

Theorem 10. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*
(1) *R is a left weakly regular.*
(2) *$A \cap B = A \circ B$ for every intuitionistic fuzzy right ideal A and every intuitionistic fuzzy left ideal B of R.*

Proof. Suppose that (1) holds. Since $A \circ B \subseteq A \cap B$ for every intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ and every intuitionistic fuzzy left ideal $B = (\mu_B, \gamma_B)$ of R by the Lemma 10. Let $x \in R$, this implies that there exist $a, b \in R$ such that $x = (ax)(bx) = (ab)(xx) = x((ab)x)$. Now

$$\begin{aligned} (\mu_A \circ \mu_B)(x) &= \vee_{x=\sum_{i=1}^n a_i b_i} \{\wedge_{i=1}^n \{\mu_A(a_i) \wedge \mu_B(b_i)\}\} \\ &\geq \mu_A(x) \wedge \mu_B((ab)x) \geq \mu_A(x) \wedge \mu_B(x) = (\mu_A \cap \mu_B)(x) \\ \text{and } (\gamma_A \circ \gamma_B)(x) &= \wedge_{x=\sum_{i=1}^n a_i b_i} \{\vee_{i=1}^n \{\gamma_A(a_i) \vee \gamma_B(b_i)\}\} \\ &\leq \gamma_A(x) \vee \gamma_B((ab)x) \leq \gamma_A(x) \vee \gamma_B(x) = (\gamma_A \cup \gamma_B)(x). \end{aligned}$$

Thus $\mu_A \cap \mu_B \subseteq \mu_A \circ \mu_B$ and $\gamma_A \cup \gamma_B \supseteq \gamma_A \circ \gamma_B$, i.e., $A \cap B \subseteq A \circ B$. Hence $A \cap B = A \circ B$, i.e., (1) $\Rightarrow$ (2). Assume that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 8 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 5. So χ_{Ra} is an intuitionistic fuzzy left ideal and $\chi_{aR \cup Ra}$ is an intuitionistic fuzzy right ideal of R, by the Proposition 2. Then by our assumption $\chi_{aR \cup Ra} \cap \chi_{Ra} = \chi_{aR \cup Ra} \circ \chi_{Ra}$, i.e., $\chi_{(aR \cup Ra) \cap Ra} = \chi_{(aR \cup Ra)Ra}$ by the Theorem 1. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in (aR)(Ra) \cup (Ra)(Ra)$. This implies that $a \in (aR)(Ra)$ or $a \in (Ra)(Ra)$. If $a \in (Ra)(Ra)$, then R is a left weakly regular. If $a \in (aR)(Ra)$, then

$$\begin{aligned} (aR)(Ra) &= ((ea)(RR))(Ra) = ((RR)(ae))(Ra) \\ &= (((ae)R)R)(Ra) = ((aR)R)(Ra) \\ &= ((RR)a)(Ra) = (Ra)(Ra). \end{aligned}$$

Hence R is a left weakly regular, i.e., (2) $\Rightarrow$ (1).

Theorem 11. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*
(1) *R is a left weakly regular.*
(2) *$A \cap I \subseteq A \circ I$ for every intuitionistic fuzzy bi-ideal A and every intuitionistic fuzzy ideal I of R.*
(3) *$B \cap I \subseteq B \circ I$ for every intuitionistic fuzzy generalized bi-ideal B and every intuitionistic fuzzy ideal I of R.*

Proof. Assume that (1) holds. Let $B = (\mu_B, \gamma_B)$ be an intuitionistic fuzzy generalized bi-ideal and $I = (\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal of R. Let $x \in R$, this means that there exist $a, b \in R$ such that $x = (ax)(bx) = (ab)(xx) = x((ab)x)$. Now

$$\begin{aligned}
(\mu_B \circ \mu_I)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{\mu_B(a_i) \wedge \mu_I(b_i)\}\} \\
&\geq \mu_B(x) \wedge \mu_I((ab)x) \\
&\geq \mu_B(x) \wedge \mu_I(x) = (\mu_B \cap \mu_I)(x). \\
&\Rightarrow \mu_B \cap \mu_I \subseteq \mu_B \circ \mu_I.
\end{aligned}$$

Similarly, $\gamma_A \cup \gamma_B \supseteq \gamma_A \circ \gamma_B$. Hence $A \cap B \subseteq A \circ B$, i.e., $(1) \Rightarrow (3)$. It is clear that $(3) \Rightarrow (2)$. Suppose that (2) holds. Then $A \cap I \subseteq A \circ I$, where A is an intuitionistic fuzzy right ideal of R. Since $A \circ I \subseteq A \cap I$, so $A \circ I = A \cap I$. Therefore R is a left weakly regular by the Theorem 10, i.e., $(2) \Rightarrow (1)$.

Theorem 12. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a left weakly regular.*

(2) *$A \cap I \cap C \subseteq (A \circ I) \circ C$ for every intuitionistic fuzzy bi-ideal A, every intuitionistic fuzzy ideal I and every intuitionistic fuzzy right ideal C of R.*

(3) *$B \cap I \cap C \subseteq (B \circ I) \circ C$ for every intuitionistic fuzzy generalized bi-ideal B, every intuitionistic fuzzy ideal I and every intuitionistic fuzzy right ideal C of R.*

Proof. Suppose that (1) holds. Let $B = (\mu_B, \gamma_B)$ be an intuitionistic fuzzy generalized bi-ideal, $I = (\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal and $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy right ideal of R. Let $x \in R$, then there exist elements $a, b \in R$ such that $x = (ax)(bx)$. Here

$$\begin{aligned}
x &= (ax)(bx) = (xb)(xa) \\
xb &= ((ax)(bx))b = ((xx)(ba))b \\
&= (b(ba))(xx) = c(xx) = x(cx) \quad \text{say } c = b(ba)
\end{aligned}$$

Now

$$\begin{aligned}
((\mu_B \circ \mu_I) \circ \mu_C)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{(\mu_B \circ \mu_I)(a_i) \wedge \mu_C(b_i)\}\} \\
&\geq (\mu_B \circ \mu_I)(xb) \wedge \mu_C(xa) \\
&\geq (\mu_B \circ \mu_I)(xb) \wedge \mu_C(x) \\
&= \vee_{xb=\sum_{i=1}^{n} p_i q_i}\{\wedge_{i=1}^{n}\{\mu_B(p_i) \wedge \mu_I(q_i)\}\} \wedge \mu_C(x) \\
&\geq \mu_B(x) \wedge \mu_I(cx) \wedge \mu_C(x) \\
&\geq \mu_B(x) \wedge \mu_I(x) \wedge \mu_C(x) = (\mu_B \cap \mu_I \cap \mu_C)(x). \\
&\Rightarrow \mu_B \cap \mu_I \cap \mu_C \subseteq (\mu_B \circ \mu_I) \circ \mu_C.
\end{aligned}$$

Similarly, $\gamma_B \cup \gamma_I \cup \gamma_C \supseteq (\gamma_B \circ \gamma_I) \circ \gamma_C$, i.e., $B \cap I \cap C \subseteq (B \circ I) \circ C$. Hence $(1) \Rightarrow (3)$. It is clear that $(3) \Rightarrow (2)$, every intuitionistic fuzzy bi-ideal of R is an intuitionistic fuzzy generalized bi-ideal of R. Assume that (2) is true. Then $A \cap I \cap R \subseteq (A \circ I) \circ R$, where A is an intuitionistic right ideal of R, i.e., $A \cap I \subseteq A \circ I$. Since $A \circ I \subseteq A \cap I$, so $A \circ I = A \cap I$. Therefore R is a left weakly regular by the Theorem 10, i.e., $(2) \Rightarrow (1)$.

Lemma 18. *Every intuitionistic fuzzy left (right) ideal of an intra-regular LA-ring R, is an intuitionistic fuzzy ideal of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of R. Let $x, y \in R$, this implies that there exist $a_i, b_i \in R$, such that $x = \sum_{i=1}^{n}(a_i x^2) b_i$. Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A(((a_i x^2) b_i) y) = \mu_A((y b_i)(a_i x^2)) \\ &\geq \mu_A(y b_i) \geq \mu_A(y) \\ \text{and } \gamma_A(xy) &= \gamma_A(((a_i x^2) b_i) y) = \gamma_A((y b_i)(a_i x^2)) \\ &\leq \gamma_A(y b_i) \leq \gamma_A(y). \end{aligned}$$

Hence A is an intuitionistic fuzzy ideal of R. Similarly, for left ideal.

Remark 13. *The concept of intuitionistic fuzzy (left, right, two-sided) ideals coincides in inta-regular LA-rings.*

Proposition 11. *Every intuitionistic fuzzy generalized bi-ideal of an intra-regular LA-ring R with left identity e, is an intuitionistic fuzzy bi-ideal of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy generalized bi-ideal of R and $x, y \in R$, this implies that there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2) b_i$. We have to show that A is an intuitionistic fuzzy LA-subring of R. Now

$$\begin{aligned} x &= (a_i x^2) b_i = (a_i x^2)(e b_i) = (a_i e)(x^2 b_i) \\ &= (a_i e)((xx) b_i) = (a_i e)((b_i x) x) = (x(b_i x))(e a_i) \\ &= (x(b_i x)) a_i = (a_i(b_i x)) x = (a_i(b_i x))(ex) \\ &= (xe)((b_i x) a_i) = (b_i x)((xe) a_i) = (b_i x)((a_i e) x) \\ &= (x(a_i e))(x b_i) = x((x(a_i e)) b_i) = xn, \quad \text{say } n = (x(a_i e)) b_i \end{aligned}$$

Thus

$$\begin{aligned} \mu_A(xy) &= \mu_A((xn)y) \geq min\{\mu_A(x), \mu_A(y)\} \\ \text{and } \gamma_A(xy) &= \gamma_A((xn)y) \leq max\{\gamma_A(x), \gamma_A(y)\}. \end{aligned}$$

Hence A is an intuitionistic fuzzy LA-subring of R.

Theorem 13. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$A \cap B \subseteq A \circ B$ for every intuitionistic fuzzy right ideal B and every intuitionistic fuzzy left ideal A of R.*

Proof. Assume that (1) holds. Let $x \in R$, then there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Now

$$\begin{aligned} x &= (a_i x^2)b_i = (a_i(xx))b_i = (x(a_i x))(eb_i) \\ &= (xe)((a_i x)b_i) = (a_i x)((xe)b_i). \end{aligned}$$

Thus

$$\begin{aligned} (\mu_A \circ \mu_B)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\mu_A(a_i) \wedge \mu_B(b_i)\} \\ &\geq min\{\mu_A(a_i x), \mu_B((xe)b_i)\} \geq min\{\mu_A(x), \mu_B(x)\} \\ &= (\mu_A \wedge \mu_B)(x) = (\mu_A \cap \mu_B)(x). \\ &\Rightarrow \mu_A \cap \mu_B \subseteq \mu_A \circ \mu_B. \end{aligned}$$

Similarly, we have $\gamma_A \cup \gamma_B \supseteq \mu_A \circ \mu_B$. Hence $A \cap B \subseteq A \circ B$, i.e., $(1) \Rightarrow (2)$. Suppose that (2) is true and $a \in R$, then Ra is a left ideal of R containing a by the Lemma 8 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 5. This means that χ_{Ra} is an intuitionistic fuzzy left ideal and $\chi_{aR\cup Ra}$ is an intuitionistic fuzzy right ideal of R, by the Lemma 2. By our supposition $\chi_{aR\cup Ra} \cap \chi_{Ra} \subseteq \chi_{Ra} \circ \chi_{aR\cup Ra}$, i.e., $\chi_{(aR\cup Ra)\cap Ra} \subseteq \chi_{(Ra)(aR\cup Ra)}$. Thus $(aR\cup Ra)\cap Ra \subseteq Ra(aR\cup Ra)$. Since $a \in (aR\cup Ra)\cap Ra$, i.e., $a \in Ra(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra)$. Now

$$\begin{aligned} (Ra)(aR) &= (Ra)((ea)(RR)) = (Ra)((RR)(ae)) \\ &= (Ra)(((ae)R)R) = (Ra)((aR)R) \\ &= (Ra)((RR)a) = (Ra)(Ra). \end{aligned}$$

This implies that

$$\begin{aligned} (Ra)(aR) \cup (Ra)(Ra) &= (Ra)(Ra) \cup (Ra)(Ra) \\ &= (Ra)(Ra) = ((Ra)a)R \\ &= ((Ra)(ea))R = ((Re)(aa))R \\ &= (Ra^2)R. \end{aligned}$$

Thus $a \in (Ra^2)R$, i.e., a is an intra regular. Therefore R is an intra-regular, i.e., $(2) \Rightarrow (1)$.

Theorem 14. *Let R be an intra-regular locally associative LA-ring. Then for every intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ of R, $A(a^n) = A(a^{2n})$ for all $a \in R$, where n is a positive integer.*

Proof. For $n = 1$. Let $a \in R$, this implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. Thus

$$\mu_A(a) = \mu_A((x_i a^2)y_i) \geq \mu_A(x_i a^2) \geq \mu_A(a^2)$$

$$= \mu_A(aa) \geq min\{\mu_A(a), \mu_A(a)\} = \mu_A(a).$$

Similarly, $\gamma_A(a) = \gamma_A(a^2)$, so $A(a) = A(a^2)$. Result is also true for $n = 2$, as $a^2 = aa = ((x_i a^2)y_i)((x_i a^2)y_i) = (x_i^2 a^4)y_i^2$. Assume that the result is true for $n = k$, i.e., $A(a^k) = A(a^{2k})$. Now $a^{k+1} = a^k a = ((x_i^k a^{2k})y_i^k)((x_i a^2)y_i) = (x_i^{k+1}a^{2(k+1)})y_i^{k+1}$. Thus

$$\begin{aligned}
\mu_A(a^{k+1}) &= \mu_A((x_i^{k+1}a^{2(k+1)})y_i^{k+1}) \geq \mu_A(x_i^{k+1}a^{2(k+1)}) \\
&\geq \mu_A(a^{2(k+1)}) = \mu_A(a^{k+1}a^{k+1}) \\
&\geq min\{\mu_A\left(a^{(k+1)}\right), \mu_A\left(a^{(k+1)}\right)\} = \mu_A\left(a^{(k+1)}\right).
\end{aligned}$$

Similarly, $\gamma_A(a) = \gamma_A(a^{2(k+1)})$, so $A(a^{k+1}) = A(a^{2(k+1)})$. Hence by induction method, the result is true for all positive integers.

Proposition 12. *Let R be an intra-regular locally associative LA-ring with left identity e. Then for every intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ of R, $A(ab) = A(ba)$ for all $a, b \in R$.*

Proof. Same as Lemma 14.

Theorem 15. *If an IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is an intuitionistic fuzzy $(1,2)$-ideal of R, then so is $\Box A = (\mu_A, \overline{\mu}_A)$ (resp. $\Diamond A = (\overline{\gamma}_A, \gamma_A)$).*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy $(1,2)$-ideal of R. We have to show that $\Box A = (\mu_A, \overline{\mu}_A)$ is also an intuitionistic fuzzy $(1,2)$-ideal of R. Now

$$\begin{aligned}
\overline{\mu}_A(x-y) &= 1 - \mu_A(x-y) \leq 1 - min\{\mu_A(x), \mu_A(y)\} \\
&= max\{1-\mu_A(x), 1-\mu_A(y)\} = max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}. \\
\overline{\mu}_A(xy) &= 1 - \mu_A(xy) \leq 1 - min\{\mu_A(x), \mu_A(y)\} \\
&= max\{1-\mu_A(x), 1-\mu_A(y)\} = max\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}. \\
\overline{\mu}_A((xa)(yz)) &= 1 - \mu_A((xa)(yz)) \leq 1 - min\{\mu_A(x), \mu_A(y), \mu_A(z)\} \\
&= max\{1-\mu_A(x), 1-\mu_A(y), 1-\mu_A(z)\} \\
&= max\{\overline{\mu}_A(x), \overline{\mu}_A(y), \overline{\mu}_A(z)\}.
\end{aligned}$$

Hence $\Box A$ is an intuitionistic fuzzy $(1,2)$-ideal of R. Similarly, for $\Diamond A = (\overline{\gamma}_A, \gamma_A)$.

Remark 14. 1. *An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is an intuitionistic fuzzy $(1,2)$-ideal of R if and only if $\Box A = (\mu_A, \overline{\mu}_A)$ (resp. $\Diamond A = (\overline{\gamma}_A, \gamma_A)$) is an intuitionistic fuzzy $(1,2)$-ideal of R.*

2. *If an IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is an intuitionistic fuzzy bi-ideal of R, then so is $\Box A = (\mu_A, \overline{\mu}_A)$ (resp. $\Diamond A = (\overline{\gamma}_A, \gamma_A)$).*

3. *An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is an intuitionistic fuzzy bi-ideal of R if and only if $\Box A = (\mu_A, \overline{\mu}_A)$ (resp. $\Diamond A = (\overline{\gamma}_A, \gamma_A)$) is an intuitionistic fuzzy bi-ideal of R.*

Theorem 16. *An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is an intuitionistic fuzzy $(1,2)$-ideal of R if and only if the fuzzy subsets μ_A and $\overline{\gamma}_A$ are fuzzy $(1,2)$-ideals of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy $(1,2)$-ideal of R, this implies that μ_A is a fuzzy $(1,2)$-ideal of R. We have to show that $\overline{\gamma}_A$ is also a fuzzy $(1,2)$-ideal of R. Now

$$\begin{aligned}
\overline{\gamma}_A(x-y) &= 1-\gamma_A(x-y) \geq 1-max\{\gamma_A(x),\gamma_A(y)\} \\
&= min\{1-\gamma_A(x), 1-\gamma_A(y)\} = min\{\overline{\gamma}_A(x),\overline{\gamma}_A(y)\}. \\
\overline{\gamma}_A(xy) &= 1-\gamma_A(xy) \geq 1-max\{\gamma_A(x),\gamma_A(y)\} \\
&= min\{1-\gamma_A(x), 1-\gamma_A(y)\} = min\{\overline{\gamma}_A(x),\overline{\gamma}_A(y)\}. \\
\overline{\gamma}_A((xa)(yz)) &= 1-\gamma_A((xa)(yz)) \geq 1-max\{\gamma_A(x),\gamma_A(y),\gamma_A(z)\} \\
&= min\{1-\gamma_A(x), 1-\gamma_A(y), 1-\gamma_A(z)\} \\
&= min\{\overline{\gamma}_A(x),\overline{\gamma}_A(y),\overline{\gamma}_A(z)\}.
\end{aligned}$$

Therefore $\overline{\gamma}_A$ is a fuzzy $(1,2)$-ideal of R.

Conversely, suppose that μ_A and $\overline{\gamma}_A$ are fuzzy $(1,2)$-ideals of R. We have to show that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $(1,2)$-ideal of R. Now

$$\begin{aligned}
1-\gamma_A(x-y) &= \overline{\gamma}_A(x-y) \geq min\{\overline{\gamma}_A(x),\overline{\gamma}_A(y)\} \\
&= min\{1-\gamma_A(x), 1-\gamma_A(y)\} \\
&= 1-max\{\gamma_A(x),\gamma_A(y)\}. \\
1-\gamma_A(xy) &= \overline{\gamma}_A(xy) \geq min\{\overline{\gamma}_A(x),\overline{\gamma}_A(y)\} \\
&= min\{1-\gamma_A(x), 1-\gamma_A(y)\} \\
&= 1-max\{\gamma_A(x),\gamma_A(y)\}. \\
1-\gamma_A((xa)(yz)) &= \overline{\gamma}_A((xa)(yz)) \geq min\{\overline{\gamma}_A(x),\overline{\gamma}_A(y),\overline{\gamma}_A(z)\} \\
&= min\{1-\gamma_A(x), 1-\gamma_A(y), 1-\gamma_A(z)\} \\
&= 1-max\{\gamma_A(x),\gamma_A(y),\gamma_A(z)\}.
\end{aligned}$$

Therefore A is an intuitionistic fuzzy $(1,2)$-ideal of R.

Remark 15. *An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is an intuitionistic fuzzy bi-ideal of R if and only if the fuzzy subsets μ_A and $\overline{\gamma}_A$ are fuzzy bi-ideals of R.*

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems, 20(1986) 87-96.

[2] K. T. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy sets and systems, 61(1994) 137-142.

[3] B. Banerjee and D. K. Basnet, Intuitionistic fuzzy subrings and ideals, J. Fuzzy Math., 11(2003) 139-155.

[4] R. J. Cho, J. Jezek and T. Kepka, Paramedial Groupoids, Czechoslovak Math. J., 49(1999) 277-290.

[5] P. S. Das, Fuzzy groups and level subgroups, J. Math. Anal. Appli., 84(1981) 264-269.

[6] K. C. Gupta and M. K. Kantroo, The intrinsic product of fuzzy subsets of a ring, Fuzzy sets and systems, 57(1993) 103-110.

[7] K. Hur, H. w. Kang and H. k. Song, Intuitionistic Fuzzy subgroups and subrings, Honam Math. J., 25(2003) 19-41.

[8] K. Hur, S. Y. Jang and H. W. Kang, Intuitionistic fuzzy ideals of a ring, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., 12 (2005) 193-209.

[9] J. Jezek and T. Kepka, Medial groupoids, Rozpravy CSAV Rada mat. a prir. ved 93/2, 1983, 93 pp.

[10] M. S. Kamran, Conditions for LA-semigroups to resemble associative structures, Ph.D. Thesis, Quaid-i-Azam University, Islamabad, 1993.

[11] M. A. Kazim and M. Naseerdin, On almost semigroups, Alig. Bull. Math., 2(1972) 1-7.

[12] N. Kuroki, Regular fuzzy duo rings, Inform. Sci., 94(1996) 119-139.

[13] W. J. Liu, Fuzzy invariant subgroups and ideals, Fuzzy sets and systems, 8(1982) 133-139.

[14] T. K. Mukherjee and M. K. Sen, On fuzzy ideals of a ring 1, Fuzzy sets and systems, 21(1987) 99-104.

[15] T. K. Mukherjee and M. K. Sen, Prime fuzzy ideals in rings, Fuzzy sets and systems, 32(1989) 337-341.

[16] P. V. Protic and N. Stevanovic, AG-test and some general properties of Abel-Grassmann's groupoids, Pure Math. Appli., 6(1995) 371-383.

[17] V. S. Ramamurthi, Weakly regular rings, Canad. Math. Bull., 16 (1973) 317-321.

[18] T. Shah, N. Kausar and I. Rehman, Intuitionistic fuzzy normal subrings over a non-associative ring, An. St. Univ. Ovidius Constanta, 1(2012) 369-386.

[19] Shah, Kausar, "Characterizations of non-associative ordered semigroups by their fuzzy bi-ideals", Theoretical Computer Science, Vol. 529 (2014) 96-110.

[20] T. Shah and I. Rehman, On LA-rings of finitely non-zero functions, Int. J. Contempt. Math. Sci., 5(2010) 209-222.

[21] U. M. Swamy and K. L. N. Swamy, Fuzzy prime ideals of rings, J. Math. Anal. Appl., 134(1988) 94-103.

[22] L. A. Zadeh, Fuzzy sets, Information and control, 8(1965) 338-363.