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β_1-paracompactness with respect to an ideal

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Abstract. The notion of β_1-paracompactness in topological spaces is introduced and studied in [1]. In this paper, we introduce and investigate the notion of β_1-paracompact spaces with respect to an ideal $\mathcal{I}$ which is a generalization of the notion of β_1-paracompact spaces. We study characterizations, subsets and subspaces of $\beta_1\mathcal{I}$-paracompact spaces. Also, we investigate the invariants of $\beta_1\mathcal{I}$-paracompact spaces by functions.

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1. Introduction and Preliminaries

In 2016, Heyam Al-Jarrah introduced and studied the concept of β_1-paracompact spaces. A space (X, τ) is said to be β_1-paracompact space [1] if every β-open cover of X has a locally finite open refinement. In this paper, we introduce a new class of spaces, called $\beta_1\mathcal{I}$-paracompact spaces and investigate their properties and their relations with other types of spaces.

The notion of ideals in topological spaces was first studied by Kuratowski [12] and Vaidyanathaswamy [23]. An ideal $\mathcal{I}$ on a set X is a nonempty collection of subsets of X which satisfies the following properties:

(i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$;

(ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $p(X)$ is the set of all subsets of X, a set operator $()^* : p(X) \to p(X)$, called a local function [10] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I} \quad \text{for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*()$ for a topology $\tau^*(\mathcal{I}, \tau)$ called $*$-topology finer than τ is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [10] and $\beta = \{U \setminus I : U \in \tau, I \in \mathcal{I}\}$ is a basis for τ^* [10]. We simply write τ^* for $\tau^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. If $\beta = \tau^*$, then we say $\mathcal{I}$ is τ-simple [10]. A sufficient condition for $\mathcal{I}$ to be simple is the following: for $A \subset X$, if for

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every $a \in A$ there exists $U \in \tau(a)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. If $(X, \tau, \mathcal{I})$ satisfies this condition, then τ is said to be compatible with respect to $\mathcal{I}$ [10] or $\mathcal{I}$ is said to be τ-local, denoted by $\mathcal{I} \sim \tau$. Given an ideal space $(X, \tau, \mathcal{I})$, we say $\mathcal{I}$ is τ-boundary [10] or $\mathcal{I}$-codense if $\mathcal{I} \cap \tau = \emptyset$. An ideal $\mathcal{I}$ is said to be weakly τ-local [11] if $A^* = \emptyset$ implies $A \in \mathcal{I}$. Some useful ideals in X are: (i) $p(A)$, where $A \subseteq X$ and (ii) $\mathcal{I}_f$, the ideal of all finite subsets of X.

By a space (X, τ), we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, we denote the closure of A and the interior of A by $cl(A)$ and $int(A)$, respectively. A subset A of (X, τ) is siad to be semi-open[13] (resp., α-open[17], regular open [21]) if $A \subset cl(int(A))$, (resp., $A \subset int(cl(int(A)))$, $A = int(cl(A))$). The family of α-sets of a space (X, τ) denoted by τ^α forms a topology on X finer than τ [17]. Abd El-Monsef et al.[8] introduced and studied the concept of β-open sets in topological spaces. A subset A of X is called β-open if $A \subset cl(int(cl(A)))$. The complement of a β-open set is said to be β-closed [8]. The family of all β-open (resp., β-closed) subsets of X is denoted by $\beta O(X, \tau)$ (resp., $\beta C(X, \tau)$). The union of all β-open subsets of X contained in A is called β-interior of A and is denoted by $\beta int(A)$ and the intersection of all β-closed subsets of X containing A is called the β-closure of A and is denoted by $\beta cl(A)$. A set A is called β-regular [19] if it is both β-open and β-closed. A space (X, τ) is said to be β-regular [14] if for each β-open set U and each $x \in U$, there exists a β-open set V such that $x \in V \subset \beta cl(V) \subset U$. For any space, $\beta O(X, \tau^\alpha) = \beta O(X, \tau)$ [2]. A collection $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ of subsets of a space (X, τ) is said to be locally finite if for each $x \in X$, there exists an open set U containing x and U intersects at most finitely many members of $\mathcal{W}$. A subset A of space X is said to be N-closed relative to X (briery, N-closed)[7] if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by open subsets of X, there exists a finite subfamily Δ_0 of Δ such that $A \subset \cup\{int(cl(int(U_\alpha))) : \alpha \in \Delta_0\}$.

Definition 1. A space (X, τ) is said to be:
(i) extremally disconnected (briefly e.d.) [24] if the closure of every open set in (X, τ) is open;
(ii) submaximal [5] if each dense subset of X is open in X.

Lemma 1. *[3] The union of a finite family of locally finite collection of sets in a space is a locally finite family of sets.*

Theorem 1. *[16] Let (X, τ) be a space, $A \subset B \subset X$ and B is β-open in (X, τ). Then A is β-open in (X, τ) if and only if A is β-open in the subspace (B, τ_B).*

Theorem 2. *[4] If $\{U_\alpha : \alpha \in \Delta\}$ is a locally finite family of subsets in a space X and if $V_\alpha \subset U_\alpha$ for each $\alpha \in \Delta$, then the family $\{V_\alpha : \alpha \in \Delta\}$ is a locally finite in X.*

Lemma 2. *[9] If $f : (X, \tau) \to (Y, \sigma)$ is a continuous surjective function and $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ is locally finite in Y, then $f^{-1}(\mathcal{U}) = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is locally finite in X.*

Lemma 3. *[18] Let $f : (X, \tau) \to (Y, \sigma)$ be almost closed surjection with N-closed point inverse. If $\{U_\alpha : \alpha \in \Delta\}$ is a locally finite open cover of X, then $\{f(U_\alpha) : \alpha \in \Delta\}$ is a locally finite cover of Y.*

Lemma 4. *[22] $\mathcal{I}$ is weakly τ-local implies $\mathcal{I}$ is τ-locally finite.*

Lemma 5. *[2] If V is open and A is semi-preopen (or β-open) then $V \cap A$ is semi-preopen (or β-open).*

2. $\beta_1\mathcal{I}$-paracompact spaces

Recall that an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-paracompact [22] (resp., $S_1\mathcal{I}$-paracompact [20]) if every open (resp., semi-open) cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ (not necessarily a cover) such that $X \setminus \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$.

Definition 2. An ideal space $(X, \tau, \mathcal{I})$ is said to be $\beta_1\mathcal{I}$-paracompact, or β_1-paracompact modulo an ideal $\mathcal{I}$ if every β-open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ (not necessarily a cover) such that $X \setminus \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. A family $\mathcal{V}$ of subsets of X such that $X \setminus \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$ is called an $\mathcal{I}$-cover of X.

It follows from the definitions that

$$\beta_1\text{-paracompact} \Rightarrow \beta_1\mathcal{I}\text{-paracompact} \Rightarrow S_1\mathcal{I}\text{-paracompact} \Rightarrow \mathcal{I}\text{-paracompact}$$

The following examples show that the converses of the above implications need not be true in general.

Example 1. *Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, X, \{0\}\}$ and $\mathcal{I} = \mathcal{I}_f$. Then (X, τ) is paracompact which implies that $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-paracompact. On the other hand $(X, \tau, \mathcal{I})$ is not $\beta_1\mathcal{I}$-paracompact. For the β-open cover $\mathcal{U} = \{\{0, x\} : x \in X, x \neq 0\}$, we can find a locally finite open refinement $\mathcal{V} = \{0\}$ of $\mathcal{U}$. But $\mathcal{V}$ does not $\mathcal{I}$-cover of X. Therefore, $(X, \tau, \mathcal{I})$ is not $\beta_1\mathcal{I}$-paracompact.*

Example 2. *Let $X = \{1, 2, 3, 4\}$ with the topology $\tau = \{\emptyset, X, \{1, 2\}, \{3, 4\}\}$ and $\mathcal{I} = \mathcal{I}_f$. Then $(X, \tau, \mathcal{I})$ is $S_1\mathcal{I}$-paracompact, since $SO(X, \tau) = \tau$, but it is not $\beta_1\mathcal{I}$-paracompact since $\mathcal{U} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ is a β-open cover of X which admits no locally finite open refinement.*

Example 3. *Consider the ideal space $(X, \tau, \mathcal{I})$ where $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1\}\}$ and $\mathcal{I} = \{A \subset X : 1 \notin A\}$. Then $\beta O(X, \tau) = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}\}$. Therefore $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact space. On the other hand, (X, τ) is not β_1- paracompact since $\mathcal{U} = \{\{1, 2\}, \{1, 3\}\}$ is a β_1-open cover of (X, τ) which admits no locally finite open refinement.*

Example 4. *Consider the ideal space $(X, \tau, \mathcal{I})$ where $X = \mathbb{R}$, the set of all real numbers, $\tau = \{\emptyset, X, \{0\}\}$ and $\mathcal{I} = \{A \subset X : 0 \notin A\}$. Then $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact space but (X, τ) is not β_1-paracompact, since the β-open cover $\mathcal{U} = \{\{0, x\} : x \in X, x \neq 0\}$ admits no locally finite open refinement.*

Corollary 1. *Let (X, τ) be a space with an ideal $\mathcal{I} = \{\emptyset\}$. Then (X, τ) is β_1-paracompact if and only if $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact.*

Corollary 2. *For an e.d. submaximal ideal space $(X, \tau, \mathcal{I})$, the following conditions are equivalent:*
(i) $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact;
(ii) $(X, \tau, \mathcal{I})$ is $S_1\mathcal{I}$-paracompact;
(iii) $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-paracompact.

Proof. This follows directly from the fact that if an ideal space $(X, \tau, \mathcal{I})$ is an e.d. submaximal space, then $\tau = SO(X, \tau) = \beta O(X, \tau)$.

Proposition 1. *If $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact, then $(X, \tau^\alpha, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact.*

Proof. Suppose $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a β-open cover of $(X, \tau^\alpha, \mathcal{I})$. Then $\mathcal{U}$ is a β-open cover of $(X, \tau, \mathcal{I})$. By hypothesis, there exist a locally finite open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $\mathcal{U}$ such that $X \cup \setminus\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Since $\tau \subset \tau^\alpha$, the family $\mathcal{V}$ is a τ^α-locally finite τ^α-open refinement of $\mathcal{U}$ and so $(X, \tau^\alpha, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact.

By replacing $\mathcal{I}$ by $\{\emptyset\}$ in Proposition 1, we have the following corollary.

Corollary 3. *[1, Theorem 2.8(2)] If (X, τ) is β_1-paracompact, then (X, τ^α) is β_1-paracompact.*

Theorem 3. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{I}$ is codense, $(X, \tau^*, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact and $\mathcal{I}$ is τ-simple, then every β-open cover of $(X, \tau, \mathcal{I})$ has τ-locally finite β-open $\mathcal{I}$-cover refinement.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a β-open cover of X. Since, $\beta O(X, \tau) \subseteq \beta O(X, \tau^*)$. Then $\mathcal{U}$ is a β-open cover of $(X, \tau^*, \mathcal{I})$. By hypothesis, there exist τ^*-locally finite τ^*-open refinement $\mathcal{V} = \{V_\lambda \setminus I_\lambda : \lambda \in \Lambda, V_\lambda \in \tau, I_\lambda \in \mathcal{I}\}$ of $\mathcal{U}$ such that $X \setminus \cup\{V_\lambda \setminus I_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. For each $x \in X$, there exists a τ^*-open set W containing x such that $W \cap (V_\lambda \setminus I_\lambda) = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. Since $\mathcal{I}$ is τ-simple, $W = U \setminus I$ for some $U \in \tau$ and $I \in \mathcal{I}$. Thus, $(U \setminus I) \cap (V_\lambda \setminus I_\lambda) = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$ which implies that $(U \cap V_\lambda) \setminus (I \cup I_\lambda) = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. Since $\mathcal{I}$ is codense, then $U \cap V_\lambda = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. Then $U \cap (V_\lambda \cap U_\alpha) = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. By Lemma 5, $\mathcal{W} = \{V_\lambda \cap U_\alpha : \lambda \in \Lambda\}$ is τ-locally finite β-open refinement of $\mathcal{U}$. Since $\mathcal{V}$ refines $\mathcal{U}$ for every $V_\lambda \setminus I_\lambda \in \mathcal{V}$, there exists $U_\alpha \in \mathcal{U}$ such that $V_\lambda \setminus I_\lambda \subset U_\alpha$. Thus, $V_\lambda \setminus I_\lambda = U_\alpha \cap (V_\lambda \setminus I_\lambda) = (V_\lambda \cap U_\alpha) \setminus I_\lambda \subset V_\lambda \cap U_\alpha \subset U_\alpha$. Then $X \setminus \cup\{V_\lambda \cap U_\alpha : \lambda \in \Lambda, \alpha \in \Delta\} \subset X \setminus \cup\{V_\lambda \setminus I_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$ which implies that $X \setminus \cup\{V_\lambda \cap U_\alpha : \lambda \in \Lambda, \alpha \in \Delta\} \in \mathcal{I}$.

Theorem 4. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{I}$ is weakly τ-local and $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact, then $(X, \tau^*, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha \setminus I_\alpha : \alpha \in \Delta, U_\alpha \in \tau, I_\alpha \in \mathcal{I}\}$ be a β-open cover of $(X, \tau^*, \mathcal{I})$. Then $\mathcal{W} = \{U_\alpha : \alpha \in \Delta\}$ is a β-open cover of X and so it has locally finite open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $X \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Now the family $\{V_\lambda \cap I_\alpha : \lambda \in \Lambda\} \subset \mathcal{I}$ is locally finite. Since, $\mathcal{I}$ is weakly τ-local, $\cup_{\lambda\in\Lambda}(V_\lambda \cap I_\alpha) \in \mathcal{I}$, by Lemma 4. Then $X \setminus \cup_{\lambda\in\Lambda}(V_\lambda \setminus I_\alpha) \subset (X \setminus \bigcup_{\lambda\in\Lambda} V_\lambda) \cup (\bigcup_{\lambda\in\Lambda} (V_\lambda \cap I_\alpha)) \in \mathcal{I}$ which implies

$X \setminus \cup_{\lambda\in\Lambda}(V_\lambda \setminus I_\alpha) \in \mathcal{I}$. Since $\mathcal{V}$ is locally finite, $\mathcal{V}_1 = \{V_\lambda \setminus I_\alpha : \lambda \in \Lambda\}$ is locally finite. Since τ^* is finer than τ, $\mathcal{V}_1$ is τ^*-locally finite τ^*-open which refines $\mathcal{U}$. Hence $(X, \tau^*, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact.

Theorem 5. *Let (X, τ) be a β-regular space. If $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact, then every β-open cover of X has a locally finite β-closed $\mathcal{I}$-cover refinement.*

Proof. Let $\mathcal{U}$ be a β-open cover of X. For each $x \in X$, let $U_x \in \mathcal{U}$ such that $x \in U_x$. Since (X, τ) is β-regular, there exists $V_x \in \beta O(X, \tau)$ such that $x \in V_x \subset \beta cl(V_x) \subset U_x$. Then the family $\mathcal{V} = \{V_x : x \in X\}$ is a β-open cover refinement of $\mathcal{U}$. By hypothesis, there exist a locally finite open refinement $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ which refine $\mathcal{V}$ such that $X \setminus \cup\{W_\alpha : \alpha \in \Delta\} \in \mathcal{I}$. The family $\beta cl(\mathcal{W}) = \{\beta cl(W_\alpha) : \alpha \in \Delta\}$ is locally finite for each $\alpha \in \Delta$. Now $X \setminus \cup\{\beta cl(W_\alpha) : \alpha \in \Delta\} \subseteq X \setminus \cup\{W_\alpha : \alpha \in \Delta\}$ implies $X \setminus \cup\{\beta cl(W_\alpha) : \alpha \in \Delta\} \in \mathcal{I}$. Hence $\beta cl(\mathcal{W})$ is $\mathcal{I}$-cover. Let $\beta cl(W_\alpha) \in \beta cl(\mathcal{W})$. Since $\mathcal{W}$ refines $\mathcal{V}$, there is some $V_x \in \mathcal{V}$ such that $W_\alpha \subset V_x$ and so $\beta cl(W_\alpha) \subset \beta cl(V_x) \subset U_x$ implies that $\beta cl(W_\alpha) \subset U_x$. Hence $\beta cl(\mathcal{W})$ refines $\mathcal{U}$. Thus, $\beta cl(\mathcal{W}) = \{\beta cl(W_\alpha) : \alpha \in \Delta\}$ is a locally finite β-closed $\mathcal{I}$-cover refinement of $\mathcal{U}$.

If $\mathcal{I} = \{\emptyset\}$ in Theorem 5, then we have the following corollary.

Corollary 4. *[1, Theorem 2.12] Let (X, τ) be a β-regular space.. If each β-open cover of the space X has a locally finite refinement, then each β-open cover of X has a locally finite β-closed refinement*

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be β-continuous [8] (resp., β-irresolute [15]) if $f^{-1}(V) \in \beta O(X, \tau)$ for each open (resp., β-open) set V in (Y, σ).

Theorem 6. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an open, β-irresolute and almost closed surjective function with N-closed point inverse. If $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact, then $(Y, \sigma, f(\mathcal{I}))$ is $\beta_1 f(\mathcal{I})$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a β-open cover of Y. Since f is β-irresolute, $\mathcal{U}_1 = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is a β-open cover of X. By hypothesis, there exists a τ-locally finite τ-open refinement $\mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\}$ of $\mathcal{U}_1$ such that $X \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Then $f(X \setminus \cup\{V_\lambda : \lambda \in \Lambda\}) \in f(\mathcal{I})$. Now, $f(X) \setminus \cup\{f(V_\lambda) : \lambda \in \Lambda\} \subseteq f(X \setminus \cup\{V_\lambda : \lambda \in \Lambda\})$ implies that $f(X) \setminus \cup\{f(V_\lambda) : \lambda \in \Lambda\} \in f(\mathcal{I})$ which implies that $Y \setminus \cup\{f(V_\lambda) : \lambda \in \Lambda\} \in f(\mathcal{I})$. Since f is open and $\mathcal{V}_1$ is τ-locally finite, $\mathcal{V} = \{f(V_\lambda) : \lambda \in \Lambda\}$ is σ-locally finite by Lemma 3. Let $f(V_\lambda) \in \mathcal{V}$. Then $V_\lambda \in \mathcal{V}_1$. Since $\mathcal{V}_1$ refines $\mathcal{U}_1$, there exists $f^{-1}(U_\alpha) \in \mathcal{U}_1$ such that $V_\lambda \subset f^{-1}(U_\alpha)$. Thus $f(V_\lambda) \subset f(f^{-1}(U_\alpha))$ implies that $f(V_\lambda) \subset U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Hence $\mathcal{V}$ refines $\mathcal{U}$. Therefore, $(Y, \sigma, f(\mathcal{I}))$ is $\beta_1 f(\mathcal{I})$-paracompact.

Since every compact set is N-closed and every closed map is almost closed, we conclude the following corollary.

Corollary 5. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an open, β-irresolute, closed surjective function with compact point inverse. If $(X, \tau, \mathcal{I})$ is $\beta_1\mathcal{I}$-paracompact, then $(Y, \sigma, f(\mathcal{I}))$ is $\beta_1 f(\mathcal{I})$-paracompact.*

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly β-continuous [1] if $f^{-1}(V) \in \tau$ for each $V \in \beta O(Y, \sigma)$.

Theorem 7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an open, strongly β-continuous, almost closed, surjective function with N-closed point inverse. If $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-paracompact, then $(Y, \sigma, f(\mathcal{I}))$ is $\beta_1 f(\mathcal{I})$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a β-open cover of Y. Since f is strongly β-continuous, $\mathcal{U}_1 = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is an open cover of X. By hypothesis, there exists a τ-locally finite τ-open refinement $\mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\}$ which refines $\mathcal{U}_1$ such that $X \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Then $f(X \setminus \cup\{V_\lambda : \lambda \in \Lambda\}) \in f(\mathcal{I})$. Now $f(X) \setminus \cup\{f(V_\lambda) : \lambda \in \Lambda\} \subseteq f(X \setminus \cup\{V_\lambda : \lambda \in \Lambda\})$ implies that $f(X) \setminus \cup\{f(V_\lambda) : \lambda \in \Lambda\} \in f(\mathcal{I})$ which implies that $Y \setminus \cup\{f(V_\lambda) : \lambda \in \Lambda\} \in f(\mathcal{I})$. Since f is open and $\mathcal{V}_1$ is τ-locally finite, $\mathcal{V} = \{f(V_\lambda) : \lambda \in \Lambda\}$ is σ-locally finite by Lemma 3. Let $f(V_\lambda) \in \mathcal{V}$. Then $V_\lambda \in \mathcal{V}_1$. Since $\mathcal{V}_1$ refines $\mathcal{U}_1$, there exists $f^{-1}(U_\alpha) \in \mathcal{U}_1$ such that $V_\lambda \subset f^{-1}(U_\alpha)$. Thus $f(V_\lambda) \subset f(f^{-1}(U_\alpha))$ implies that $f(V_\lambda) \subset U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Hence $\mathcal{V}$ refines $\mathcal{U}$. Therefore, $(Y, \sigma, f(\mathcal{I}))$ is $\beta_1 f(\mathcal{I})$-paracompact.

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be pre β-open[15] if for every β-open set V of (X, τ), $f(V)$ is β-open in (Y, σ).

Theorem 8. *Let $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ be a pre β-open, continuous, bijective function. If $(Y, \sigma, \mathcal{J})$ is a $\beta_1\mathcal{J}$-paracompact, then $(X, \tau, f^{-1}(\mathcal{J}))$ is $\beta_1 f^{-1}(\mathcal{J})$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a β-open cover of X. Since f is a pre β-open, $f(\mathcal{U}) = \{f(U_\alpha) : \alpha \in \Delta\}$ is a β-open cover of Y and so it has a σ-locally finite σ-open refinement $\mathcal{W} = \{W_\lambda : \lambda \in \Lambda\}$ of $f(\mathcal{U})$ such that $Y \setminus \cup\{W_\lambda : \lambda \in \Lambda\} \in \mathcal{J}$. Let $Y \setminus \cup\{W_\lambda : \lambda \in \Lambda\} = J \in \mathcal{J}$. This implies $Y = (\cup\{W_\lambda : \lambda \in \Lambda\}) \cup J$. Then $f^{-1}(Y) = (\cup\{f^{-1}(W_\lambda) : \lambda \in \Lambda\}) \cup f^{-1}(J)$ which implies $X = (\cup\{f^{-1}(W_\lambda) : \lambda \in \Lambda\}) \cup f^{-1}(J)$. It follows that $X \setminus \cup\{f^{-1}(W_\lambda) : \lambda \in \Lambda\} \in f^{-1}(\mathcal{J})$. Since f is continuous, by Lemma 2, $\mathcal{V} = \{f^{-1}(W_\lambda) : \lambda \in \Lambda\}$ is is τ-open, τ-locally finite. Let $f^{-1}(W_\lambda) \in \mathcal{V}$. Then $W_\lambda \in \mathcal{W}$. Since $\mathcal{W}$ refines $f(\mathcal{U})$, there exists $f(U_\alpha) \in f(\mathcal{U})$ such that $W_\lambda \subset f(U_\alpha)$. Thus $f^{-1}(W_\lambda) \subset f^{-1}(f(U_\alpha))$ implies that $f^{-1}(W_\lambda) \subset U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Hence $\mathcal{V}$ refines $\mathcal{U}$. Therefore, $(X, \tau, f^{-1}(\mathcal{J}))$ is $\beta_1 f^{-1}(\mathcal{J})$-paracompact.

If $\mathcal{I} = \{\emptyset\}$ in Theorem 8, then we have the following corollary.

Corollary 6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a pre β-open, continuous, bijective function. If (Y, σ) is β_1-paracompact, then (X, τ) is β_1-paracompact.*

3. $\beta_1\mathcal{I}$-paracompact subsets

In this section, we define the subsets and subspaces of $\beta_1\mathcal{I}$-paracompact and study some of their properties.

Definition 3. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\beta_1\mathcal{I}$-paracompact relative to X ($\beta_1\mathcal{I}$-paracompact subset) if each cover $\mathcal{U}$ of A by β-open sets of X, there exists a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $A \setminus \cup\{V : V \in \mathcal{V}\} \in \mathcal{I}$. A is said to be $\beta 1_A \mathcal{I}_A$-paracompact ($\beta 1_A \mathcal{I}_A$-paracompact subspace) if $(A, \tau_A, \mathcal{I}_A)$ is $\beta 1_A \mathcal{I}_A$-paracompact as a subspace, where τ_A is the usual subspace topology and $\mathcal{I}_A = \{A \cap I : I \in \mathcal{I}\}$.

A subset A of a space (X, τ) is said to be βg-closed [6] if $\beta cl(A) \subseteq U$ whenever $A \subset U$ and U is any β-open set in (X, τ).

Theorem 9. *Every βg-closed subset of a $\beta_1\mathcal{I}$-paracompact is $\beta_1\mathcal{I}$-paracompact.*

Proof. Let A be a βg-closed subset of $(X, \tau, \mathcal{I})$ and $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A by β-open sets of X. Since $A \subseteq \cup_{\alpha\in\Delta} U_\alpha$ and A is a βg-closed, we have $\beta cl(A) \subseteq \cup_{\alpha\in\Delta} U_\alpha$. Then $\mathcal{U}_1 = \mathcal{U} \cup \{X \setminus \beta cl(A)\}$ is a β-open cover of X. By hypothesis, there exist a locally finite open family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}$ which refines $\mathcal{U}_1$ ($V_\lambda \subset U_\alpha$ for some $\alpha \in \Delta$ and $V \subset X \setminus \beta cl(A)$) such that $X \setminus \cup[\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}] \in \mathcal{I}$. Then $\beta cl(A) \setminus \cup\{V_\lambda : \lambda \in \Lambda\} = \beta cl(A) \setminus [V \cup (\cup\{V_\lambda : \lambda \in \Lambda\})] \subset X \setminus \cup[\{V_\lambda : \lambda \in \Lambda\} \cup \{V\}] \in \mathcal{I}$. Since $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \subset \beta cl(A) \setminus \cup\{V_\lambda : \lambda \in \Lambda\}$, $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$, by heredity property of $\mathcal{I}$. Since $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}$ is a locally finite, the family $\mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\}$ is locally finite. Thus, the family $\mathcal{V}_1$ is locally finite open and $\mathcal{V}_1$ refines $\mathcal{U}$. Therefore, A is $\beta\mathcal{I}$-paracompact.

Theorem 10. *Every regular open subset of a $\beta_1\mathcal{I}$-paracompact is $\beta 1_A \mathcal{I}_A$-paracompact.*

Proof. Let A be a regular open in (X, τ) and $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ be a β-open cover of A in $(A, \tau_A, \mathcal{I}_A)$. Since A is open in $(X, \tau, \mathcal{I})$, W_α is a β-open set in $(X, \tau, \mathcal{I})$ for each $\alpha \in \Delta$, by Theorem 1. Then $\mathcal{U} = \{W_\alpha : \alpha \in \Delta\} \cup \{X \setminus A\}$ is a β-open cover of the $\beta_1\mathcal{I}$-paracompact $(X, \tau, \mathcal{I})$ and so it has a locally finite open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $X \setminus \cup\{V_\lambda : \lambda \in \Lambda\} = I \in \mathcal{I}$. Then $A \subset A \cap [(\cup\{V_\lambda : \lambda \in \Lambda\}) \cup I] = (\cup\{V_\lambda \cap A : \lambda \in \Lambda\}) \cup (I \cap A) = (\cup\{V_\lambda \cap A : \lambda \in \Lambda\}) \cup I_A$ which implies that $A \setminus \cup\{V_\lambda \cap A : \lambda \in \Lambda\} \in \mathcal{I}_A$. Let $x \in A$. Since $\mathcal{V}$ is locally finite, there exists $V \in \tau(x)$ such that $V_\lambda \cap V = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. Then $(V_\lambda \cap V) \cap A = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$ and so $(V_\lambda \cap A) \cap (V \cap A) = \emptyset$. Therefore, $\mathcal{V}_A = \{V_\lambda \cap A : \lambda \in \Lambda\}$ is τ_A-locally finite. Let $V_\lambda \cap A \in \mathcal{V}_A$. Since $\mathcal{V}$ refines $\mathcal{U}$, there is some $W_\alpha \in \mathcal{U}$ such that $V_\lambda \subset W_\alpha$ which implies $V_\lambda \cap A \subset W_\alpha$. Therefore, $\mathcal{V}_A$ refines $\mathcal{W}$. Hence A is $\beta 1_A \mathcal{I}_A$-paracompact.

Corollary 7. *Every clopen subset of a $\beta_1\mathcal{I}$-paracompact is $\beta 1_A \mathcal{I}_A$-paracompact.*

If $\mathcal{I} = \{\emptyset\}$ in Theorem 9 and Theorem 10, then we have the following corollary.

Corollary 8. *[1, Theorem 3.5] Let (X, τ) be a β_1-paracompact space. Then:*
(i) If A is regular open subset of (X, τ), then (A, τ_A) is $\beta 1_A$-paracompact;
(ii) If A is a βg-closed subset of (X, τ), then A is a β_1-paracompact.

Theorem 11. *Let A and B be subsets of an ideal space $(X, \tau, \mathcal{I})$ such that $A \subset B \subset X$. Then the following conditions hold.*
(i) If A is $\beta\mathcal{I}$-paracompact and B is β-open in (X, τ), then A is $\beta 1_B \mathcal{I}_B$-paracompact.
(ii) If A is $\beta 1_B \mathcal{I}_B$-paracompact and B is open in (X, τ), then A is $\beta_1\mathcal{I}$-paracompact.

Proof. (i) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A such that $U_\alpha \in \beta O(B, \tau_B)$. Since $B \in \beta O(X, \tau)$, $\mathcal{U}$ is a β-open cover of A in (X, τ), by Theorem 1. By hypothesis, there exists a locally finite open family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ refines $\mathcal{U}$ such that $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} = I \in \mathcal{I}$. Then $A \subseteq (\cup\{V_\lambda : \lambda \in \Lambda\}) \cup I$ and $A = A \cap B \subseteq [\cup\{V_\lambda : \lambda \in \Lambda\} \cup I] \cap B = \cup\{V_\lambda \cap B : \lambda \in \Lambda\} \cup (I \cap B)$ implies $A \setminus \cup\{V_\lambda \cap B : \lambda \in \Lambda\} \in \mathcal{I}_B$. Let $x \in B$. Since $\mathcal{V}$ is locally finite, there exits $U \in \tau(x)$ such that $U \cap V_\lambda = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. This implies $(U \cap V_\lambda) \cap B = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$ which implies $(U \cap B) \cap (V_\lambda \cap B) = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. Therefore, the family $\mathcal{V}_B = \{V_\lambda \cap B : \lambda \in \Lambda\}$ is τ_B-locally finite τ_B-open . Let $V_\lambda \cap B \in \mathcal{V}_B$. Since $\mathcal{V}$ refines $\mathcal{U}$ there exits $U_\alpha \in \mathcal{U}$ such that $V_\lambda \subset U_\alpha$ and so $V_\lambda \cap B \subset U_\alpha$. Hence $\mathcal{V}_A$ refines $\mathcal{U}$. Therefore A is ${\beta_1}_B\mathcal{I}_B$-paracompact.
(ii) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A by β-open subsets of X. Then the family $\mathcal{U}_1 = \{B \cap U_\alpha : \alpha \in \Delta\}$ is a β-open cover of A in $(B, \tau_B, \mathcal{I}_B)$. By hypothesis, exists τ_B-locally finite τ_B-open family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ refines $\mathcal{U}_1$ such that $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}_B$, where $\mathcal{I}_B = I \cap B$. It follows that $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Since B is open in X. Then by Theorem 1, $\mathcal{V}$ is a locally finite open refinement of $\mathcal{U}$. Therefore, A is $\beta_1\mathcal{I}$-paracompact.

If $\mathcal{I} = \{\emptyset\}$ in Theorem 11, then we have the following corollary.

Corollary 9. *[1, Theorem 3.6] Let A and B be subsets of an ideal space (X, τ) such that $A \subset B \subset X$. Then:*
(i) If A is β-paracompact and B is β-open in (X, τ), then A is ${\beta_1}_B$-paracompact.
(ii) If A is ${\beta_1}_B$-paracompact and B is open in (X, τ), then A is β_1-paracompact.

Theorem 12. *Let A be a clopen subspace of an ideal space $(X, \tau, \mathcal{I})$. Then A a is ${\beta_1}_A\mathcal{I}_A$-paracompact if and only if it is $\beta_1\mathcal{I}$-paracompact.*

Proof. To prove necessity, let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A by β-open subsets of the ideal subspace $(A, \tau_A, \mathcal{I}_A)$. Since A is open, $\mathcal{U}$ is a cover of A by β-open subsets of X and so it has a locally finite open refinement, say $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} = I \in \mathcal{I}$. Then $A \subseteq (\cup\{V_\lambda : \lambda \in \Lambda\}) \cup I$. Now $A \subseteq A \cap [(\cup\{V_\lambda : \lambda \in \Lambda\}) \cup I] = \cup\{V_\lambda \cap A : \lambda \in \Lambda\} \cup (A \cap I)$. It follows that $A \setminus \cup\{V_\lambda \cap A : \lambda \in \Lambda\} \in \mathcal{I}_A$. Let $x \in A$. Since $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is locally finite, there exists $W \in \tau(x)$ such that $V_\lambda \cap W = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. Then $(V_\lambda \cap W) \cap A = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$ which implies $(V_\lambda \cap A) \cap (W \cap A) = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. Thus the family $\mathcal{V}_A = \{V_\lambda \cap A : \lambda \in \Lambda\}$ is τ_A-locally finite τ_A-open refitment of $\mathcal{U}$. Hence A is ${\beta_1}_A\mathcal{I}_A$-paracompact.
To prove sufficiency, let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of A by β-open subsets of an ideal space $(X, \tau, \mathcal{I})$. Then $\mathcal{U}_1 = \{A \cap U_\alpha : \alpha \in \Delta\}$ is a β-open cover of the ${\beta_1}_A\mathcal{I}_A$-paracompact ideal subspace $(A, \tau_A, \mathcal{I}_A)$ and so it has a τ_A-locally finite τ_A-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}_A$. Then $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. But A is an open set in X, so V_λ is an open set for every $\lambda \in \Lambda$. Now $\tau_A \subseteq \tau$ and $X \setminus A$ is an open set in X which intersects no member of $\mathcal{V}$. Therefore $\mathcal{V}$ is locally finite and refines $\mathcal{U}$. Thus A is a $\beta_1\mathcal{I}$-paracompact.

If $\mathcal{I} = \{\emptyset\}$ in Theorem 12, then we have the following corollary.

Corollary 10. *[1, Theorem 3.8] Let A be a clopen subspace of a space (X, τ). Then A is a ${\beta_1}_A$-paracompact if and only if it is β_1-paracompact.*

Theorem 13. *If $(X, \tau, \mathcal{I})$ is T_2 space and A is $\beta_1\mathcal{I}$-paracompact relative to X, then A is closed in (X, τ^*).*

Proof. Let $x \in X \setminus A$. For each $y \in A$, there exists $U \in \tau$ such that $y \in U_y$ and $x \notin cl(U_y)$. Therefore, the family $\mathcal{U} = \{U_y : y \in A\}$ is an open cover of A which is $\beta_1\mathcal{I}$-paracompact relative to X. Since $\mathcal{U}$ is a β-open cover of A and so it has a locally finite open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of $\mathcal{U}$ such that $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Now $x \notin cl(V_\lambda)$ for each λ implies that $x \notin \cup\{cl(V_\lambda) : \lambda \in \Lambda\}$. Since the locally finite family $\mathcal{V}$ is closure-preserving, $x \notin \cup\{cl(V_\lambda) : \lambda \in \Lambda\} = cl(\cup\{V_\lambda : \lambda \in \Lambda\})$. Let $U = X \setminus cl(\cup\{V_\lambda : \lambda \in \Lambda\})$ and $J = A \setminus cl(\cup\{V_\lambda : \lambda \in \Lambda\}) \subset A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} = I_1$, where $I_1 \in \mathcal{I}$. Then $U \setminus J \in \tau^*(x)$ and $(U \setminus J) \cap A = \emptyset$ which implies $x \notin A^*$. Hence $A^* \subset A$. This shows that A is closed in (X, τ^*).

If $\mathcal{I} = \{\emptyset\}$ in Theorem 13, then we conclude the following corollary.

Corollary 11. *Let A be a β_1-paracompact relative subset of a T_2 space (X, τ). Then A is closed in (X, τ).*

Theorem 14. *In an ideal space $(X, \tau, \mathcal{I})$, if A and B are $\beta_1\mathcal{I}$-paracompact, then $A \cup B$ is $\beta_1\mathcal{I}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A \cup B$ by β-open sets in X. Then $\mathcal{U}$ is a β-open cover of A and B. By hypothesis, there exist locally finite families $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ of A and $\mathcal{W} = \{W_\gamma : \gamma \in \Lambda_0\}$ of B which refines $\mathcal{U}$ such that $A \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$ and $B \setminus \cup\{W_\gamma : \gamma \in \Lambda_0\} \in \mathcal{I}$. Then $A \cup B \subset (\cup\{V_\lambda : \lambda \in \Lambda\} \cup I_1) \cup (\cup\{W_\gamma : \gamma \in \Lambda_0\} \cup I_2)$, where $I_1, I_2 \in \mathcal{I}$ which implies that $A \cup B \subset (\cup\{V_\lambda \cup W_\gamma : \lambda \in \Lambda, \gamma \in \Lambda_0\}) \cup (I_1 \cup I_2)$. It follows that $(A \cup B) \setminus \cup\{V_\lambda \cup W_\gamma : \lambda \in \Lambda, \gamma \in \Lambda_0\} \in \mathcal{I}$. Since the families $\mathcal{V}$ and $\mathcal{W}$ are locally finite the family $\mathcal{V}' = \{V_\lambda \cup W_\gamma : \lambda \in \Lambda, \gamma \in \Lambda_0\}$ is locally finite, by Lemma 1 which refines $\mathcal{U}$. Therefore, $A \cup B$ is $\beta_1\mathcal{I}$-paracompact.

Theorem 15. *In an ideal space $(X, \tau, \mathcal{I})$, if A is $\beta_1\mathcal{I}$-paracompact and B is a β-closed subset of X, then $A \cap B$ is $\beta_1\mathcal{I}$-paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A \cap B$ by β-open subsets of X. Then $\mathcal{U}_A = \mathcal{U} \cup \{X \setminus B\}$ is a cover of A by β-open sets in X. By hypothesis, there exists a locally finite open refinement $\mathcal{V}_A = \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}$ of $\mathcal{U}_A$, where $V_\lambda \subset U_\alpha$ and $V \subset X \setminus B$ such that $A \setminus [(\cup\{V_\lambda : \lambda \in \Lambda\}) \cup \{V\}] \in \mathcal{I}$. Let $A \setminus [(\cup\{V_\lambda : \lambda \in \Lambda\}) \cup \{V\}] = I$. Then $I \cap B = A \setminus [(\cup(\{V_\lambda : \lambda \in \Lambda\}) \cup \{V\}] \cap B = A \cap (X \setminus [(\cup\{V_\lambda : \lambda \in \Lambda\}) \cup \{V\}]) \cap B$ implies that $I \cap B = A \cap [X \setminus (\cup\{V_\lambda : \lambda \in \Lambda\}) \cap \{X \setminus V\} \cap B]$. It follows that $I \cap B = (A \cap B) \setminus \cup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. Since $V_\lambda \subset V_\lambda \cup V$, $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is locally finite open by Theorem 2 which refines $\mathcal{U}$. Hence $A \cap B$ is $\beta_1\mathcal{I}$-paracompact.

Corollary 12. *Let $f : (X, \tau) \to (Y, \sigma, \mathcal{J})$ be a pre β-open, continuous, bijective function. If A is $\beta_1\mathcal{J}$-paracompact relative to Y, then $f^{-1}(A)$ is $\beta_1 f^{-1}(\mathcal{J})$-paracompact relative to X.*

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