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Value-at-Risk Modeling with Conditional Copulas in Euclidean Space Framework

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Abstract. This paper aims to establish an analytic relation between a time-varying conditional copula and the value at risk modeled by the underlying. Specically, under the assumption that the space is euclidean we use scalar product to clarify a link between the conditional copula varying with time and norms. It is then established a new expression on the geometric yield.

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1. Introduction

Modeling the risk of portfolio is to highlight the different methods or protocols to minimize the loss of values of a portfolio. The use of multivariate copulas in this modeling, is a contemporary approach, to develop indicators for the evaluation of the dependence between the different assets of this portfolio.

In multivariate theory of probability a pioneer theorem (Sklar,1959). Abe SKlar showed that the copula function enables to capture and to piece together the univariate models (Sklar's Theorem). Therefore, every n-dimensional continuous distribution H can be canonically parameterized by its univariate marginal $H_1; ...; H_n$ using $[0,1]^n$ a copula C defined on the unit cube [0; 1] , such as

$$H(x_1, ..., x_n) = C[H(x_1), ..., H_n(x_n)]; \; with \; (x_1, ..., x_n) \in \bar{\mathbb{R}}^n = [-\infty, +\infty]^n . \qquad (1)$$

Under additional assumptions, differentiating the formula (1) shows that the density function of the copula is equal to the ratio of the joint density h of H to the product of n marginal densities h_i such as, for all $(u_1, ..., u_n) \in [0,1]^n$,

$$c(u_1, ..., u_n) = \frac{\partial^n C(u_1, ..., u_n)}{\partial u_1 \times ... \times \partial u_n} = \frac{h\left[H_n^{-1}(u_1), ..., H_n^{-1}(u_n)\right]}{h_1\left[H_1^{-1}(u_1) \times ... \times H_n^{-1}(u_n)\right]}. \qquad (2)$$

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where H_i^{-1} is the quantile function of H_i; that is, $H_i^{-1}(u) = inf\{x \in \mathbb{R}, H(x_i) \geq u\}$. In stochastic financial analysis, from the definition in the univariate case we know that the quantile function provides a point that accumulates a probability for the left tail and for the right tail. The univariate quantile function $Q_X(\alpha)$ is used in the risk theory to define the univariate risk measure : the value at risk is defined.

More generally in multivariate study, for a random vector X satisfying the regularity conditions, we define the multidimensional VaR at probability level α by :

$$VaR_\alpha(X) = \mathbb{E}[X \mid X \in \partial L(\alpha)] \tag{3}$$

where $\partial L(\alpha)$ is the boundary of the $\alpha - level$ set of F, the univariate component of the vector : $VaR_\alpha(X)$ are such as, for all portfolio X_i;$VaR_\alpha(X) = \{z/F_{X_i}(z) \geq \alpha\} = F_{X_i}^{-1}(\alpha)$, $F_{X_i}^{-1}$ being the right continuous inverse of F_X.

In this paper, it is matter of the notion of multivariate risk coupled with that of conditional copula varying with time. We have established, subject of being in a Euclidean space, a relation between the conditional copula varying with time and the scalar product or norm. We are inspired by Patton's work on conditional copulas varying with time. Indeed, from proposition and Sklar's theorem adapted to the conditional copula (which are all from Patton) we use the relation (5) to obtain our different results. Finally, a new expression on the geometric yield is established.

2. Preliminaries

In this section we have grouped together the different notion definitions, propositions and theorems which will be useful thereafter. We need Sklar's theorem and its adaptation in the conditional case proposed by Patton (2006) to elaborate the different results we found.

2.1. A survey of Conditional Copulas

In copulas theory Joe (1997) or Nelsen (2007) provide detailed and readable introductions to copulas and their statistical and mathematical foundations while Bouy et al. (2000) or Cherubini et al. (2004) deal with applications of copulas to different levels of financial issues and derivatives pricing.

A $n - dimensional$ copula is a multivariate distribution function $C\ :\ [0,1]^n \longrightarrow [0,1]$ satisfying the following properties

i) Grounded : $C(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_n) = 0$ for all i and all $(u_1, ..., u_{i-1}, u_{i+1}, ..., u_n) \in [0,1]^{n-1}$.

ii) copula marginal : $C(u_1, ..., u_{i-1}, 1, u_{i+1}, ..., u_n)$ is an $(n-1)$ copula for all $i \in \{1, ..., n\}$.

iii) $n - increasingness$: the volume V_B of any rectangle $B = [a, b] \subseteq [0,1]^n$ is positive,

$$V_B = \int_B dC(u_1, ..., u_n) =_{i_1 \overset{[}{=} 1]2} \sum ..._{i_n \overset{[}{=} 1]2} \sum (-1)^{i_1 + ... + i_n} C(u_{i_1}, ..., u_{i_n}) \geq 0. \tag{4}$$

Using the above relation (4) (positiveness of the volume of any hyper-rectangle of $\bar{\mathbb{R}}^n$) Barro et all.(2012) provide the following result by extending a proposition of Patton (2002) both to space-varying case and to higher dimensional framework.

Proposition 1. *Let* F_{t,w_t} *denote the joint distribution of* $\left(\tilde{X}_{t,n-1}, W_t\right)$, $t \in T$ *with* $\tilde{X}_{t,n-1} = (X_{t,1}, ..., X_{t,n-1})$ *,then the conditional time-varying distribution of* $\left(\tilde{X}_{t,n-1}, W_t\right)$ *is given, for all* $\tilde{y}_t \in \left(\bar{\mathbb{R}}\right)^{n-1}$ *by*

$$H_{t,w_t}(x_t/w_t) = f_w^{-1}(w_t) \frac{\partial H_{t,w_t}(x_{t,1}, ..., x_{t,n-1}, w_t)}{\partial w_t} \tag{5}$$

where f_w *is the spacial density of the law of* W_t*. Moreover, the following properties are satisfied*

(i) $H_{t,w_t}(x_{t,1}, ..., -\infty, ..., x_{t,n-1}, w_t) = 0$ *for all* $\tilde{y}_t \in \left(\bar{\mathbb{R}}\right)^{n-1}$.

(ii) $H(\infty, ..., \infty/w_t) = 1$ *for all* $\tilde{y}_t \in \left(\bar{\mathbb{R}}\right)^{n-1}$.

(iii) For all $\tilde{x}_t^{(1)} = \left(x_{t,1}^{(1)}, ..., x_{t,n-1}^{(1)}\right) \in \left(\bar{\mathbb{R}}\right)^{n-1}$ *and* $\tilde{y}_t^{(1)} = \left(x_{t,1}^{(2)}, ..., x_{t,n-1}^{(2)}\right) \in \left(\bar{\mathbb{R}}\right)^{n-1}$ *such as* $x_{t,j}^{(1)} \leq x_{t,j}^{(2)}$ *then*

$$\sum_{(i_1,...,i_n)\in\{1,2\}^n} (-1)^{[j=1]n\sum j_j} H_{n-1,w_t}\left(x_{t,1}^{(i_1)}, ..., x_{t,n-1}^{(i_{n-1})}, w_t\right) \geq 0. \tag{6}$$

Let's consider a linear portfolio of consisting of n different financial instruments (risks, actions) $X = (X_1, ..., X_n)$. Further, let $p_0 = (p_{0,1}, ..., p_{0,n})$ the initial value of the portfolio is given by $V_0 =[i= 1]n\sum x_i p_{0,i}$ for a realization $x = (x_1, ..., x_n)$ of X. At the next date t the uncertain Profit and Loss function of the portfolio is given by

$$F_t(x_1, ..., x_n) =[i=1]n\sum x_i (p_{0,i} - p_{t,i}) =[i=1]n\sum x_i p_{t,i} \left(e^{z_{t,i}} - 1\right). \tag{7}$$

where $Z_t = (z_{t,i}, ..., z_{t,n})$ is the Log Price vector such as $z_{t,i} = logp_{t,i}$. Particularly, from the integral probability transforms we can associate to F_t, a parametric copula C_t such as, for all $(u_{t,1}, ..., u_{t,n}) \in [0, 1]^n$,

$$C_t(u_{t,1}, ..., u_{t,n}) = P(F_{t,1}(X_1) \leq u_{t,1}, ..., F_{t,n}(X_n) \leq u_{t,n}) \tag{8}$$

2.2. Scalar product and copulas application on VaR

According to Karl Friedrich Siburg et al. (1975) , the restrictions of $\langle,\rangle$, $\|\|$ and d to $\mathfrak{C}_n$ are called the sobolev scalar product, the Sobolev norm and the Sobolev distance function on $\mathfrak{C}_n$, respectively. But we suppose a new norm in a space a Euclidean vector space is a prehilbert space of finite dimension. It is complete.

Definition 1. *Consider $E = \mathbb{R}^n$ a vector space with the scalar product $\langle , \rangle$. In the following we consider ourselves in finite dimension and $(E, \langle , \rangle)$ is Euclidean space. The application*

$$x \longmapsto \|x\| = \sqrt{\langle x, x \rangle}$$

defines on E a norm, called euclidean norm and noted $\|\|$. $\forall\,(x, y) \in E^2$ we have :

- Cauchy-Schwartz inequality :

$$|\langle x, y \rangle| \leq \|x\|\,\|y\| \tag{9}$$

- Cauchy-Schwartz equality (in the case where (x, y) do not form a free family):

$$|\langle x, y \rangle| = \|x\|\,\|y\| \tag{10}$$

- Polarization identity :

$$\langle x, y \rangle = \frac{1}{4}\left(\|x + y\|^2 + \|x - y\|^2\right) \tag{11}$$

In Euclidean space there is an orthogonal basis and the Gram-Schimdt process allows to build it. In a Euclidean space, any orthogonal family can be completed in an orthogonal basis. Let an orthogonal base of a euclidean vector space $(E, \langle , \rangle)$ and u an endomorphism from E:

3. Main Results

Let's consider a linear portfolio of consisting of n different financial instruments (risks, actions) $X = (X_1, ..., X_n)$ and let $p_t = (p_{1,t}, p_{2,t}, ..., p_{n,t})$ at a given date measured at given time t. Further, let $p_0 = (p_{0,1}, ..., p_{0,n})$ the initial value of the portfolio is given by $V_0 = \overset{[}{i} = 1]n\sum x_i p_{0,i}$ for a realization $x = (x_1, ..., x_n)$ of X.

$$p_t = (p_{1,t}, ..., p_{n,t}) = \overset{[}{i=1}]n\sum \langle p_{i,t}, e_i \rangle\, e_i, \;\; \textit{and} \;\; \|p_t\| = \sqrt{\overset{[}{i=1}]n\sum p_{i,t}^2} = \sqrt{\overset{[}{i=1}]n\sum \langle p_t, e_i \rangle^2}.$$

Consider

$$P_t = (p_{1,t}e^{z_{t,1}}, ..., p_{n,t}e^{z_{n,1}}) = \overset{[}{i=1}]n\sum \langle p_{i,t}e^{z_{t,i}}, e_i \rangle\, e_i,$$

and

$$\|P_t\| = \sqrt{\overset{[}{i=1}]n\sum p_{i,t}^2 e^{2z_{t,i}}} = \sqrt{\overset{[}{i=1}]n\sum \langle P_t, e_i \rangle^2}.$$

The concept of scalar product that allowed us to highlight a link between the conditional copula and the notion of norms in the metric spaces. Let assume that we are in an euclidean space and that all necessary conditions are fulfilled. We use the characteristic elements of metric spaces (euclidean space), to establish with the notion of scalar product or norm a relation between the VaR and the conditional time-dependent copula.

3.1. Scalar product and copulas applications on the VaR

The following sub-section proposal was inspired by the Gram-Schmidt process. We think it is necessary to depend the proposition 2 afterwards.

Proposition 2. *Let E be a Euclidean space and $(e_1, ..., e_n)$ be a base of E in which. Then, there is a only base $\xi = (\xi_1, ..., \xi_n)$ such that; if*

$$VaR_u(X) = (VaR_{u_1}(X_1), ..., VaR_{u_n}(X_n)) = \overset{[}{i=1}]n\sum \langle VaR_{u_i}, e_i\rangle e_i$$

then

$$\langle VaR_u(X), \xi\rangle - \overset{[}{i=2}]n\sum \frac{VaR_{u_i}\vartheta_i}{\|\vartheta_i\|} = VaR_{u_1}\frac{e_1}{\|e_1\|} \tag{12}$$

with $\xi_1 = \frac{e_1}{\|e_1\|}$ *and* $\forall i \in \{1, ..., n-1\}$, *and*

$$\xi_{i+1} = \frac{\vartheta_{i+1}}{\|\vartheta_{i+1}\|} \quad with\ \vartheta_{i+1} = e_{i+1} - \overset{[}{k=1}]i\sum \langle e_{i+1}, \xi_k\rangle \xi_k.$$

Proof. By assumption E is a Euclidean space and let $(e_1, e_2, ..., e_n)$ be a base of E such that

$$VaR_u(X) = (VaR_{u_1}(X_1), ..., VaR_{u_n}(X_n)) = \overset{[}{i=1}]n\sum \langle VaR_{u_i}, e_i\rangle e_i.$$

So, it follows that,

$$\|VaR_u(X)\| = \sqrt{\overset{[}{i=1}]n\sum VaR^2_{u_i}} = \sqrt{\overset{[}{i=1}]n\sum \langle VaR_u, e_i\rangle^2}$$

The orthogonalization process of Gram-Schimdt (1875) allows us to say that there is only one base $(\xi_1, ..., \xi_n)$ as

$$\xi_1 = \frac{e_1}{\|e_1\|} \quad and \quad \forall i \in \{1, ..., n-1\}, \xi_{i+1} = \frac{\vartheta_{i+1}}{\|\vartheta_{i+1}\|} \quad with \vartheta_{i+1} = e_{i+1} - \overset{[}{k=1}]i\sum \langle e_{i+1}, \xi_k\rangle \xi_k. \tag{13}$$

Futhermore, it comes that :

$$\langle VaR_u(X), \xi\rangle = \overset{[}{i=1}]n\sum VaR_{u_i}\xi_i$$

$$\langle VaR_u(X), \xi\rangle - \overset{[}{i=2}]n\sum \frac{VaR_{u_i}\vartheta_i}{\|\vartheta_i\|} = VaR_{u_1}\frac{e_1}{\|e_1\|}$$

Let's consider a linear portfolio of consisting of n different financial instruments (risks, actions) $X = (X_1, ..., X_n)$ and let $p_t = (p_{1,t}, p_{2,t}, ..., p_{n,t})$ at a given date measured at given time t. Further, let $p_0 = (p_{0,1}, ..., p_{0,n})$ the initial value of the portfolio is given by $V_0 = \overset{[}{i} = 1]n\sum x_i p_{0,i}$.

Theorem 1. *For a realization $x = (x_1, ..., x_n)$ of X, at the next date t the uncertain Profit and Loss function of the portfolio is given by*

$$F_t(x_1, ..., x_n) = \overset{[}{i=1}]n \sum x_i (p_{0,i} - p_{t,i}) = \overset{[}{i=1}]n \sum x_i p_{t,i} (e^{z_{t,i}} - 1);$$

then

$$C_t(u_1, ..., u_n) = \|VaR_u(X)\| \, \|P_t - p_t\|. \tag{14}$$

Furthermore,

$$C_t(u_1, ..., u_n) = \frac{1}{4}\left(\|VaR_u(X) + P_t - p_t\|^2 + \|VaR_u(X) - (P_t - p_t)\|^2\right) \tag{15}$$

and where $P_t = (p_{1,t}e^{z_{t,1}}, ..., p_{n,t}e^{z_{n,1}})$,

$$VaR_u(X) = (VaR_{u_1}(X_1), ..., VaR_{u_n}(X_n)).$$

is a Value at risk of the X and $\|\cdot\|$ euclidean norm and noted.

Proof. Consider the following relation

$$C_t(u_1, ..., u_n) = F_{W_t}\left(F_{1,W_t}^{(-1)}(u_1), ..., F_{n,W_t}^{(-1)}(u_n)\right).$$

Ff we consider relation (7) we obtain,

$$C_t(u_1, ..., u_n) = \overset{[}{i=1}]n \sum VaR_{u_i}(X_i)(p_{t,i}e^{z_{t,i}} - p_{t,i}).$$

Consider $P_t = (p_{1,t}e^{z_{t,1}}, ..., p_{n,t}e^{z_{n,1}})$ and $VaR_u(X) = (VaR_{u_1}(X_1), ..., VaR_{u_n}(X))$. Then,

$$C_t(u_1, ..., u_n) = \langle VaR_u(X), \, P_t - p_t \rangle$$

Value at risk is intrinsically linked to the portfolio and therefore to the initial amount and the amount at a given time t. We will suppose linked vector $VaR_u(X)$ and vector $p_t - P_t$. The relation 10 we give

$$|\langle VaR_u(X), \, P_t - p_t \rangle| = \|VaR_u(X)\| \, \|P_t - p_t\|.$$

Then,

$$C_t(u_1, ..., u_n) = \|VaR_u(X)\| \, \|P_t - p_t\|$$

and equality (11) given

$$C_t(u_1, ..., u_n) = \frac{1}{4}\left(\|VaR_u(X) + P_t - p_t\|^2 + \|VaR_u(X) - (P_t - p_t)\|^2\right)$$

The following result allows us to obtain;

Proposition 3. *Let $p_t = (p_{1,t}, p_{2,t}, ..., p_{n,t})$ at a given date measured at given time t. And suppose these risks represent potential losses in dependent lines of business for example an insurance company. Then*

$$C\left(u_1, ..., u_n/w_t\right) = f_w^{-1}\left(w_t\right)\left\langle \frac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right), \left(P_t - p_t\right)\right\rangle \tag{16}$$

and where $P_t = \left(p_{1,t}e^{z_{t,1}}, ..., p_{n,t}e^{z_{n,1}}\right)$, $VaR_u\left(X/w_t\right)$ is a Value at risk of the X such that w_t and $\|\|$ euclidean norm and noted.

Proof. Let F_{W_t} be any conditional time-varying conditional distribution with marginal $\{F_{i,W_t}; 1 \leq i \leq n\}$. Then there exists a only copula $C : [0,1]^n \longrightarrow [0,1]$ such as

$$C\left(u_1, ..., u_n/w_t\right) = F_{W_t}\left(F_{1,W_t}^{(-1)}\left(u_1/w\right), ..., F_{n,W_t}^{(-1)}\left(u_n/w\right)\right) \tag{17}$$

where $F_{i,W_t}^{(-1)}\left(u_i/w\right) = inf\left\{x : F_{i,W_t}\left(x/w\right) \geq u_i\right\}$ for each u_i and $w_t \in W_t$. Then, it follows that :

$$C\left(u_1, ..., u_n/w_t\right) = \frac{\partial F_{W_t}\left(F_{1,W_t}^{(-1)}\left(u_1/w_t\right), ..., F_{n-1,W_t}^{(-1)}\left(u_{n-}/w_t\right), w_t\right)}{\partial w_t}.$$

By considering the equality (7) int the following relation

$$C\left(u_1, ..., u_n/w_t\right) = F_{W_t}\left(F_{1,W_t}^{(-1)}\left(u_1/w_t\right), ..., F_{n,W_t}^{(-1)}\left(u_n/w_t\right)/w_t\right)$$

are obtains :

$$\begin{aligned} C\left(u_1, ..., u_n/w_t\right) &= = f_w^{-1}\left(w_t\right) \times \left[\overset{[}{i=1}]n\sum \frac{\partial\left(VaR_{u_i}(X_i/w_t)\right)}{\partial w_t} p_{i,t}e^{z_{t,i}} \right. \\ &- \left. \overset{[}{i=1}]n\sum \frac{\partial\left(VaR_{u_i}(X_i/w_t)\right)}{\partial w_t} p_{i,t} \right] \end{aligned}$$

Consider $P_t = \left(p_{1,t}e^{z_{t,1}}, ..., p_{n,t}e^{z_{n,1}}\right)$, we have :

$$C\left(u_1, ..., u_n/w_t\right) = f_w^{-1}\left(w_t\right) \times \left\langle \frac{\partial}{\partial w_t} VaR_u\left(X/w_t\right), P_t\right\rangle - f_w^{-1}\left(w_t\right) \times \left\langle \frac{\partial}{\partial w_t} VaR_u\left(X/w_t\right), p_t\right\rangle$$

$$C\left(u_1, ..., u_n/w_t\right) = \left\langle \frac{\partial}{\partial w_t} VaR_u\left(X/w_t\right), f_w^{-1}\left(w_t\right)\left(P_t - p_t\right)\right\rangle$$

if $f_w^{-1} \in \mathcal{C}^1$ we can write, then

$$C\left(u_1, ..., u_n/w_t\right) = f_w^{-1}\left(w_t\right)\left\langle \frac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right), \left(P_t - p_t\right)\right\rangle$$

Proposition 4. *Let these risks represent potential losses in dependent lines of business for an insurance company for example. In the following we consider ourselves in finite dimension and $(E, \langle , \rangle)$ is Euclidean space. Further, the conditional copula of is given by*

$$C\left(u_1,...,u_n/w_t\right)=\left|f_w^{-1}\left(w_t\right)\right|\left\|\frac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)\right\|\left\|P_t-p_t\right\| \tag{18}$$

Furthermore

$$\begin{aligned} C\left(u_1,...,u_n/w_t\right) &= \tfrac{1}{4}f_w^{-1}\left(w_t\right)\left(\left\|\tfrac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)+\left(P_t-p_t\right)\right\|^2\right. \\ &+ \left.\left\|\tfrac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)-\left(P_t-p_t\right)\right\|^2\right) \end{aligned}. \tag{19}$$

Proof. For this proof, consider the following relation :

$$C\left(u_1,...,u_n/w_t\right)=f_w^{-1}\left(w_t\right)\left\langle\frac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right),P_t-p_t\right\rangle;$$

Moreover considering equality (10) it came that;

$$\left|f_w^{-1}\left(w_t\right)\left\langle\frac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right),P_t-p_t\right\rangle\right|=\left|f_w^{-1}\left(w_t\right)\right|\left\|\frac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)\right\|\left\|P_t-p_t\right\|$$

so

$$C\left(u_1,...,u_n/w_t\right)=\left|f_w^{-1}\left(w_t\right)\right|\left\|\frac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)\right\|\left\|P_t-p_t\right\|$$

and if we take it the relation (11)

$$\begin{aligned} \left\langle VaR_u\left(X/w_t\right),P_t-p_t\right\rangle &= \tfrac{1}{4}\left(\left\|\tfrac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)+\left(P_t-p_t\right)\right\|^2\right. \\ &+ \left.\left\|\tfrac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)-\left(P_t-p_t\right)\right\|^2\right) \end{aligned}.$$

Then, it follows that :

$$\begin{aligned} f_w^{-1}\left(w_t\right)\left\langle\tfrac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right),P_t-p_t\right\rangle &= \tfrac{1}{4}\tfrac{\partial}{\partial w_t}\left(f_w^{-1}\left(w_t\right)\right)\left(\left\|\tfrac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)+\left(P_t-p_t\right)\right\|^2\right. \\ &+ \left.\left\|\tfrac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)-\left(p_t-P_t\right)\right\|^2\right) \end{aligned}$$

hence the result :

$$\begin{aligned} C\left(u_1,...,u_n/w_t\right) &= \tfrac{1}{4}f_w^{-1}\left(w_t\right)\left(\left\|\tfrac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)+\left(P_t-p_t\right)\right\|^2\right. \\ &+ \left.\left\|\tfrac{\partial}{\partial w_t}\left(VaR_u\left(X/w_t\right)\right)-\left(P_t-p_t\right)\right\|^2\right) \end{aligned}.$$

3.2. The CVaR in an Euclidean space

The Tail-VaR (TVaR) is derivative coherent risk measure of the VaR. For a given confidence level $\alpha \in]0,1[$, it follows that

$$TVaR_\alpha\left(X\right)=\frac{1}{1-\alpha}\int_\alpha^1 VaR_\xi\left(X\right)d\xi=\frac{1}{1-\alpha}\left[\mathbb{E}\left[X\right]-\int_0^\alpha VaR_\xi\left(X\right)d\xi\right] \tag{20}$$

The XTVaR is the average amount of ruins beyond the VaR;

$$XTVaR_\alpha\left(X\right)=TVaR_\alpha\left(X\right)-VaR_\alpha\left(X\right) \tag{21}$$

the relation 21 we get the following proposition.

Proposition 5. *The Tail-VaR (TVaR) is derivative coherent risk measure of the VaR. For a given confidence level* $\alpha \in]0,1[$*, it follows that*

$$\begin{aligned} C\left(u_1,...,u_n/w_t\right) &= f_w^{-1}\left(w_t\right)\left\langle\tfrac{\partial}{\partial w_t}\left(TVaR_u\left(X/w_t\right)\right),\left(P_t-p_t\right)\right\rangle \\ &- f_w^{-1}\left(w_t\right)\left\langle\tfrac{\partial}{\partial w_t}\left(XTVaR_u\left(X/w_t\right)\right),\left(P_t-p_t\right)\right\rangle \end{aligned} \tag{22}$$

Furthermore

$$C\left(u_1,...,u_n/w_t\right)=f_w^{-1}\left(w_t\right)\frac{1}{2}\left(\left\|\frac{\partial}{\partial w_t}\left(TVaR_u\left(X/w_t\right)\right)\right\|^2-\left\|\frac{\partial}{\partial w_t}\left(XTVaR_u\left(X/w_t\right)\right)\right\|^2\right) \tag{23}$$

Proof. For relation (21) and the Proposition (3) we obtain the following equality :

$$\begin{aligned} C\left(u_1,...,u_n/w_t\right) &= f_w^{-1}\left(w_t\right)\left[\left\langle\tfrac{\partial}{\partial w_t}\left(TVaR_u\left(X/w_t\right)\right),\left(P_t-p_t\right)\right\rangle\right. \\ &- \left.\left\langle\tfrac{\partial}{\partial w_t}\left(XTVaR_u\left(X/w_t\right)\right),\left(P_t-p_t\right)\right\rangle\right] \end{aligned}$$

Then, it follows that :

$$\begin{aligned} C\left(u_1,...,u_n/w_t\right) &= f_w^{-1}\left(w_t\right)\tfrac{1}{4}\left[\left\|\tfrac{\partial}{\partial w_t}\left(TVaR_u\left(X/w_t\right)\right)+\left(P_t-p_t\right)\right\|^2\right.\\ &- \left.\left\|\tfrac{\partial}{\partial w_t}\left(XTVaR_u\left(X/w_t\right)\right)+\left(P_t-p_t\right)\right\|^2\right]\\ &= f_w^{-1}\left(w_t\right)\tfrac{1}{4}\left[2\left(\left\|\tfrac{\partial}{\partial w_t}\left(TVaR_u\left(X/w_t\right)\right)\right\|^2+\left\|P_t-p_t\right\|^2\right)\right.\\ &- \left.2\left(\left\|XTVaR_u\left(X/w_t\right)\right\|^2+\left\|\left(p_t-P_t\right)\right\|^2\right)\right] \end{aligned}$$

it comes that :

$$C\left(u_1,...,u_n/w_t\right)=f_w^{-1}\left(w_t\right)\frac{1}{2}\left(\left\|\frac{\partial}{\partial w_t}\left(TVaR_u\left(X/w_t\right)\right)\right\|^2-\left\|\frac{\partial}{\partial w_t}\left(XTVaR_u\left(X/w_t\right)\right)\right\|^2\right).$$

3.3. Performance Measures and the Distribution of L&P of a Portfolio

Proposition 6. *Let $w=\left(w_1,...,w_n\right)^T\in\mathbb{R}^n$ a portfolio consisting of n capital (the allocation of capital) and $S_t=\left(S_{1,t},...,S_{n,t}\right)^T$ the non-negative random vector representing the capital at the moment t. Then geometric yield*

$$\begin{aligned} R_t &= log\left(\frac{\left(\left[\left(\sigma(S_t)\right)^2+\left(\mathbb{E}(S_t)\right)^2\right]\right)^{1/2}}{\left(\left[\left(\sigma(S_{t-1})\right)^2+\left(\mathbb{E}(S_{t-1})\right)^2\right]\right)^{1/2}}\right.\\ &+ \left.\frac{n\times\Delta_t}{\left(\left[\left(\sigma(w)\right)^2+\left(\mathbb{E}(w)\right)^2\right]\right)^{1/2}\times\left(\left[\left(\sigma(S_{t-1})\right)^2+\left(\mathbb{E}(S_{t-1})\right)^2\right]\right)^{1/2}}\right) \end{aligned} \tag{24}$$

where $\sigma\left(\cdot\right)$ is a standard deviation and $\mathbb{E}\left(\cdot\right)$ is a mean and with Δ_t all the interim payments obtained between the dates $t-1$ and t.

The distribution of $\left(P_{t+\tau}-P_t\right)$ is called profit distribution loss that expresses the change in the value of the portfolio.

Proof. The P_t value of the portfolio is given by :

$$P_t=\sum_{j=1}^{n}w_jS_{j,t}. \tag{25}$$

The profits and losses associated with holding the asset are then defined by the difference :

$$P\&L = P_t + \Delta_t - P_{t-1} = \sum_{j=1}^{n} w_j \left(S_{j,t} - S_{j,t-1}\right) + \Delta_t. \tag{26}$$

The equality 26 becomes using the scalar product: :

$$P\&L = \langle w, S_t - S_{t-1} \rangle = \|w\| \, \|S_t - S_{t-1}\| + \Delta_t \tag{27}$$

Indeed on the same types of considerations of the section 3.1 (w and S_t are linked). Then :

$$P\&L = \left(\sum_{j=1}^{n} w_j^2\right)^{1/2} \times \left(\sum_{j=1}^{n} \left(S_{j,t} - S_{j,t-1}\right)^2\right)^{1/2} + \Delta_t \tag{28}$$

These losses and profits are expressed in the form of a geometric return noted R_t :

$$R_t = log\left(\frac{P_t + \Delta_t}{P_{t-1}}\right) = log\left(\frac{\left(\sum\limits_{j=1}^{n} w_j^2\right)^{1/2} \times \left(\sum\limits_{j=1}^{n} S_{j,t}^2\right)^{1/2} + \Delta_t}{\left(\sum\limits_{j=1}^{n} w_j^2\right)^{1/2} \times \left(\sum\limits_{j=1}^{n} S_{j,t-1}^2\right)^{1/2}}\right) \tag{29}$$

if we use the following relation;

$$\mathbb{E}\left(S_t^2\right) = \left[\frac{1}{n}\left(\sigma\left(S_t\right)\right)^2 + \left(\mathbb{E}\left(S_t\right)\right)^2\right]^{1/2}.$$

It comes that:

$$\begin{aligned} R_t &= log\left(\frac{\left(\left[(\sigma(S_t))^2+(\mathbb{E}(S_t))^2\right]\right)^{1/2}}{\left(\left[(\sigma(S_{t-1}))^2+(\mathbb{E}(S_{t-1}))^2\right]\right)^{1/2}}\right. \\ &+ \left.\frac{n\times\Delta_t}{\left(\left[(\sigma(w))^2+(\mathbb{E}(w))^2\right]\right)^{1/2}\times\left(\left[(\sigma(S_{t-1}))^2+(\mathbb{E}(S_{t-1}))^2\right]\right)^{1/2}}\right) \end{aligned} \tag{30}$$

It is assumed that the return on the date t, i.e. R_t , is a real random variable.

Corollary 1. *Let $w = (w_1, ..., w_n)^T \in \mathbb{R}^n$ a portfolio consisting of n capital (the allocation of capital) and $S_t = (S_{1,t}, ..., S_{n,t})^T$ the non-negative random vector representing the capital at the moment t. Then geometric yield*

$$R_t \simeq \frac{n \times \Delta_t}{\left(\left[\left(\sigma\left(w\right)\right)^2 + \left(\mathbb{E}\left(w\right)\right)^2\right]\right)^{1/2} \times \left(\left[\left(\sigma\left(S_t\right)\right)^2 + \left(\mathbb{E}\left(S_t\right)\right)^2\right]\right)^{1/2}} \tag{31}$$

where $\sigma(\cdot)$ is a standard deviation and $\mathbb{E}(\cdot)$ is a mean and with Δ_t all the interim payments obtained between the dates $t-1$ and t.

Proof. By taking the relationship (24) and as S_t and S_{t-1} are of the same nature we have :

$$\sigma(S_t) \simeq \sigma(S_{t-1}) \ \ and \ \mathbb{E}(S_t) \simeq \mathbb{E}(S_{t-1}).$$

and view the limited developmental formula in the neighborhood of zero $log(1+x) \simeq x$ it comes that

$$R_t = log\left(1 + \frac{n \times \Delta_t}{\left(\left[(\sigma(w))^2 + (\mathbb{E}(w))^2\right]\right)^{1/2} \times \left(\left[(\sigma(S_t))^2 + (\mathbb{E}(S_t))^2\right]\right)^{1/2}}\right)$$

Hence the result.

3.4. Conclusion and Discussion

We thought to introduce the notion of product in the stochastic modeling of the copula and the value at risk. Because it allows us to combine precisely these two notions (Copula, VaR). One of the characteristics of the scalar product is the fact that it makes it possible to move the diferential from one component to another. The different relations that we obtained in this document allow to establish a close link between copula and the value at risk through an analytical expression with the norms. It becomes quite easy to calculate the values of VaR when we know the value of the copula and the type of copula.

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