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On a Class of Parameter Dependent Series Generalizing Euler's Constant

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Abstract. Series which depend on a parameter and generalize the constant discovered by Euler are introduced and studied. Convergence results are established. An infinite series expansion is obtained from these generalized formulas which can be used to evaluate the generalized constant. Euler's constant can be obtained as a special case. Some asymptotic results are formulated and limits of some closely related sequences are given.

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1. Introduction

A sequence τ_n obtained by subtracting $\log n$ from the harmonic numbers H_n is known to converge to a real number called γ. Euler discovered the constant γ named after him and is defined by the limit

$$\gamma = \lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \log\, n\Big). \tag{1.1}$$

This is a fundamental constant which is now called Euler's constant. It has connections with values of the gamma function and Riemann zeta function and as well there are its close relatives e^{γ} and $e^{-\gamma}$. There are many unsolved problems concerning the nature of the constant, such as the question of the irrationality of Euler's constant. The study of Euler's constant continues to attract the interest of many investigators and will likely continue to do so [1, 2].

In a paper written in 1731, Euler summed the harmonic series in terms of zeta values and computed Euler's constant to five decimal places. Since then, this number has assumed a tremendously important role in mathematics and many of its applications [3]-[6].

Proposition (Euler) [7] The limit

$$\gamma = \lim_{n\to\infty} (H_n - \log\, n)$$

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exists. It is given by the conditionally convergent series

$$\gamma = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n}.$$

□

Euler reports that this series converges conditionally since it is an alternating series with decreasing terms and finds that $\gamma \approx 0.577218$.

Many estimates are also known for γ as well as upper and lower bounds for $\tau_n - \gamma$. For example, a recent estimate for this difference which has appeared [8] is for $n \in \mathbb{N}$,

$$\frac{1}{2n + \frac{2}{5}} < \tau_n - \gamma < \frac{1}{2n + \frac{1}{3}}.$$

The intention here is to provide and then study some reasonable generalizations of this sequence τ_n for Euler's constant and to obtain some results which yield a powerful tool for the high precision calculation of Euler's constant itself [9], [10].

2. Generalizations of Euler's Constant

Define the sequence $\gamma_n(a)$ which depends on the real variable $a \in [0, \infty)$ as

$$\gamma_n(a) == \sum_{k=1}^{n} [\frac{1}{k+a} - \log(a+k+1) + \log(a+k)] \tag{2.1}$$

$$= \sum_{k=1}^{n} (\frac{1}{k+a}) - \log(a+n+1) + \log(a+1).$$

Setting $a = 0$ in (2.1) and defining $\gamma_n = \gamma_n(0)$, equation (2.1) becomes

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1). \tag{2.2}$$

Up to a term which goes to zero with increasing n, this sequence is the same as the sequence τ_n, and Euler's constant is obtained in the limit

$$\gamma = \lim_{n \to \infty} \gamma_n. \tag{2.3}$$

Equation (2.1) leads to one generalization of Euler's constant. It can be generalized further to depend on two parameters a, b by writing

$$\gamma_n(a, b) = \sum_{k=1}^{n} \frac{1}{k+b} - \log(n+a). \tag{2.4}$$

The sequence in (2.4) can be easily related to that in (2.1) by writing

$$\gamma_n(a,b) = \sum_{k=1}^{n} \frac{1}{k+b} - \log(b+n+1) + \log(b+1) + \log(b+n+1) - \log(a+n) - \log(b+1)$$

$$= \gamma_n(b) + \log\big(\frac{b+n+1}{a+n}\big) - \log(b+1). \tag{2.5}$$

Since the second term on the right of (2.5) goes to zero as n gets large, existence of the limit $\lim_{n\to\infty}\gamma_n(b)$ in (2.5) implies that $\lim_{n\to\infty}\gamma_n(a,b) = \gamma(a,b)$ exists as well by the usual limit rules. Moreover, the limit $\gamma(a,b)$ can be calculated in terms of $\gamma(b)$ by means of

$$\gamma(a,b) = \gamma(b) - \log(b+1). \tag{2.6}$$

3. Convergence of the Sequences

The expression for $\gamma_n(a)$ in (2.1) can be written in an equivalent form which is more useful for establishing convergence of the sequence,

$$\gamma_n(a) = \sum_{k=1}^{n} \big(\frac{1}{a+k} - \int_k^{k+1} \frac{dx}{a+x}\big). \tag{3.1}$$

The integration variable is changed in (3.1) by means of the linear transformation $x = k+t$. This serves to transform the integration interval to $(0,1)$ and $\gamma_n(a)$ takes the form

$$\gamma_n(a) = \sum_{k=1}^{n} \big(\frac{1}{a+k} - \int_k^{k+1} \frac{dx}{a+x}\big)$$

$$= \sum_{k=1}^{n} \big(\int_0^1 \frac{dt}{a+k} - \int_0^1 \frac{dt}{a+k+t}\big) = \sum_{k=1}^{n} \int_0^1 \frac{t\,dt}{(a+k)(a+k+t)}. \tag{3.2}$$

Theorem 1. *Let a be real and not equal to a negative integer. Then the sequence $\gamma_n(a)$ converges to a continuous function of a.*

Proof Clearly for $a \geq 0$ the following upper bound holds for the k-th term in (3.2)

$$\int_0^1 \frac{t}{(a+k)(a+k+t)}\,dt \leq \frac{1}{2(a+k)^2} \leq \frac{1}{2k^2}.$$

Since the series of inverse squares converges, this sequence provides the required bounding sequence which allows us to conclude that

$$\sum_{k=1}^{\infty} \int_0^1 \frac{1}{(a+k)(a+k+t)}\,dt \leq \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

By applying the Weierstrass test at this point allows us to conclude series (3.2) converges absolutely and uniformly when $a \geq 0$. Since the convergence is uniform and the terms in the series are continuous functions of a, the limit $\gamma(a)$ is a continuous function of a.

If $a < 0$ and not equal to a negative integer, it suffices to study the remainder series

$$\sum_{k=N_0+1}^{\infty} \int_0^1 \frac{t}{(a+k)(a+k+t)}\, dt \tag{3.3}$$

Choose $N_0 > -a$, then the following upper bound for the k-th term holds since $a + N_0 > 0$ and $j \in \mathbb{N}$,

$$\int_0^1 \frac{t}{(a+k)(a+k+t)}\, dt|_{k=N_0+j} \leq \frac{1}{2j^2}.$$

Consequently, there is the bound

$$\sum_{k=N_0+1}^{\infty} \int_0^1 \frac{1}{(a+k)(a+k+t)}\, dt < \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

Thus the remainder series (3.3) converges uniformly by the Weierstrass theorem as in the case where a is positive. Hence the limit $\gamma(a)$ exists and is again continuous, since the remainder series differs form the whole series by a finite number of terms. □

4. Expansion of the Remainder

As a consequence of Theorem 1, we can write

$$\gamma(a) = \gamma_N(a) + \sum_{k=N+1}^{\infty} (\frac{1}{a+k} - \int_k^{k+1} \frac{dx}{a+x}) = \gamma_N(a) + \sum_{k=N+1}^{\infty} \int_0^1 \frac{t}{(a+k)(a+k+t)}\, dt. \tag{4.1}$$

It will prove useful to denote the remainder of the series in (4.1) as

$$R_N(a) = \sum_{k=N+1}^{\infty} \int_0^1 \frac{t}{(a+k)(a+k+t)}\, dt. \tag{4.2}$$

Thus (4.1) can be summarized concisely in the form

$$\gamma(a) = \gamma_N(a) + R_N(a).$$

It is possible to generate an expansion for (4.2) by proceeding in the following way. First add and subtract the following quantity to (4.2).

$$\int_0^1 t\, dt \sum_{k=N+1}^{\infty} \frac{1}{(a+k)(a+k+1)}. \tag{4.3}$$

This will not alter the value of (4.2). The sum (4.3) is independent of t and converges when a is not a negative integer. The result is obtained upon evaluating the sum (4.3) in closed form

$$R_N(a) = \sum_{k=N+1}^{\infty} \int_0^1 \left(\frac{1}{(a+k)(a+k+1)} - \frac{1}{(a+k)(a+k+1)}\right) dt + \int_0^1 t\,dt \sum_{k=N+1}^{\infty} \frac{1}{(a+k)(a+k+1)}$$

$$= \sum_{k=N+1}^{\infty} \int_0^1 \frac{t(1-t)}{(a+k)(a+k+1)(a+k+t)}\,dt + \int_0^1 tdt\,\frac{1}{a+N+1}. \qquad (4.4)$$

Define the first in a sequence of such constants α_1 as the integral,

$$\alpha_1 = \int_0^1 t\,dt = \frac{1}{2}. \qquad (4.5)$$

Substituting (4.5) into (4.4), $R_N(a)$ can be summarized as follows

$$R_N(a) = \sum_{k=N+1}^{\infty} \int_0^1 \frac{t(1-t)}{(a+k)(a+k+1)(a+k+t)}\,dt + \frac{\alpha_1}{a+N+1}.$$

Taking the first term on the right side of $R_N(a)$ and repeating the process, it is determined that

$$\sum_{k=N+1}^{\infty} \int_0^1 \frac{t(1-t)}{(a+k)(a+k+1)(a+k+t)}\,dt$$

$$= \sum_{k=N+1}^{\infty} \int_0^1 \frac{t(1-t)(2-t)}{(a+k)(a+k+1)(a+k+2)(a+k+t)}\,dt + \frac{1}{2}\int_0^1 t(1-t)\,dt\,\frac{1}{(a+N+1)(a+N+2)}. \qquad (4.6)$$

Defining the constant α_2 to be

$$\alpha_2 = \frac{1}{2}\int_0^1 t(1-t)\,dt = \frac{1}{12},$$

remainder $R_N(a)$ in (4.2) can be expressed as

$$R_N(a) = \sum_{k=N+1}^{\infty} \int_0^1 \frac{t(1-t)(2-t)}{(a+k)(a+k+1)(a+k+2)(a+k+t)}\,dt + \sum_{k=1}^{2} \frac{\alpha_k}{(a+N+1)\cdots(a+N+k)}. \qquad (4.7)$$

Formula (4.7) leads to the statement of a Theorem. For the proof, the following Lemma will be required.

Lemma 1. *Suppose a is not equal to zero or a negative integer. Then for all integers $m \geq 1$, the following sum can be expressed in closed form,*

$$\sum_{k=N+1}^{\infty} \frac{1}{(a+k)(a+k+1)\cdots(a+k+m)} = \frac{1}{m\cdot(a+N+1)\cdots(a+N+m)}. \qquad (4.8)$$

Proof The proof is by induction on m. Suppose first $m = 1$ and a is as in the statement, then denoting the sum by $f(a,m)$, we have

$$f(a,1) = \sum_{k=N+1}^{\infty} \frac{1}{(a+k)(a+k+1)} = \sum_{k=N+1}^{\infty} (\frac{1}{a+k} - \frac{1}{a+k+1}) = \frac{1}{a+N+1}.$$

So the result holds for $m = 1$ and all a in the stated domain. Suppose (4.8) holds up to the integer $m-1$ and write the difference

$$f(a,m-1) - f(a+1,m-1)$$

$$= \sum_{k=N+1}^{\infty} (\frac{1}{(a+k)\cdots(a+k+m-1)} - \frac{1}{(a+k+1)\cdots(a+k+m)})$$

$$= \sum_{k=N+1}^{\infty} (\frac{a+k+m}{(a+k)(a+k+1)\cdots(a+k+m)} - \frac{a+k}{(a+k)\cdots(a+k+m)})$$

$$= m \sum_{k=N+1}^{\infty} \frac{1}{(a+k)\cdots(a+k+m)} = mf(a,m). \tag{4.9}$$

Next, replace m by $m-1$ in (4.8) and calculate the difference explicitly

$$f(a,m-1) - f(a+1,m-1)$$

$$= \frac{a+N+m}{(m-1)(a+N+1)\cdots(a+N+m)} - \frac{a+N+1}{(m-1)(a+N+1)\cdots(a+N+m)}$$

$$= \frac{1}{(a+N+1)\cdots(a+N+m)}. \tag{4.10}$$

Equating the results (4.9) and (4.10), the sum in (4.8) is obtained for the case m. Therefore, the principle of mathematical induction implies (4.8) holds for all positive integers m and a in the domain as stated. □

It should be noted that by continuity, (4.8) can be extended to the case $a = 0$, since both sides are continuous functions of a. It is now possible to use this to prove a key result which is stated now.

Theorem 2. *Let $m \in \mathbb{N}$ such that $m \geq 2$ and a is not a negative integer. Define $R_{N,m}(a)$ to be*

$$R_{N,m}(a) = \sum_{k=N+1}^{\infty} \int_0^1 \frac{t(1-t)\cdots(m-t)}{(a+k)(a+k+1)\cdots(a+k+m)(a+k+t)}\, dt. \tag{4.11}$$

Then remainder (4.2) admits the following expansion

$$R_N(a) = R_{N,m}(a) + \sum_{k=1}^{m} \frac{\alpha_k}{(a+N+1)\cdots(a+N+k)}. \tag{4.12}$$

The constants α_k which appear in (4.12) *are defined to be*

$$\alpha_1 = \frac{1}{2},$$

$$\alpha_k = \frac{1}{k} \int_0^1 t(1-t)\cdots(k-1-t)\,dt, \qquad k \geq 2. \tag{4.13}$$

Proof The cases $m = 1, 2$ have already been worked out and may serve as the basis for a proof by induction. Equation (4.12) may be taken as the induction hypothesis with m replaced by $m-1$. It will be shown using Lemma 2 that this case implies (4.12) with m. Writing out $R_{N,m-1}(a)$ it is found that

$$R_{N,m-1}(a) - \sum_{k=N+1}^{\infty} \int_0^1 \frac{t(1-t)\cdots(m-1-t)}{(a+k)\cdots(a+k+m-1)(a+k+m)}\,dt$$

$$+ \int_0^1 t(1-t)\cdots(m-1-t)\,dt \sum_{k=N+1}^{\infty} \frac{1}{(a+k)\cdots(a+k+m)}$$

$$= R_{N,m}(a) + \frac{1}{m} \int_0^1 t(1-t)\cdots(m-1-t)\,dt \sum_{k=N+1}^{\infty} \frac{1}{(a+k)\cdots(a+k+m)}$$

$$= R_{N,m}(a) + \frac{\alpha_k}{(a+N+1)\cdots(a+N+m)}. \tag{4.14}$$

Substituting (4.14) into (4.12), it may be concluded that case $m-1$ implies m and so (4.12) is proved by means of the principle of mathematical induction. The α_k are determined by means of (4.13). □

Lemma 2. *The following upper and lower bounds hold for the sequence α_m when $m \geq 2$*

$$\frac{1}{6m}(m-2)! \leq \alpha_m \leq \frac{1}{6m}(m-1)!. \tag{4.15}$$

Proof The bounds (4.15) are an immediate consequence of the observation that

$$\frac{1}{m} \int_0^1 t(1-t)\,dt \cdot 1 \cdot 2 \cdot 3 \cdots (m-2) \leq \alpha_m \leq \frac{1}{m} \int_0^1 t(1-t)\,dt \cdot 2 \cdot 3 \cdots (m-1). \tag{4.16}$$

Substituting the numerical value of the integral which is $1/6$ into (4.16), we arrive at (4.15). □

Lemma 2 can be used to study the behavior of $R_{N,m}(a)$ when a is positive in (4.11) as $m \to \infty$.

Theorem 3. *Suppose $a \geq 0$ and $N \in \mathbb{N}$ is fixed, then the following limit holds*

$$\lim_{m \to \infty} R_{N,m}(a) = 0. \tag{4.17}$$

Proof Clearly we see that $R_{N,m}(a) > 0$ for all $a \geq 0$, and so the following bounds for $R_{N,m}(a)$ hold

$$0 < R_{N,m}(a) \leq R_{N,m}(0) = \sum_{k=N+1}^{\infty} \int_0^1 \frac{t(1-t)\cdots(m-t)}{k(k+1)\cdots(k+m)(k+t)}\, dt$$

$$< \alpha_{m+1} \sum_{k=N+1}^{\infty} \left(\frac{1}{(k-1)k\cdots(k+m-1)} - \frac{1}{k(k+1)\cdots(k+m)}\right)$$

$$= \frac{\alpha_{m+1}}{N(N+1)\cdots(N+m)} = \alpha_{m+1}\frac{(N-1)!}{(N+m)!}. \tag{4.18}$$

Using Lemma 2 in (4.18), the following bounds result

$$0 < R_{N,m}(a) < \frac{m!}{6(m+1)} \cdot \frac{(N-1)!}{(N+m)!} = \frac{1}{6N(m+1)} \cdot \frac{1}{\binom{N+m}{m}}. \tag{4.19}$$

Letting $m \to \infty$ in (4.19) and applying the squeeze theorem, the result stated in (4.17) follows. $\square$

5. Generalizations of Euler's Constant

Theorem 4. *Let $a \in [0, \infty)$, then a form of generalized Euler's constant can be defined and calculated from*

$$\gamma(a) = \gamma_N(a) + \sum_{k=1}^{\infty} \frac{\alpha_k}{(a+N+1)\cdots(a+N+k)}. \tag{5.1}$$

Moreover, Euler's constant (1.1) *is obtained as*

$$\gamma = \sum_{k=1}^{N} \frac{1}{k} - \log(N+1) + \sum_{k=1}^{\infty} \frac{\alpha_k}{(N+1)\cdots(N+k)}. \tag{5.2}$$

Proof To obtain (5.1), substitute (4.12) for $R_N(a)$ into (4.1) to obtain

$$\gamma(a) = \gamma_N(a) + R_{N,m}(a) + \sum_{k=1}^{m} \frac{\alpha_k}{(a+N+1)\cdots(a+N+k)}. \tag{5.3}$$

Letting $m \to \infty$ in (5.3) and using (4.17), equation (5.1) is obtained. Finally, set $a = 0$ in (5.1) and we get result (5.2) for the constant found by Euler. It is also worth noting that when (2.1) is substituted into (5.1), the generalized constant takes the form

$$\gamma(a) = \sum_{k=1}^{N} \frac{1}{a+k} - \log(a+N+1) + \log(a+1) + \sum_{k=1}^{\infty} \frac{\alpha_k}{(a+N+1)\cdots(a+N+k)}. \tag{5.4}$$

□

Setting $a = 0$ in (5.4) and substituting the definition $(N+1)_k = (N+1)\cdots(N+k)$, Euler's constant (5.2) can be put into the form,

$$\gamma = \sum_{k=1}^{N} \frac{1}{k} - \log(N+1) + N! \sum_{k=1}^{\infty} \frac{\alpha_k}{(N+k)!} = \sum_{k=1}^{N} \frac{1}{k} - \log(N+1) + \sum_{k=1}^{\infty} \frac{\alpha_k}{(N+1)_k}. \quad (5.5)$$

A formula similar to (5.5) for γ has appeared before [11].

6. Asymptotic Results and Some Limits

Theorem 5. *The limit function* $\gamma(a) : [0,\infty) \to [0,\infty)$ *defined in* (5.4)

$$\gamma(a) = \lim_{N\to\infty} \gamma_N(a) = \lim_{N\to\infty} \Big(\frac{1}{a+1} + \cdots + \frac{1}{a+N} - \log(\frac{a+N+1}{a+1})\Big), \quad (6.1)$$

for $a \in (0,\infty)$ *is a strictly decreasing function of* a.

Proof Let $N \in \mathbb{N}$ with $h_N : (0,\infty) \to (0,\infty)$ defined to be the function

$$h_N(a) = \frac{1}{a+1} + \frac{1}{a+2} + \cdots + \frac{1}{a+N} - \log(\frac{a+N+1}{a+1}). \quad (6.2)$$

Differentiate (6.2) term by term with respect to a to obtain

$$h_N'(a) = -\sum_{k=1}^{N} \frac{1}{(a+k)^2} + \frac{N}{(a+1)(a+N+1)} < -\sum_{k=1}^{N} \frac{1}{(a+k)(a+k+1)} + \frac{N}{(a+1)(a+N+1)}$$

$$= -\frac{N}{(a+1)(a+N+1)} + \frac{N}{(a+1)(a+N+1)} = 0.$$

Therefore, at each $a \in (0,\infty)$, the derivative of $h_N(a)$ satisfies $h_N'(a) < 0$ and so it follows that the function $h_N(a)$ is strictly decreasing for all integers N. Hence it follows from this that the function $\gamma(a)$ is also strictly decreasing on $(0,\infty)$ as well. □

Theorem 6. *Let* $a \in (0,\infty)$ *and for* $N \in \mathbb{N}$, *define three sequences* λ_N, μ_N *and* σ_N *as follows*

$$\lambda_N(a) = \frac{1}{a+1} + \frac{1}{a+2} + \cdots + \frac{1}{a+N} - \log(\frac{a+N}{a+1} + \frac{1}{2(a+1)}), \quad (6.3)$$

$$\mu_N(a) = \lambda_N(a) - \frac{1}{24(a+N+\frac{1}{2})^2}, \quad (6.4)$$

$$\sigma_N(a) = \lambda_N(a) - \frac{1}{24(a+N)^2}. \quad (6.5)$$

Then the following statements hold:

(i) $\gamma(a) < \lambda_{N+1}(a) < \lambda_N(a)$ *for* $N \in \mathbb{N}$ *and*

$$\lim_{N\to\infty} N^2\left(\lambda_N(a) - \gamma(a)\right) = \frac{1}{24}. \tag{6.6}$$

(ii) $\mu_N(a) < \mu_{N+1}(a) < \gamma(a)$ *for each* $N \in \mathbb{N}$ *and*

$$\lim_{N\to\infty} N^4\left(\gamma(a) - \mu_N(a)\right) = \frac{7}{960}. \tag{6.7}$$

(iii) $\gamma(a) < \sigma_N(a) < \sigma_{N+1}(a)$ *for each* $N \in \mathbb{N}$ *and*

$$\lim_{N\to\infty} N^3\left(\gamma(a) - \sigma_N(a)\right) = \frac{1}{24}. \tag{6.8}$$

(iv) For each $N \in \mathbb{N}$*, the following bounds hold*

$$\frac{1}{24(a+N)^2} < \lambda_N(a) - \gamma(a) < \frac{1}{24(a+N-\frac{1}{2})^2}. \tag{6.9}$$

Proof *(i)* Clearly for $a \in (0,\infty)$ and $n \geq 1$ it is the case that

$$\lambda_n(a) - \lambda_{n+1}(a) = \log(\frac{2a+2n+3}{2a+2n+1}) - \frac{1}{a+n+1}. \tag{6.10}$$

To study this difference between terms in the sequence as a function of n and a, let us define the function $f(a,x)$ for $x \geq 0$ which agrees with (6.10) when $x = n$ as

$$f(a,x) = \log(\frac{2a+2x+3}{2a+2x+1}) - \frac{1}{a+x+1}. \tag{6.11}$$

It is the case that $f(a,x) \to 0^+$ as $x \to \infty$. Differentiate $f(a,x)$ with respect to x to give

$$f_x(a,x) = -\frac{2a+2x+1}{(2a+2x+1)^2(2a+2x+3)(a+x+1)^3} < 0,$$

Moreover, $f(0,0) > 0$ and $f_a(a,0)$ at fixed $x = 0$ remains positive while decreasing monotonically to zero for all a in the interval. Therefore $f(a,x)$ must remain positive as it decreases to zero monotonically, which implies from (6.10) that $\lambda_n(a) > \lambda_{n+1}(a)$. Thus $\lambda_n(a)$ satisfies

$$\gamma_n(a) = \lambda_n(a) + \log(\frac{2a+2n+1}{2a+2n+2}) < \lambda_n(a),$$

since the logarithm is negative, and so $\gamma_n(a)$ approaches $\gamma(a)$ from above as $n \to \infty$. The Césaro-Stoltz theorem permits the evaluation of the limit from

$$n^2\left(\lambda_n(a) - \gamma(a)\right) = \frac{\lambda_{n+1}(a) - \lambda_n(a)}{\dfrac{1}{(n+1)^2} - \dfrac{1}{n^2}} = \frac{\dfrac{1}{a+n+1} + \log(\dfrac{2a+2n+1}{2a+2n+3})}{\dfrac{1}{(n+1)^2} - \dfrac{1}{n^2}}$$

$$= \frac{1}{24} - \frac{1}{8\,n}(a + \frac{1}{2}) + O(\frac{1}{n^2}).$$

This development implies the required limit as stated.

(*ii*)

$$\mu_{n+1}(a) - \mu_n(a) = \lambda_{n+1}(a) - \lambda_n(a) - \frac{1}{24(a+n+\frac{1}{2})^2} + \frac{1}{24(a+n+\frac{1}{2})^2}$$

$$= \frac{1}{a+n+1} - \log(\frac{2a+2n+3}{2a+2n+1}) - \frac{1}{24(a+n+\frac{3}{2})^2} + \frac{1}{24(a+n+\frac{1}{2})^2} \qquad (6.12)$$

As in (*i*), define $g(a,x)$ to be the right side of (6.12) with n replaced by x. Differentiating $g(a,x)$ with respect to x gives

$$g_x(a,x) = -\frac{28(a+x)^2 + 56(a+x) + 25}{3(a+x+1)^2(2a+2x+3)^3(2a+2x+1)^3} < 0,$$

and $g(a,x) \to 0^+$ as $x \to \infty$. Since $g(0,0) > 0$ and $g_a(a,0) < 0$, the function $g(a,0)$ remains positive while decreasing monotonically to zero. Consequently (6.12) implies the inequalities $\gamma(a) \geq \mu_{n+1}(a) > \mu_n(a)$. The Césaro-Stoltz theorem gives the required limit by writing

$$n^4\left(\gamma(a) - \mu_n(a)\right) = \frac{\gamma(a) - \mu_{n+1}(a) - \gamma(a) + \mu_n(a)}{\dfrac{1}{(n+1)^4} - \dfrac{1}{n^4}} = \frac{\mu_n(a) - \mu_{n+1}(a)}{\dfrac{1}{(n+1)^4} - \dfrac{1}{n^4}}$$

$$= \frac{7}{960} - \frac{7}{192\,n}(a + \frac{1}{2}) + O(\frac{1}{n^2}).$$

(*iii*)

$$\sigma_{n+1}(a) - \sigma_n(a) = \lambda_{n+1}(a) - \lambda_n(a) - \frac{1}{24(a+n+1)^2} + \frac{1}{24(a+n)^2}$$

$$= \frac{1}{a+n+1} - \log(\frac{2a+2n+3}{2a+2n+1}) - \frac{1}{24(a+n+1)^2} + \frac{1}{24(a+n)^2}. \qquad (6.13)$$

As in (*i*) and (*ii*), define the function $h(a,x)$ to be the right-hand side of (6.13) with n replaced by x. Differentiating $h(a,x)$ with respect to x, we have

$$h_x(a,x) = -\frac{24(a+x)^3 + 37(a+x)^2 + 17(a+x) + 3}{12(a+x)^3(a+x+1)^3(2a+2x+1)(2a+2x+3)}.$$

Therefore $h(a,x)$ is strictly decreasing in x and approaches zero as $x \to \infty$. Also $f(0,1) > 0$ and $f_a(a,1) < 0$ so $h(a,1)$ is always positive and decreasing at fixed $x = 1$, so we conclude that $h(a,x)$ remains positive. Applying this to (6.13), the required inequalities for $\sigma_n(a)$ are obtained. The Césaro-Stoltz theorem provides the means for determining the limit

$$n^3\left(\gamma(a) - \sigma_n(a)\right) = \frac{\sigma_n(a) - \sigma_{n+1}(a)}{\dfrac{1}{(n+1)^3} - \dfrac{1}{n^3}}$$

$$= \frac{-\frac{1}{a+n+1} + \log(\frac{2a+2n+3}{2a+2n+1}) + \frac{1}{24(a+n+1)} - \frac{1}{24(a+n)^2}}{\frac{1}{(n+1)^3} - \frac{1}{n^3}}$$

$$= \frac{1}{24} - \frac{1}{6\,n}(a + \frac{23}{120}) + O(\frac{1}{n^2}).$$

(*iv*) This follows as a straightforward consequence of the previous results. Since $\mu_n(a) < \gamma(a)$, it follows that

$$\lambda_n(a) - \frac{1}{24(a+n-\frac{1}{2})^2} - \gamma(a) < 0.$$

This implies the upper bound,

$$\lambda_n(a) - \gamma(a) < \frac{1}{24(a+n-\frac{1}{2})^2}.$$

In a similar way, since $\sigma_n(a) > \gamma(a)$ the lower bound follows directly as well. □

Theorem 7. (*i*) *Let v_n be any sequence which converges to Euler's constant. Then Euler's constant is also given by the limit*

$$\gamma = \lim_{n\to\infty} \frac{\sum_{k=0}^{\infty} (\frac{n^k}{k!})^p \, v_k}{\sum_{k=0}^{\infty} (\frac{n^k}{k!})^p}. \tag{6.14}$$

(*ii*) *Let H_n denote the harmonic sequence and suppose $p > 0$. Euler's constant is given by the limit*

$$\gamma = \lim_{n\to\infty} \frac{\sum_{k=0}^{\infty} (\frac{n^k}{k!})^p (H_k - \log(k+1))}{\sum_{k=0}^{\infty} (\frac{n^k}{k!})^p}. \tag{6.15}$$

Proof (*i*) Suppose the sequence v_n converges to Euler's constant, $\lim_{n\to\infty} v_n = \gamma$. Define sequences a_n and b_n with $p > 0$ as follows,

$$a_n = \frac{v_n}{(n!)^p}, \qquad b_n = \frac{1}{(n!)^p}. \tag{6.16}$$

Then $b_n > 0$ and it can be verified using the Ratio test that $\sum_{n=0}^{\infty} b_n t^n$ converges for t real and $p > 0$, hence (6.15) satisfies

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \gamma.$$

Then it follows by a theorem [12] that the series $\sum_{n=0}^{\infty} a_n t^n$ converges and that

$$\lim_{t\to\infty} \frac{\sum_{k=0}^{\infty} \dfrac{v_k}{(k!)^p} t^k}{\sum_{k=0}^{\infty} \dfrac{1}{(k!)^p} t^k} = \gamma. \tag{6.17}$$

As far as the limit is concerned, the variable t in (6.17) may be replaced by n^p, since the limit exists by the theorem and is unique. Replacing $t \to \infty$ with $n \to \infty$ on the limit, then (6.17) immediately implies (6.14).

(*ii*) Taking $v_n = \gamma_n$ defined in (2.2), then v_n converges to γ and can be used in (6.14). This completes the proof. □

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