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On Determining Initial Conditions of Equations Flexural-Torsinal Vibrations of a Bar

Aysel T. Ramazanova

University Duisburg-Essen, Germany

Abstract. The problem of finding the initial conditions in the boundary-value problem for the system of flexural-torsional vibrations of a bar with additional conditions on the straight line is reduced to an optimal control problem and studied by the methods of optimal control theory. The gradient of the functional is calculated and using the gradient expression a necessary and sufficient optimality condition are proved.

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1. Introduction

It is known that some problems of mathematical physics, mechanics, are described by fourth order partial equations. A tuning fork, a bar vibrations equation, a rotary shaft, oscillating motions equation and plate vibrations equation are among these equations give some references. It is imperative optimal control problems in processes described by these equations. The control connected with flexural-torsional vibrations of a bar has a great signifficance in dynamics of aircraft constructions. Therefore, the study of bar vibrations problems controls described by differential equations is necessary both from practical and theoretical point of view.

2. Problem statement

We consider a boundary value problem for equations of flexural-torsional vibrations of a bar, described by the system of two differential equations in the domain $Q = \{0 < x < l,\ 0 < t < T\}$ with boundary and initial conditions

$$\frac{\partial^2}{\partial x^2}\left(E\left(x\right)I\left(x\right)\frac{\partial^2 y}{\partial x^2}\right)+\rho\left(x\right)A\left(x\right)\frac{\partial^2 y}{\partial t^2}-\rho\left(x\right)A\left(x\right)e\left(x\right)\frac{\partial^2 \theta}{\partial t^2}=f_1\left(x,t\right), \qquad (1)$$

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Email addresses: ramazanova-aysel@mail.ru

$$\frac{\partial^2}{\partial x^2}\left(E\left(x\right)C_w\left(x\right)\frac{\partial^2\theta}{\partial x^2}\right)-G\left(x\right)C\left(x\right)\frac{\partial^2\theta}{\partial x^2}-\rho\left(x\right)A\left(x\right)e\left(x\right)\frac{\partial^2 y}{\partial t^2}+$$

$$+\rho\left(x\right)\left(I\left(x\right)+A\left(x\right)e^2\left(x\right)\right)\frac{\partial^2\theta}{\partial t^2}=f_2\left(x,t\right),\left(x,t\right)\in Q, \tag{2}$$

$$y_{x=0}=y|_{x=l}=0, \qquad \left.\frac{\partial y}{\partial x}\right|_{x=0}=\left.\frac{\partial y}{\partial x}\right|_{x=l}=0, \qquad 0\le \mathrm{t}\le \mathrm{T}, \tag{3}$$

$$\theta|_{x=0}=\theta|_{x=l}=0, \qquad \left.\frac{\partial \theta}{\partial x}\right|_{x=0}=\left.\frac{\partial \theta}{\partial x}\right|_{x=l}=0, \qquad 0\le \mathrm{t}\le \mathrm{T}, \tag{4}$$

$$y|_{t=0}=0, \qquad \left.\frac{\partial y}{\partial t}\right|_{t=0}=v_1\left(x\right), \quad 0\le \mathrm{x}\le \mathrm{l}, \tag{5}$$

$$\theta|_{t=0}=0, \qquad \left.\frac{\partial \theta}{\partial t}\right|_{t=0}=w_1\left(x\right), \quad 0\le \mathrm{x}\le \mathrm{l}, \tag{6}$$

where $l>0,\ T>0$ are given numbers, $y(x,t)$ is the lateral displacement of the bar, $\theta(x,t)$ is the turning angle of the bar cross-section, $E(x)$ is the Young modulus, $I(x)$ is a polar inertia moment of the cross section with respect to its gravity center, $\rho(x)$ is a density of the bar material, $A(x)$ is the area cross section, $e(x)$ is the distance from the gravity center to the center of torsion, $C_w(x)$ is the sectional moment of inertia of the cross-section, $G(x)$ a shear modulus, $C(x)$ is geometrical rigidity of free torsion, $E(x)C_w(x)$ is the rigidity of flexural functions, $G(x)C(x)$ is the rigidity of free torsion, the functions $\left(v_1\left(x\right),w_1\left(x\right)\right)\in L_2\left(0,l\right)\times L_2\left(0,l\right)$- to be defined.

Note that for each fixed vector function $\left(v_1\left(x\right),w_1\left(x\right)\right)\in L_2\left(0,l\right)\times L_2\left(0,l\right)$ problem (1)-(6) has a unique generalized solution from the spaces $W_2^{2,1}\left(Q\right)$ [3,5,6].

To determine $v\left(x\right)=\left(v_1\left(x\right),w_1\left(x\right)\right)$, we give the additional conditions

$$y\left(x,T;v\right)=\varphi_1\left(x\right),\ 0\le x\le l, \tag{7}$$

$$\theta\left(x,T;v\right)=\varphi_2\left(x\right),\ 0\le x\le l, \tag{8}$$

where $\varphi_1\left(x\right),\varphi_2\left(x\right)$ – are given functions.

We reduce this problem to the following optimal control problem: it is required to find such a vector-function $\left(v_1\left(x\right),w_1\left(x\right)\right)\in L_2\left(0,l\right)\times L_2\left(0,l\right)$, that minimizes the functional

$$J_0\left(v\right)=\frac{1}{2}\int_0^l\left[\left(\left(y\left(x,T;v\right)-\varphi_1\left(x\right)\right)^2+\left(\theta\left(x,T;v\right)-\varphi_2\left(x\right)\right)^2\right]dx \tag{9}$$

together with the solution of boundary value problem (1)-(6).
The function $v\left(x\right)=\left(v_1\left(x\right),w_1\left(x\right)\right)$- is called a control. We call problem (1)-(6),(9) a reduced problem.The problem (1) - (6), (9) is regularized as follows. We introduce the functional

$$J_\alpha(v) = J_0(v) + \frac{\alpha}{2}\left(\|v_1\|^2_{L_2(0,l)} + \|w_1\|^2_{L_2(0,l)}\right), \ \alpha = const > 0 \tag{10}$$

Now for a class of admissible controls we take a convex, closed set $U_{ad} \in L_2(0,l) \times L_2(0,l)$ of vector-functions $v(x) = (v_1(x), w_1(x))$.

Suppose that data of problem (1)-(6) satisfy the following conditions:

1) $E(x), \ I(x), \ \rho(x), A(x), \ e(x), C_\omega(x), \ G(x), \ \ C(x)$, are mesaurable, bounded and positive functions on the interval $[0,l]$;

2) $f_1, f_2 \in L_2(Q), \ \ \varphi_1, \varphi_2 \in L_2(0,l)$- are given functions.

3. Differentiability of functional (10)

We show that the functional (10) is differentiable in $L_2(0,l) \times L_2(0,l)$.

Introduce the following problem adjoint to the problem (1)-(6), (10):

$$\frac{\partial^2}{\partial x^2}\left(E(x) I(x) \frac{\partial^2 \Psi_1}{\partial x^2}\right) + \rho(x) A(x) \frac{\partial^2 \Psi_1}{\partial t^2} - \rho(x) A(x) e(x) \frac{\partial^2 \Psi_2}{\partial t^2} = 0,$$

$$(x,t) \in Q, \tag{11}$$

$$\frac{\partial^2}{\partial x^2}\left(E(x) C_w(x) \frac{\partial^2 \Psi_2}{\partial x^2}\right) - G(x) C(x) \frac{\partial^2 \Psi_2}{\partial x^2} - \rho(x) A(x) e(x) \frac{\partial^2 \Psi_1}{\partial t^2} +$$

$$+\rho(x)\left(I(x) + A(x) e^2(x)\right) \frac{\partial^2 \Psi_2}{\partial t^2} = 0, (x,t) \in Q, \tag{12}$$

$$\Psi_1|_{x=0} = \Psi_1|_{x=l} = 0, \quad \left.\frac{\partial \Psi_1}{\partial x}\right|_{x=0} = \left.\frac{\partial \Psi_1}{\partial x}\right|_{x=l} = 0, \quad 0 \le t \le T, \tag{13}$$

$$\Psi_2|_{x=0} = \Psi_2|_{x=l} = 0, \quad \left.\frac{\partial \Psi_2}{\partial x}\right|_{x=0} = \left.\frac{\partial \Psi_2}{\partial x}\right|_{x=l} = 0, \ 0 \le t \le T,$$

$$\Psi_1|_{t=T} = 0, \qquad \Psi_2|_{t=T} = 0, 0 \le x \le l, \tag{14}$$

$$\left.\frac{\partial \Psi_1}{\partial t}\right|_{t=T} =$$

$$= -\frac{(I(x) + A(x) e^2(x))(\varphi_1(x) - y(x,T;v) + A(x) e(x)(\varphi_2(x) - \theta(x,T;v))}{\rho(x) A(x) I(x)},$$

$$\left.\frac{\partial \Psi_2}{\partial t}\right|_{t=T} = -\frac{(\varphi_1(x) - y(x,T;v) e(x) + \varphi_2(x) - \theta(x,T;v))}{\rho(x) I(x)}. \tag{15}$$

We take the two admissible controls and assign them the increments $\delta v_1 \in L_2(0,l)$ and $\delta w_1 \in L_2(0,l)$ in such a way that, $(v_1(x) + \delta v_1(x), w_1(x) + \delta w_1(x) \in U_{ad})$,

$$v(x) = (v_1(x), w_1(x)), v(x) + \delta v(x) =$$
$$= (v_1(x) + \delta v_1(x), w_1(x) + \delta w_1(x)) \in L_2(0,l).$$

Then the increment of the functional (10) is computed as

$$\Delta J_\alpha(v) = J_\alpha(v + \delta v) - J_\alpha(v) =$$

$$= \frac{1}{2}\int_0^l \Big[(y(x,T;v_1 + \delta v_1, w_1 + \delta w_1) - \varphi_1(x))^2 -$$
$$-(y(x,T;v_1,w_1) - \varphi_1(x))^2 + (\theta(x,T;v_1 + \delta v_1, w_1 + \delta w_1) - \varphi_2(x))^2 -$$
$$-(\theta(x,T;v_1,w_1) - \varphi_2(x))^2\Big]dx +$$
$$+\frac{\alpha}{2}\left(\|v_1 + \delta v_1\|^2_{L_2(0,l)} - \|v_1\|^2_{L_2(0,l)}\right) + \frac{\alpha}{2}\left(\|w_1 + \delta w_1\|^2_{L_2(0,l)}\right), \quad (16)$$

where

$$y(x,t;v(x) + \delta v(x)) = y(x,t;v) + \delta y(x,t)$$
$$\theta(x,t;v(x) + \delta v(x)) = \theta(x,t;v) + \delta\theta(x,t).$$

Hence it follows that

$$\Delta J_\alpha(v) = \int_0^l [(y(x,T;v) - \varphi_1(x))\,\delta y(x,T) + (\theta(x,T;v) - \varphi_2(x))\,\delta\theta(x,T)]\,dx +$$
$$+\alpha\int_0^l (v_1\delta v_1 + w_1\delta w_1)dx + R, \quad (17)$$

where

$$\mathrm{R} = \frac{1}{2}\int_0^l \Big[(\delta y(x,T))^2 + (\delta\theta(x,T))^2\Big]dx + \frac{\alpha}{2}\left(\int_0^l ((\delta v_1)^2 + (\delta w_1)^2)dx\right)$$

and

$(\delta y(x,t), \delta\theta(x,t)) \in W_2^{2,1}(Q) \times W_2^{2,1}(Q)$ is the generalized solution of the

$$\frac{\partial^2}{\partial x^2}\left(E(x)\,I(x)\,\frac{\partial^2 \delta y}{\partial x^2}\right) + \rho(x)\,A(x)\,\frac{\partial^2 \delta y}{\partial t^2} -$$
$$-\rho(x)\,A(x)\,e(x)\,\frac{\partial^2 \delta\theta}{\partial t^2} = 0, (x,t) \in Q, \quad (18)$$

$$\frac{\partial^2}{\partial x^2}\left(E(x)\,C_w(x)\,\frac{\partial^2\delta\theta}{\partial x^2}\right)-G(x)\,C(x)\,\frac{\partial^2\delta\theta}{\partial x^2}-$$

$$-\rho(x)\,A(x)\,e(x)\,\frac{\partial^2\delta y}{\partial t^2}+\rho(x)\left(I(x)+A(x)\,e^2(x)\right)\frac{\partial^2\delta\theta}{\partial t^2}=0,(x,t)\in Q\ , \tag{19}$$

$$\delta y|_{x=0}=\delta y|_{x=l}=0,\ \left.\frac{\partial\delta y}{\partial x}\right|_{x=0}=\left.\frac{\partial\delta y}{\partial x}\right|_{x=l}=0,\ 0\le t\le T, \tag{20}$$

$$\delta\theta|_{x=0}=\delta\theta|_{x=l}=0,\ \left.\frac{\partial\delta\theta}{\partial x}\right|_{x=0}=\left.\frac{\partial\delta\theta}{\partial x}\right|_{x=l}=0,\ 0\le t\le T, \tag{21}$$

$$\delta y|_{t=0}=0,\ \left.\frac{\partial\delta y}{\partial t}\right|_{t=0}=\delta v_1(x)\,,$$

$$\delta\theta|_{t=0}=0,\ \ \frac{\partial\delta\theta}{\partial t}\,|_{t=0}=\delta w_1(x)\,,\ \ 0\le x\le l \tag{22}$$

i.e. for any function

$$\forall\ \eta_1=\eta_1(x,t)\,,\ \ \eta_2=\eta_2(x,t)\in W_2^{2,1}(Q)\,,$$

$$\eta_1|_{x=0}=\eta_1|_{x=l}=0,\quad \left.\frac{\partial\eta_1}{\partial x}\right|_{x=0}=\left.\frac{\partial\eta_1}{\partial x}\right|_{x=l}=0,\quad 0\le t\le T,$$

$$\eta_2|_{x=0}=\eta_2|_{x=l}=0,\quad \left.\frac{\partial\eta_2}{\partial x}\right|_{x=0}=\left.\frac{\partial\eta_2}{\partial x}\right|_{x=l}=0,\quad 0\le t\le T,$$

the following integral identities are fulfilled

$$\iint_Q\left(E(x)\,I(x)\,\frac{\partial^2\delta y}{\partial x^2}\frac{\partial^2\eta_1}{\partial x^2}-\rho(x)\,A(x)\,\frac{\partial\delta y}{\partial t}\frac{\partial\eta_1}{\partial t}+\right.$$

$$\left.\rho(x)\,A(x)\,e(x)\,\frac{\partial\delta\theta}{\partial t}\frac{\partial\eta_1}{\partial t}\right)dxdt+$$

$$+\int_0^l\rho(x)\,A(x)\,\frac{\partial\delta y}{\partial t}\,\eta_1|_0^T dx-\int_0^l\rho(x)\,A(x)\,e(x)\,\frac{\partial\delta\theta}{\partial t}\,\eta_1|_0^T dx=0, \tag{23}$$

$$\iint_Q\left(E(x)\,C_w(x)\,\frac{\partial^2\delta\theta}{\partial x^2}\frac{\partial^2\eta_2}{\partial x^2}-G(x)\,C(x)\,\frac{\partial^2\eta_2}{\partial x^2}\delta\theta+\right.$$

$$\left.+\rho(x)\,A(x)\,e(x)\,\frac{\partial\delta y}{\partial t}\frac{\partial\eta_2}{\partial t}-\rho(x)\left(I(x)+A(x)\,e^2(x)\right)\frac{\partial\delta\theta}{\partial t}\frac{\partial\eta_2}{\partial t}\right)dxdt-$$

$$\int_0^l\rho(x)\,A(x)\,e(x)\,\frac{\partial\delta y}{\partial t}\,\eta_2|_0^T dx+$$

$$+\int_0^l \rho(x)\left(I(x)+A(x)e^2(x)\right)\frac{\partial\delta\theta}{\partial t}\eta_2|_0^T dx=0. \tag{24}$$

As the functions $(\Psi_1(x,t), \Psi_2(x,t))$are the generalized solutions of problem (11)- (15), for any functions $g_1, g_2 \in W_2^{2,1}(Q)$,

$$g_1|_{x=0}=g_1|_{x=l}=0, \quad \left.\frac{\partial g_1}{\partial x}\right|_{x=0}=\left.\frac{\partial g_1}{\partial x}\right|_{x=l}=0, \quad 0\le t\le T,$$

$$g_2|_{x=0}=g_2|_{x=l}=0, \quad \left.\frac{\partial g_2}{\partial x}\right|_{x=0}=\left.\frac{\partial g_2}{\partial x}\right|_{x=l}=0, \quad 0\le t\le T,$$

the following integral identities are fulfilled

$$\iint_Q \left(E(x)l(x)\frac{\partial^2\Psi_1}{\partial x^2}\frac{\partial^2 g_1}{\partial x^2}-\rho(x)A(x)\frac{\partial\Psi_1}{\partial t}\frac{\partial g_1}{\partial t}+\right.$$

$$\left.+\rho(x)A(x)e(x)\frac{\partial\Psi_2}{\partial t}\frac{\partial g_1}{\partial t}\right)dxdt+\int_0^l \rho(x)A(x)\frac{\partial\Psi_1}{\partial t}g_1|_0^T dx-$$

$$-\int_0^l \rho(x)A(x)e(x)\frac{\partial\Psi_2}{\partial t}g_1|_0^T dx=0, (x,t)\in Q, \tag{25}$$

$$\iint_Q \left(E(x)C_w(x)\frac{\partial^2\Psi_2}{\partial x^2}\frac{\partial^2 g_2}{\partial x^2}-G(x)C(x)\frac{\partial^2\Psi_2}{\partial x^2}g_2+\right.$$

$$\left.+\rho(x)A(x)e(x)\frac{\partial\Psi_1}{\partial t}\frac{\partial g_2}{\partial t}-\rho(x)\left(I(x)+A(x)e^2(x)\right)\frac{\partial\Psi_2}{\partial t}\frac{\partial g_2}{\partial t}\right)dxdt-$$

$$-\int_0^l \rho(x)A(x)e(x)\frac{\partial\Psi_1}{\partial t}g_2|_0^T dx+$$

$$+\int_0^l \rho(x)\left(I(x)+A(x)e^2(x)\right)\frac{\partial\Psi_2}{\partial t}g_2|_0^T dx=0, (x,t)\in Q. \tag{26}$$

In equalities (23) and (24) instead of $\eta_1(x,t)$ and $\eta_2(x,t)$ we take $\Psi_1(x,t)$ and $\Psi_2(x,t)$, in the identities (25) and (26) instead of $g_1(x,t)$ and $g_2(x,t)$ we take $\delta y(x,t)$ and $\delta\theta(x,t)$ respectively, subtract the obtained relations and sum them. Then we have,

$$\int_0^l [(y(x,T;v)-\varphi_1(x))\delta y(x,T)dx+(\theta(x,T;v)-\varphi_2(x))\delta\theta(x,T)dx=$$

$$=\int_0^l [\rho(x)A(x)\ \Psi_1(x,0)-\rho(x)A(x)e(x)\ \Psi_2(x,0)]\delta v_1 dx+ \tag{27}$$

$$+\int_0^l [-\rho(x)A(x)e(x)\Psi_1(x,0)+\rho(x)\ \left(I(x)+A(x)e^2(x)\right)\ \Psi_2(x,0)]\delta w_1 \mathrm{dx}$$

Therefore from formulas (17) and (27) it follows that

$$\Delta J_\alpha(v) = \int_0^l \left[\rho(x) A(x)\ \Psi_1(x,0) - \rho(x) A(x) e(x) \Psi_2(x,0) + \alpha v_1\right] \delta v_1 dx +$$

$$+ \int_0^l \left[-\rho(x) A(x) e(x) \Psi_1(x,0) + \rho(x)\left(I(x) + A(x) e^2(x)\right) \cdot\right.$$

$$\left. \cdot\, \Psi_2(x,0) + \alpha w_1\right] \delta w_1 dx + R \tag{28}$$

Next we show that

$$\|\delta y(x,T)\|^2_{L_{2(0,l)}} \leq c\|\delta v_1\|^2_{L_2(0,l)}, \tag{29}$$

$$\|\delta\theta(x,T)\|^2_{L_{2(0,l)}} \leq c\|\delta w_1\|^2_{L_2(0,l)}. \tag{30}$$

For this purpose first we show that

$$\|\delta y(x,t)\|^2_{W_2^{2,1}(Q)} \leq c\|\delta v_1\|^2_{L_2(0,l)}, \tag{31}$$

$$\|\delta\theta(x,t)\|^2_{W_2^{2,1}(Q)} \leq c\|\delta w_1\|^2_{L_2(0,l)} \tag{32}$$

For proving estimations (31) and (32) we apply the Faedo-Galerkin method. Let $\{\omega_i(x)\}_{i=1}^{\infty}$ be a fundamental system in $\overset{0}{W}{}_2^2(0,l)$ and

$$\int_0^l \omega_i(x)\,\omega_k(x)\,dx = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

We look for approximate solutions$\left(\delta y^N(x,t), \delta\theta^N(x,t)\right)$of problem (18), (19) in the form $\delta y^N(x,t) = \sum_{i=1}^N c_{1i}^N(t)\,\omega_i(x)$ and $\delta\theta^N(x,t) = \sum_{i=1}^N c_{2i}^N(t)\,\omega_i(x)$ from the following relations

$$\int_0^l \left(E(x) I(x) \frac{\partial^2 \delta y^N}{\partial x^2}\frac{d^2\omega_p(x)}{\partial x^2} + \rho(x) A(x) \frac{\partial^2 \delta y^N}{\partial t^2}\omega_p(x) - \right.$$

$$\left. - \rho(x) A(x) e(x) \frac{\partial^2 \delta\theta^N}{\partial t^2}\omega_p(x)\right) dx = 0, \quad p = \overline{1,N}, \tag{33}$$

$$\int_0^l \left(E(x) C_w(x) \frac{\partial^2 \delta\theta^N}{\partial x^2}\frac{d^2\omega_p(x)}{\partial x^2} - G(x) C(x)\, \delta\theta^N \frac{d^2\omega_p(x)}{\partial x^2} - \right.$$

$$-\rho(x) A(x) e(x) \frac{\partial^2 \delta y^N}{\partial t^2}\omega_p(x) +$$

$$\left.\rho(x)\left(I(x) + A(x) e^2(x)\right) \frac{\partial^2 \delta\theta^N}{\partial t^2}\omega_p(x)\right) dx = 0\ p = \overline{1,N}, \tag{34}$$

$$c_{1i}^N\big|_{t=0}=0,\ \left.\frac{dc_{1i}^N(t)}{dt}\right|_{t=0}=(v_1,\omega_i), \tag{35}$$

$$c_{2i}^N\big|_{t=0}=0,\ \left.\frac{dc_{2i}^N(t)}{dt}\right|_{t=0}=(w_1,\omega_i),\ i=\overline{1,N}. \tag{36}$$

Equalities (33) and (34) are the system of linear ordinary differential equations of second order with the unknowns $c_{1i}^N(t)$ and $c_{2i}^N(t)$, $i=\overline{1,N}$ solved with respect to $d^2c_{1i}^N/dt^2$ and $d^2c_{2i}^N/dt^2$. Under the conditions on the problem data, this system is uniquely solvable under initial conditions (35) and (36), moreover $d^2c_{1i}^N/dt^2$, $d^2c_{2i}^N/dt^2\in L_2(0,T)$, $i=\overline{1,N}$.

Multiplying each of the equalities (33) and (34) by its own dc_{1p}^N/dt, dc_{2p}^N/dt and summing over p from 1 to N we come to the equalities

$$\int_0^l \Big(E(x)I(x)\frac{\partial^2\delta y^N}{\partial x^2}\frac{\partial^3\delta y^N}{\partial x^2\partial t}+\rho(x)A(x)\frac{\partial^2\delta y^N}{\partial t^2}\frac{\partial\delta y^N}{\partial t}-$$

$$-\rho(x)A(x)e(x)\frac{\partial^2\delta\theta^N}{\partial t^2}\frac{\partial\delta y^N}{\partial t}\Big)dx=0, \tag{37}$$

$$\int_0^l \Big(E(x)C_w(x)\frac{\partial^2\delta\theta^N}{\partial x^2}\frac{\partial^3\delta\theta^N}{\partial x^2\partial t}-G(x)C(x)\delta\theta^N\frac{\partial^3\delta\theta^N}{\partial x^2\partial t}-$$

$$\rho(x)A(x)e(x)\frac{\partial^2\delta y^N}{\partial t^2}\frac{\partial\delta\theta^N}{\partial t}+$$

$$+\rho(x)\big(I(x)+A(x)e^2(x)\big)\frac{\partial^2\delta\theta^N}{\partial t^2}\frac{\partial\delta\theta^N}{\partial t}\Big)dx=0 \tag{38}$$

Suppose that $G(x)C(x)$ are independent of x.
Then it follows that

$$\frac{1}{2}\frac{d}{dt}\int_0^l\Bigg[\Bigg(E(x)I(x)\left(\frac{\partial^2\delta y^N}{\partial x^2}\right)^2+\rho(x)A(x)\left(\frac{\partial\delta y^N}{\partial t}\right)^2+E(x)C_w(x)\left(\frac{\partial^2\delta\theta^N}{\partial x^2}\right)^2+$$

$$+GC\left(\frac{\partial\delta\theta^N}{\partial x}\right)^2+\rho(x)\big(I(x)+A(x)e^2(x)\big)\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2-$$

$$-2\rho(x)A(x)e(x)\left(\frac{\partial\delta y^N}{\partial t}\frac{\partial\delta\theta^N}{\partial t}\right)\Bigg]dx=0. \tag{39}$$

We integrate the last equality with respect to t from 0 to t :

$$\int_0^l\Bigg[\Bigg(E(x)I(x)\left(\frac{\partial^2\delta y^N}{\partial x^2}\right)^2+\rho(x)A(x)\left(\frac{\partial\delta y^N}{\partial t}\right)^2+E(x)C_w(x)\left(\frac{\partial^2\delta\theta^N}{\partial x^2}\right)^2+$$

$$+GC\left(\frac{\partial\delta\theta^N}{\partial x}\right)^2+\rho(x)\left(I(x)+A(x)e^2(x)\right)\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2-$$

$$-2\rho(x)A(x)e(x)\left(\frac{\partial\delta y^N}{\partial t}\frac{\partial\delta\theta^N}{\partial t}\right)\Bigg]dx=$$

$$=\int_0^l\left[\rho(x)A(x)\left(\frac{\partial\delta y^N(x,0)}{\partial t}\right)^2+\rho(x)\left(I(x)+A(x)e^2(x)\right)\left(\frac{\partial\delta\theta^N(x,0)}{\partial t}\right)^2-\right.$$

$$\left.-2\rho(x)A(x)e(x)\left(\frac{\partial\delta y^N(x,0)}{\partial t}\frac{\partial\delta\theta^N(x,0)}{\partial t}\right)\right]dx. \tag{40}$$

In equality (40) we make some transformations:

$$\int_0^l\left[(E(x)I(x)\left(\frac{\partial^2\delta y^N}{\partial x^2}\right)^2+\rho(x)A(x)\left(\frac{\partial\delta y^N}{\partial t}\right)^2+E(x)C_w(x)\left(\frac{\partial^2\delta\theta^N}{\partial x^2}\right)^2+\right.$$

$$+GC\left(\frac{\partial\delta\theta^N}{\partial x}\right)^2+\rho(x)\left(I(x)+A(x)e^2(x)\right)\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2-$$

$$\left.-\rho(x)A(x)e(x)\left(\left(\frac{\partial\delta y^N}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2\right)\right]dx\le$$

$$\le\int_0^l\left[\rho(x)A(x)\left(\frac{\partial\delta y^N(x,0)}{\partial t}\right)^2+\rho(x)\left(I(x)+A(x)e^2(x)\right)\left(\frac{\partial\delta\theta^N(x,0)}{\partial t}\right)^2+\right.$$

$$\left.+\rho(x)A(x)e(x)\left(\left(\frac{\partial\delta y^N(x,0)}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N(x,0)}{\partial t}\right)^2\right)\right]dx. \tag{41}$$

It's clear that,

$$\int_0^l\left|\frac{\partial\delta y^N(x,0)}{\partial t}\right|^2dx\le\int_0^l\left|\sum_{i=1}^N(\delta v_1,\omega_i)\,\omega_i(x)\right|^2dx\le$$

$$\le c\sum_{i=1}^N|\delta v_{1i}|^2\le c\sum_{i=1}^\infty|\delta v_{1i}|^2\le c\,\|\delta v_1\|^2_{L_2(0,l)}, \tag{42}$$

$$\int_0^l\left|\frac{\partial\delta\theta^N(x,0)}{\partial t}\right|^2dx\le\int_0^l\left|\sum_{i=1}^N(\delta w_1,\omega_i)\,\omega_i(x)\right|^2dx\le$$

$$\leq c\sum_{i=1}^{N}|\delta w_{1i}|^2 \leq c\sum_{i=1}^{\infty}|\delta w_{1i}|^2 \leq c\left\|\delta w_1\right\|^2_{L_2(0,l)}, \tag{43}$$

where $\delta v_{1i}=\int_0^l \delta v_1(x)\,\omega_i(x)\,dx$, $\delta w_{1i}=\int_0^l \delta w_1(x)\,\omega_i(x)\,dx$ are Fourier coefficients of the functions $\delta v_1(x)$, $\delta w_1(x)$. From (41), (53), (43) we get:

$$\int_0^l\left[(E(x)I(x)\left(\frac{\partial^2\delta y^N}{\partial x^2}\right)^2+\rho(x)A(x)\left(\frac{\partial\delta y^N}{\partial t}\right)^2+E(x)C_w(x)\left(\frac{\partial^2\delta\theta^N}{\partial x^2}\right)^2+\right.$$

$$+GC\left(\frac{\partial\delta\theta^N}{\partial x}\right)^2+\rho(x)\big(I(x)+A(x)e^2(x)\big)\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2-$$

$$\left.-\rho(x)A(x)e(x)\left(\left(\frac{\partial\delta y^N}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N}{\partial t}\right)^2\right)\right]dx\leq$$

$$\leq\int_0^l\left[\rho(x)A(x)\left(\frac{\partial\delta y^N(x,0)}{\partial t}\right)^2+\rho(x)(I(x)+\right.$$

$$+A(x)e^2(x)\left(\frac{\partial\delta\theta^N(x,0)}{\partial t}\right)^2+$$

$$\left.+\rho(x)A(x)e(x)\left(\left(\frac{\partial\delta y^N(x,0)}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N(x,0)}{\partial t}\right)^2\right)\right]dx\leq$$

$$\leq c\left(\left\|\delta v_1\right\|^2_{L_2(0,l)}+\left\|\delta w_1\right\|^2_{L_2(0,l)}\right) \tag{44}$$

Assume that $1-e(x)\geq\alpha_0>0, I(x)+A(x)e(x)(e(x)-1)\geq\alpha_1>0$, $\forall x\in[0,l]$, where $\alpha_0,\alpha_1>0$- are the given numbers.
Since $E(x)$, $I(x), A(x), C_\omega(x), \rho(x)$ are positive functions on the segment $[0,l]$, by equivalence of the norms in the space $\overset{0}{W}{}_2^2(0,l)$, from the last inequality by means of elementary transformations we get:

$$\int_0^l\left[\left(\delta y^N(x,t)\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta y^N(x,t)}{\partial x^2}\right)^2+\right.$$

$$\left.+\left(\delta\theta^N(x,t)\right)^2+\left(\frac{\partial\delta\theta^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta\theta^N(x,t)}{\partial x^2}\right)^2\right]dx\leq$$

$$\leq c\int_0^t\int_0^l\left[\left(\delta y^N(x,s)\right)^2+\left(\frac{\partial\delta y^N(x,s)}{\partial t}\right)^2+\left(\frac{\partial\delta y^N(x,s)}{\partial x}\right)^2+\left(\frac{\partial^2\delta y^N(x,s)}{\partial x^2}\right)^2+\right.$$

$$\left.+\left(\delta\theta^N(x,s)\right)^2+\left(\frac{\partial\delta\theta^N(x,s)}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N(x,s)}{\partial x}\right)^2+\left(\frac{\partial^2\delta\theta^N(x,s)}{\partial x^2}\right)^2\right]dxds+$$

$$+c\left(\|\delta v_1\|^2_{L_2(0,l)}+\|\delta w_1\|^2_{L_2(0,l)}\right). \tag{45}$$

Applying the Gronuoll lemma, we have:

$$\int_0^l\left[\left(\delta y^N(x,t)\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta y^N(x,t)}{\partial x^2}\right)^2+\right.$$

$$\left.+\left(\delta\theta^N(x,t)\right)^2+\left(\frac{\partial\delta\theta^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta\theta^N(x,t)}{\partial x^2}\right)^2\right]dx\leq$$

$$\leq c\left(\|\delta v_1\|^2_{L_2(0,l)}+\|\delta w_1\|^2_{L_2(0,l)}\right)\ \forall t\in[0,T]. \tag{46}$$

From the last inequality it follows that

$$\int_0^T\int_0^l\left[\left(\delta y^N(x,t)\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta y^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta y^N(x,t)}{\partial x^2}\right)^2+\right.$$

$$\left.+\left(\delta\theta^N(x,t)\right)^2+\left(\frac{\partial\delta\theta^N(x,t)}{\partial t}\right)^2+\left(\frac{\partial\delta\theta^N(x,t)}{\partial x}\right)^2+\left(\frac{\partial^2\delta\theta(x,t)}{\partial x^2}\right)^2\right]dx\leq$$

$$\leq c\left(\|\delta v_1\|^2_{L_2(0,l)}+\|\delta w_1\|^2_{L_2(0,l)}\right). \tag{47}$$

From the sequence $\left(\delta y^N,\delta\theta^N\right)$ we can choose a subsequence weakly convergent in $W_2^{2,1}(Q)\times W_2^{2,1}(Q)$ to some element $(\delta y,\delta\theta)\in W_2^{2,1}(Q)\times W_2^{2,1}(Q)$. By virtue of weak lower semicontinuity of the norm in the Hilbert space, we get from (47) that for $\delta y(x,t)$ and $\delta\theta(x,t)$ the following estimation is valid

$$\|\delta y\|^2_{W_2^{2,1}(Q)}+\|\delta\theta\|^2_{W_2^{2,1}(Q)}\leq c\left(\|\delta v_1\|^2_{L_2(0,l)}+\|\delta w_1\|^2_{L_2(0,l)}\right).$$

Hence the estimate (31) and (32).
Since $W_2^{2,1}(Q)$ is boundedly imbedded in $L_2(0,T)$ [6, pp. 73-74], hence it follows that

$$\|\delta y(x,T)\|^2_{L_2(0,l)} \le c_1 \|\delta y\|^2_{W_2^{2,1}(Q)} \le c\left(\|\delta v_1\|^2_{L_2(0,l)} + \|\delta w_1\|^2_{L_2(0,l)}\right), \quad (48)$$

$$\|\delta \theta(x,T)\|^2_{L_2(0,l)} \le c_2 \|\delta \theta\|^2_{W_2^{2,1}(Q)} \le c\left(\|\delta v_1\|^2_{L_2(0,l)} + \|\delta w_1\|^2_{L_2(0,l)}\right).$$

It is easy to show that $(\delta y, \delta\theta)$ is the generalized solution of problem (18)-(19) [6, pp. 210-215].

Thus, from (48) we find that

$$\mathrm{R} = \frac{1}{2}\int_0^l \left[(\delta y(x,T))^2 + (\delta\theta(x,T))^2\right]dx + \frac{\alpha}{2}\left(\int_0^l ((\delta v_1)^2 + (\delta w_1)^2)dx\right) \le$$

$$\le c\left(\|\delta v_1\|^2_{L_2(0,l)} + \|\delta w_1\|^2_{L_2(0,l)}\right). \quad (49)$$

Thus, from (28) and (49) it follows that the differential of the functional $J(v)$ is equal to

$$\left\langle J'(v), \delta v\right\rangle = \int_0^l \left[\rho(x)A(x)\ \Psi_1(x,0) - \rho(x)A(x)e(x)\ \Psi_2(x,0) + \alpha v_1\right]\delta v_1 \mathrm{dx} +$$

$$+\left[-\rho(x)A(x)e(x)\ \Psi_1(x,0) + \rho(x)\left(I(x) + A(x)e^2(x)\right)\ \Psi_2(x,0) + \alpha w_1\right]\delta w_1 \mathrm{dx}.$$

3. Necessary and sufficient condition of optimality

Theorem 1. For the control $v(x) = (v_1^0(x),\ w_1^0(x))$ to be an optimal control in problem (1)-(6), (10) it is necessary and sufficient that

$$\int_0^l \left[\rho(x)A(x)\ \Psi_1(x,0) - \rho(x)A(x)e(x)\Psi_2(x,0) + \alpha v_1\right]\left(v_1(x) - v_1^0(x)\right)\mathrm{dx} +$$

$$+\left[\int_0^l -\rho(x)A(x)e(x)\Psi_1(x,0) + \rho(x)\left(I(x) + A(x)e^2(x)\right)\ \Psi_2(x,0) + \alpha w_1\right]\times$$

$$\times\left(w_1(x) - w_1^0(x)\right)\mathrm{dx} \ge 0,\ \forall v = (v_1, w_1) \in U_{ad}. \quad (50)$$

Proof. Let $v_0(x) = \left(v_1^0(x), w_1^0(x)\right)$- **to be an optimal control in problem** (1)-(6), (10). As U_{ad}- is a convex set in $L_2(0,l) \times L_2(0,l)$, by virtue of the known theorem from [7, pp. 28],

$$\left\langle J'(v), v - v_0\right\rangle \ge 0, \forall v \in U_{ad}.$$

From the last inequality we get necessity. As problem (1)-(6), (10) is a linear-quadratic, the obtained condition is a sufficient condition as well for the optimality of the control $v_0(x)$.

Conclusion: In this paper, the inverse problem of determining the right-hand sides of the flexural-torsional vibrations of a rod is considered. This problem is reduced to the problem of optimal control. The gradient of the functional is calculated and, using the gradient expression, a necessary and sufficient optimality condition is proved.

Example 1. We consider a boundary value problem for equations of flexural-torsional vibrations of a bar, described by the system of two differential equations in the domain $Q = \{0 < x < 1,\, 0 < t < 1\}$

$$\frac{\partial^4 y}{\partial x^4} + 4\frac{\partial^2 y}{\partial t^2} - 2\frac{\partial^2 \theta}{\partial t^2} = f_1(x,t), \tag{51}$$

$$\frac{\partial^4 \theta}{\partial x^4} - \frac{\partial^2 \theta}{\partial x^2} - 2\frac{\partial^2 y}{\partial t^2} + 3\frac{\partial^2 \theta}{\partial t^2} = f_2(x,t), \quad (x,t)\in\mathrm{Q} \tag{52}$$

$$y_{x=0} = y|_{x=1} = 0, \qquad \left.\frac{\partial y}{\partial x}\right|_{x=0} = \left.\frac{\partial y}{\partial x}\right|_{x=1} = 0, \qquad 0\leq\mathrm{t}\leq\mathrm{T}, \tag{53}$$

$$\theta|_{x=0} = \theta|_{x=1} = 0, \qquad \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = \left.\frac{\partial \theta}{\partial x}\right|_{x=1} = 0, \qquad 0\leq\mathrm{t}\leq\mathrm{T}, \tag{54}$$

$$y|_{t=0} = 0, \ \left.\frac{\partial y}{\partial t}\right|_{t=0} = v_1(x), \tag{55}$$

$$\theta|_{t=0} = 0, \ \left.\frac{\partial \theta}{\partial t}\right|_{t=0} = v_2(x), \tag{56}$$

$$f_1(x,t) = 24t, \quad f_2(x,t) = 44t + 24tx - 24tx^2.$$

In the special case, the coefficients of equations (51)-(52) were taken in the from:$E = \frac{1}{2},\ I = 2,\ \rho = 1,\ e = \frac{1}{2},\ A = 4,\ C_w = 2,\ G = 1,\ C = 1.$
In order to determine $v(x) = (v_1(x), v_2(x))$, we give the additional conditions:

$$y\left(x, \frac{1}{2}, v\right) = \frac{x^2(1-x)^2}{2}$$

$$\theta\left(x, \frac{1}{2}, v\right) = x^2(1-x)^2.$$

In this special case the functional (9) has the form

$$J_0(v) = \frac{1}{2}\int_0^1 \left[\left(y\left(x,\frac{1}{2},v\right) - \frac{x^2(1-x)^2}{2}\right)^2 + \left(\theta\left(x,\frac{1}{2},v\right) - x^2(1-x)^2\right)^2\right] dx.$$

It is easy to verify that

$$y\,(x,t) = tx^2(1-x)^2,\ \theta\,(x,t) = 2tx^2(1-x)^2$$

and

$$\inf_{v\in L_2(0,1)\times L_2(0,1)} J\,(v) = \min_{v\in L_2(0,1)\times L_2(0,1)} J\,(v) = 0,$$

and the minimum of the functional$J\,(v)$ is attained for $v = v_0\,(x) = \left(v_1^0\,(x)\,, v_2^0\,(x)\right)$ $= \left(x^2(1-x)^2, 2x^2(1-x)^2\right)$.

In this case necessary and sufficient condition (28) is fulfilled by itself, when $\alpha = 0$.

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