β₁-paracompactness with respect to an ideal

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Abstract. The notion of β₁-paracompactness in topological spaces is introduced and studied in [1]. In this paper, we introduce and investigate the notion of β₁-paracompact spaces with respect to an ideal I which is a generalization of the notion of β₁-paracompact spaces. We study characterizations, subsets and subspaces of β₁I-paracompact spaces. Also, we investigate the invariants of β₁I-paracompact spaces by functions.

2010 Mathematics Subject Classification: 54A05, 54A08, 54D10
Key Words and Phrases: Ideal, β-open set, β₁-paracompact, β₁-paracompact modulo I, locally finite collection, β- irresolute, β-continuous, strongly β-continuous.

1. Introduction and Preliminaries

In 2016, Heyam Al-Jarrah introduced and studied the concept of β₁-paracompact spaces. A space (X,τ) is said to be β₁-paracompact space [1] if every β-open cover of X has a locally finite open refinement. In this paper, we introduce a new class of spaces, called β₁I-paracompact spaces and investigate their properties and their relations with other types of spaces.

The notion of ideals in topological spaces was first studied by Kuratowski [12] and Vaidyanathaswamy [23]. An ideal I on a set X is a nonempty collection of subsets of X which satisfies the following properties:

(i) A ∈ I and B ⊂ A implies B ∈ I;
(ii) A ∈ I and B ∈ I implies A ∪ B ∈ I.

Given a topological space (X,τ) with an ideal I on X and if p(X) is the set of all subsets of X, a set operator (·)∗ : p(X) → p(X), called a local function [10] of A with respect to τ and I is defined as follows: for A ⊂ X, A∗(I,τ) = {x ∈ X : U ∩ A /∈ I for every U ∈ τ(x)} where τ(x) = {U ∈ τ : x ∈ U}. A Kuratowski closure operator cl∗(·) for a topology τ∗(I,τ) called *-topology finer than τ is defined by cl∗(A) = A ∪ A∗(I,τ) [10] and β = {U \ I : U ∈ τ, I ∈ I} is a basis for τ∗ [10]. We simply write τ* for τ∗(I,τ). If I is an ideal on X, then (X,τ,I) is called an ideal space. If β = τ*, then we say I is τ-simple [10]. A sufficient condition for I to be simple is the following: for A ⊂ X, if for
every $a \in A$ there exists $U \in \tau(a)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. If $(X, \tau, \mathcal{I})$ satisfies this condition, then $\tau$ is said to be compatible with respect to $\mathcal{I}$ [10] or $\mathcal{I}$ is said to be $\tau$-local, denoted by $\mathcal{I} \sim \tau$. Given an ideal space $(X, \tau, \mathcal{I})$, we say $\mathcal{I}$ is $\tau$-boundary [10] or $\mathcal{I}$-codense if $\mathcal{I} \cap \tau = \emptyset$. An ideal $\mathcal{I}$ is said to be weakly $\tau$-local [11] if $A^* = \emptyset$ implies $A \in \mathcal{I}$. Some useful ideals in $X$ are: (i) $p(A)$, where $A \subseteq X$ and (ii) $\mathcal{I}_f$, the ideal of all finite subsets of $X$.

By a space $(X, \tau)$, we always mean a topological space $(X, \tau)$ with no separation properties assumed. If $A \subseteq X$, we denote the closure of $A$ and the interior of $A$ by $cl(A)$ and $int(A)$, respectively. A subset $A$ of $(X, \tau)$ is said to be semi-open [13] (resp., $\alpha$-open [17], regular open [21]) if $A \subseteq cl(int(A))$ (resp., $A \subseteq int(cl(A))$), $A = int(cl(A))$. The family of $\alpha$-sets of a space $(X, \tau)$ denoted by $\tau^\alpha$ forms a topology on $X$ finer than $\tau$ [17]. Abd El-Monsef et al. [8] introduced and studied the concept of $\beta$-open sets in topological spaces. A subset $A$ of $X$ is called $\beta$-open if $A \subseteq cl(int(cl(A)))$. The complement of a $\beta$-open set is said to be $\beta$-closed [8]. The family of all $\beta$-open (resp., $\beta$-closed) subsets of $X$ is denoted by $\beta O(X, \tau)$ (resp., $\beta C(X, \tau)$). The union of all $\beta$-open subsets of $X$ contained in $A$ is called $\beta$-interior of $A$ and is denoted by $\beta int(A)$ and the intersection of all $\beta$-closed subsets of $X$ containing $A$ is called the $\beta$-closure of $A$ and is denoted by $\beta cl(A)$. A set $A$ is called $\beta$-regular [19] if it is both $\beta$-open and $\beta$-closed. A space $(X, \tau)$ is said to be $\beta$-regular [14] if for each $\beta$-open set $U$ and each $x \in U$, there exists a $\beta$-open set $V$ such that $x \in V \subseteq cl(V) \subseteq U$. For any space, $\beta O(X, \tau^\alpha) = \beta O(X, \tau)$ [2]. A collection $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ of subsets of a space $(X, \tau)$ is said to be locally finite if for each $x \in X$, there exists an open set $U$ containing $x$ and $U$ intersects at most finitely many members of $\mathcal{W}$. A subset $A$ of $X$ is said to be $N$-closed relative to $X$ (briery, $N$-closed) [7] if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of $A$ by open subsets of $X$, there exists a finite subfamily $\Delta_0$ of $\Delta$ such that $A \subseteq \bigcup(int(cl(int(U_\alpha))) : \alpha \in \Delta_0)$.  

**Definition 1.** A space $(X, \tau)$ is said to be:

(i) extremally disconnected (briefly e.d.) [24] if the closure of every open set in $(X, \tau)$ is open;
(ii) submaximal [5] if each dense subset of $X$ is open in $X$.

**Lemma 1.** [3] The union of a finite family of locally finite collection of sets in a space is a locally finite family of sets.

**Theorem 1.** [16] Let $(X, \tau)$ be a space, $A \subseteq B \subseteq X$ and $B$ is $\beta$-open in $(X, \tau)$. Then $A$ is $\beta$-open in $(X, \tau)$ if and only if $A$ is $\beta$-open in the subspace $(B, \tau_B)$.

**Theorem 2.** [4] If $\{U_\alpha : \alpha \in \Delta\}$ is a locally finite family of subsets in a space $X$ and if $V_\alpha \subseteq U_\alpha$ for each $\alpha \in \Delta$, then the family $\{V_\alpha : \alpha \in \Delta\}$ is a locally finite in $X$.

**Lemma 2.** [9] If $f : (X, \tau) \to (Y, \sigma)$ is a continuous surjective function and $U = \{U_\alpha : \alpha \in \Delta\}$ is locally finite in $Y$, then $f^{-1}(U) = \{f^{-1}(U_\alpha) : \alpha \in \Delta\}$ is locally finite in $X$.

**Lemma 3.** [18] Let $f : (X, \tau) \to (Y, \sigma)$ be almost closed surjection with $N$-closed point inverse. If $\{U_\alpha : \alpha \in \Delta\}$ is a locally finite open cover of $X$, then $\{f(U_\alpha) : \alpha \in \Delta\}$ is a locally finite cover of $Y$. 

Definition 2. An ideal space \((X, \tau, \mathcal{I})\) is said to be \(\mathcal{I}\)-paracompact \([22]\) (resp., \(S_1\mathcal{I}\)-paracompact \([20]\)) if every open (resp., semi-open) cover \(U\) of \(X\) has a locally finite open refinement \(V\) (not necessarily a cover) such that \(X \setminus \bigcup \{V : V \in \mathcal{V}\} \in \mathcal{I}\).

Lemma 4. \([22]\) \(\mathcal{I}\) is weakly \(\tau\)-local implies \(\mathcal{I}\) is \(\tau\)-locally finite.

Lemma 5. \([22]\) If \(V\) is open and \(A\) is semi-preopen (or \(\beta\)-open) then \(V \cap A\) is semi-preopen (or \(\beta\)-open).

2. \(\beta_1\mathcal{I}\)-paracompact spaces

Recall that an ideal space \((X, \tau, \mathcal{I})\) is said to be \(\mathcal{I}\)-paracompact \([22]\) (resp., \(S_1\mathcal{I}\)-paracompact \([20]\)) if every open (resp., semi-open) cover \(U\) of \(X\) has a locally finite open refinement \(V\) (not necessarily a cover) such that \(X \setminus \bigcup \{V : V \in \mathcal{V}\} \in \mathcal{I}\).

Definition 2. An ideal space \((X, \tau, \mathcal{I})\) is said to be \(\beta_1\mathcal{I}\)-paracompact, or \(\beta_1\)-paracompact modulo an ideal \(\mathcal{I}\) if every \(\beta\)-open cover \(U\) of \(X\) has a locally finite open refinement \(V\) (not necessarily a cover) such that \(X \setminus \bigcup \{V : V \in \mathcal{V}\} \in \mathcal{I}\). A family \(\mathcal{V}\) of subsets of \(X\) such that \(X \setminus \bigcup \{V : V \in \mathcal{V}\} \in \mathcal{I}\) is called an \(\mathcal{I}\)-cover of \(X\).

It follows from the definitions that

\[ \beta_1\text{-paracompact} \Rightarrow \beta_1\mathcal{I}\text{-paracompact} \Rightarrow S_1\mathcal{I}\text{-paracompact} \Rightarrow \mathcal{I}\text{-paracompact} \]

The following examples show that the converses of the above implications need not be true in general.

Example 1. Let \(X = \mathbb{R}\) with the topology \(\tau = \{\emptyset, X, \{0\}\}\) and \(\mathcal{I} = \mathcal{I}_f\). Then \((X, \tau)\) is paracompact which implies that \((X, \tau, \mathcal{I})\) is \(\mathcal{I}\)-paracompact. On the other hand \((X, \tau, \mathcal{I})\) is not \(\beta_1\mathcal{I}\)-paracompact. For the \(\beta\)-open cover \(U = \{\{0, x\} : x \in X, x \neq 0\}\) , we can find a locally finite open refinement \(V = \{0\}\) of \(U\). But \(V\) does not \(\mathcal{I}\)-cover of \(X\). Therefore, \((X, \tau, \mathcal{I})\) is not \(\beta_1\mathcal{I}\)-paracompact.

Example 2. Let \(X = \{1, 2, 3, 4\}\) with the topology \(\tau = \{\emptyset, X, \{1, 2\}, \{3, 4\}\}\) and \(\mathcal{I} = \mathcal{I}_f\). Then \((X, \tau, \mathcal{I})\) is \(S_1\mathcal{I}\)-paracompact, since \(SO(X, \tau) = \tau\), but it is not \(\beta_1\mathcal{I}\)-paracompact since \(U = \{\{1\}, \{2\}, \{3\}, \{4\}\}\) is a \(\beta\)-open cover of \(X\) which admits no locally finite open refinement.

Example 3. Consider the ideal space \((X, \tau, \mathcal{I})\) where \(X = \{1, 2, 3\}\), \(\tau = \{\emptyset, X, \{1\}\}\) and \(\mathcal{I} = \{A \subseteq X : 1 \notin A\}\). Then \(\beta O(X, \tau) = \{\emptyset, X, \{1\}, \{1, 2\}, \{1, 3\}\}\). Therefore \((X, \tau, \mathcal{I})\) is \(\beta_1\mathcal{I}\)-paracompact space. On the other hand, \((X, \tau)\) is not \(\beta_1\)-paracompact since \(U = \{\{1, 2\}, \{1, 3\}\}\) is a \(\beta\)-open cover of \((X, \tau)\) which admits no locally finite open refinement.

Example 4. Consider the ideal space \((X, \tau, \mathcal{I})\) where \(X = \mathbb{R}\), the set of all real numbers, \(\tau = \{\emptyset, X, \{0\}\}\) and \(\mathcal{I} = \{A \subseteq X : 0 \notin A\}\). Then \((X, \tau, \mathcal{I})\) is \(\beta_1\mathcal{I}\)-paracompact space but \((X, \tau)\) is not \(\beta_1\)-paracompact, since the \(\beta\)-open cover \(U = \{\{0, x\} : x \in X, x \neq 0\}\) admits no locally finite open refinement.

Corollary 1. Let \((X, \tau)\) be a space with an ideal \(\mathcal{I} = \{\emptyset\}\). Then \((X, \tau)\) is \(\beta_1\)-paracompact if and only if \((X, \tau, \mathcal{I})\) is \(\beta_1\mathcal{I}\)-paracompact.
Corollary 2. For an e.d. submaximal ideal space \((X, \tau, I)\), the following conditions are equivalent:
(i) \((X, \tau, I)\) is \(\beta_I\)-paracompact;
(ii) \((X, \tau, I)\) is \(S_I\)-paracompact;
(iii) \((X, \tau, I)\) is \(I\)-paracompact.

Proof. This follows directly from the fact that if an ideal space \((X, \tau, I)\) is an e.d. submaximal space, then \(\tau = SO(X, \tau) = \beta O(X, \tau)\).

Proposition 1. If \((X, \tau, I)\) is \(\beta_I\)-paracompact, then \((X, \tau^a, I)\) is \(\beta_I\)-paracompact.

Proof. Suppose \((X, \tau, I)\) is \(\beta_I\)-paracompact. Let \(U = \{U_\alpha : \alpha \in \Delta\}\) be a \(\beta\)-open cover of \((X, \tau^a, I)\). Then \(U\) is a \(\beta\)-open cover of \((X, \tau, I)\). By hypothesis, there exist a locally finite open refinement \(V = \{V_\lambda : \lambda \in \Lambda\}\) of \(U\) such that \(X \cup \{V_\lambda : \lambda \in \Lambda\} \in I\).

Proof. Let \(U = \{U_\alpha : \alpha \in \Delta\}\) be a \(\beta\)-open cover of \(X\). Since \(\beta O(X, \tau) \subseteq \beta O(X, \tau^a)\).

Theorem 3. Let \((X, \tau, I)\) be an ideal space. If \(I\) is codense, \((X, \tau^a, I)\) is \(\beta_I\)-paracompact and \(I\) is \(\tau\)-simple, then every \(\beta\)-open cover of \((X, \tau, I)\) has \(\tau\)-locally finite \(\beta\)-open \(I\)-cover refinement.

Proof. Let \(U = \{U_\alpha : \alpha \in \Delta\}\) be a \(\beta\)-open cover of \(X\). Since \(\beta O(X, \tau) \subseteq \beta O(X, \tau^a)\).

Theorem 4. Let \((X, \tau, I)\) be an ideal space. If \(I\) is weakly \(\tau\)-local and \((X, \tau, I)\) is \(\beta_I\)-paracompact, then \((X, \tau^*, I)\) is \(\beta_I\)-paracompact.

Proof. Let \(U = \{U_\alpha : \alpha \in \Delta, U_\alpha \in \tau, I_\alpha \in I\}\) be a \(\beta\)-open cover of \((X, \tau^*, I)\). Then \(W = \{U_\alpha : \alpha \in \Delta\}\) is a \(\beta\)-open cover of \(X\) and so it has locally finite open refinement \(V = \{V_\lambda : \lambda \in \Lambda\}\) such that \(X \cup \{V_\lambda : \lambda \in \Lambda\} \in I\). Now the family \(\{V_\lambda \cap I_\alpha : \lambda \in \Lambda\} \subseteq I\) is locally finite. Since, \(I\) is weakly \(\tau\)-local, \(\cup_{\lambda \in \Lambda}(V_\lambda \cap I_\alpha) \in I\), by Lemma 4. Then \(X \cup \cup_{\lambda \in \Lambda}(V_\lambda \cap I_\alpha) \subseteq (X \cup \cup_{\lambda \in \Lambda} V_\lambda) \cup (\cup_{\lambda \in \Lambda}(V_\lambda \cap I_\alpha)) \in I\) which implies...
Since $V$ is locally finite, $V_1 = \{ V_\lambda \setminus I_\alpha : \lambda \in \Lambda \}$ is locally finite. Since $\tau^*$ is finer than $\tau$, $V_1$ is $\tau^*$-locally finite $\tau^*$-open which refines $\mathcal{U}$. Hence $(X, \tau^*, \mathcal{I})$ is $\beta_1 \mathcal{I}$-paracompact.

**Theorem 5.** Let $(X, \tau)$ be a $\beta$-regular space. If $(X, \tau, \mathcal{I})$ is $\beta_1 \mathcal{I}$-paracompact, then every $\beta$-open cover of $X$ has a locally finite $\beta$-closed $\mathcal{I}$-cover refinement.

**Proof.** Let $\mathcal{U}$ be a $\beta$-open cover of $X$. For each $x \in X$, let $U_x \in \mathcal{U}$ such that $x \in U_x$. Since $(X, \tau)$ is $\beta$-regular, there exists $V_x \subset \beta O(X, \tau)$ such that $x \in V_x \subset \beta cl(V_x) \subset U_x$. Then the family $V = \{ V_x : x \in X \}$ is a $\beta$-open cover refinement of $\mathcal{U}$. By hypothesis, there exist a locally finite open refinement $W = \{ W_\alpha : \alpha \in \Delta \}$ which refines $V$ such that $X \setminus \bigcup \{ W_\alpha : \alpha \in \Delta \} \in \mathcal{I}$. The family $\beta cl(W) = \{ \beta cl(W_\alpha) : \alpha \in \Delta \}$ is locally finite for each $\alpha \in \Delta$. Now $X \setminus \bigcup \{ \beta cl(W_\alpha) : \alpha \in \Delta \} \subset X \setminus \bigcup \{ W_\alpha : \alpha \in \Delta \}$ implies $X \setminus \bigcup \{ \beta cl(W_\alpha) : \alpha \in \Delta \} \in \mathcal{I}$. Hence $\beta cl(W)$ is $\mathcal{I}$-cover. Let $\beta cl(W_\alpha) \in \beta cl(W)$. Since $W$ refines $V$, there is some $V_x \in V$ such that $W_\alpha \subset V_x$ and so $\beta cl(W_\alpha) \subset \beta cl(V_x) \subset U_x$ implies that $\beta cl(W_\alpha) \subset U_x$. Hence $\beta cl(W)$ refines $\mathcal{U}$. Thus, $\beta cl(W) = \{ \beta cl(W_\alpha) : \alpha \in \Delta \}$ is a locally finite $\beta$-closed $\mathcal{I}$-cover refinement of $\mathcal{U}$.

If $\mathcal{I} = \{ \emptyset \}$ in Theorem 5, then we have the following corollary.

**Corollary 4.** [1, Theorem 2.12] Let $(X, \tau)$ be a $\beta$-regular space. If each $\beta$-open cover of the space $X$ has a locally finite refinement, then each $\beta$-open cover of $X$ has a locally finite $\beta$-closed $\mathcal{I}$-cover refinement.

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\beta$-continuous [8] (resp., $\beta$-irresolute [15]) if $f^{-1}(V) \in \beta O(X, \tau)$ for each open (resp., $\beta$-open) set $V$ in $(Y, \sigma)$.

**Theorem 6.** Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an open, $\beta$-irresolute and almost closed surjective function with $\mathcal{I}$-closed point inverse. If $(X, \tau, \mathcal{I})$ is $\beta_1 \mathcal{I}$-paracompact, then $(Y, \sigma, f(\mathcal{I}))$ is $\beta_1 f(\mathcal{I})$-paracompact.

**Proof.** Let $\mathcal{U} = \{ U_\alpha : \alpha \in \Delta \}$ be a $\beta$-open cover of $Y$. Since $f$ is $\beta$-irresolute, $\mathcal{U}_1 = \{ f^{-1}(U_\alpha) : \alpha \in \Delta \}$ is a $\beta$-open cover of $X$. By hypothesis, there exists a $\tau$-locally finite $\tau$-open refinement $\mathcal{V}_1 = \{ V_\lambda : \lambda \in \Lambda \}$ of $\mathcal{U}_1$ such that $X \setminus \bigcup \{ V_\lambda : \lambda \in \Lambda \} \in \mathcal{I}$. Then $f(X) \setminus \bigcup \{ f(V_\lambda) : \lambda \in \Lambda \} \in f(\mathcal{I})$. Now, $f(X) \setminus \bigcup \{ f(V_\lambda) : \lambda \in \Lambda \} \in f(X) \setminus \bigcup \{ f(V_\lambda) : \lambda \in \Lambda \}$ implies that $f(X) \setminus \bigcup \{ f(V_\lambda) : \lambda \in \Lambda \} \in f(\mathcal{I})$ which implies that $Y \setminus \bigcup \{ f(V_\lambda) : \lambda \in \Lambda \} \in f(\mathcal{I})$. Since $f$ is open and $\mathcal{V}_1$ is $\tau$-locally finite, $V = \{ f(V_\lambda) : \lambda \in \Lambda \}$ is $\sigma$-locally finite by Lemma 3. Let $f(V_\lambda) \in \mathcal{V}$. Then $V_\lambda \in \mathcal{V}_1$. Since $\mathcal{V}_1$ refines $\mathcal{U}_1$, there exists $f^{-1}(U_\alpha) \in \mathcal{U}_1$ such that $V_\lambda \subset f^{-1}(U_\alpha)$. Thus $f(V_\lambda) \subset f(f^{-1}(U_\alpha))$ implies that $f(V_\lambda) \subset U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Hence $\mathcal{V}$ refines $\mathcal{U}$. Therefore, $(Y, \sigma, f(\mathcal{I}))$ is $\beta_1 f(\mathcal{I})$-paracompact.

Since every compact set is $\mathcal{I}$-closed and every closed map is almost closed, we conclude the following corollary.

**Corollary 5.** Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be an open, $\beta$-irresolute, closed surjective function with $\mathcal{I}$-closed point inverse. If $(X, \tau, \mathcal{I})$ is $\beta_1 \mathcal{I}$-paracompact, then $(Y, \sigma, f(\mathcal{I}))$ is $\beta_1 f(\mathcal{I})$-paracompact.
Therefore, \( f : (X, \tau) \to (Y, \sigma) \) is said to be strongly \( \beta \)-continuous \([1]\) if \( f^{-1}(V) \in \tau \) for each \( V \in \beta O(Y, \sigma) \).

**Theorem 7.** Let \( f : (X, \tau, I) \to (Y, \sigma) \) be an open, strongly \( \beta \)-continuous, almost closed, surjective function with \( N \)-closed point inverse. If \((X, \tau, I)\) is \( I \)-paracompact, then \((Y, \sigma, f(I))\) is \( \beta_1 f(I) \)-paracompact.

**Proof.** Let \( U = \{ U_\alpha : \alpha \in \Delta \} \) be a \( \beta \)-open cover of \( Y \). Since \( f \) is strongly \( \beta \)-continuous, \( U_1 = \{ f^{-1}(U_\alpha) : \alpha \in \Delta \} \) is an open cover of \( X \). By hypothesis, there exists a \( \tau \)-locally finite \( \tau \)-open refinement \( \mathcal{V}_1 = \{ V_\lambda : \lambda \in \Lambda \} \) which refines \( U_1 \) such that \( X \setminus \bigcup \{ V_\lambda : \lambda \in \Lambda \} \in \mathcal{I} \). Then \( f(X) \setminus \bigcup \{ V_\lambda : \lambda \in \Lambda \} \subseteq f(X \setminus \bigcup \{ V_\lambda : \lambda \in \Lambda \}) \) implies that \( f(X) \setminus \bigcup \{ V_\lambda : \lambda \in \Lambda \} \subseteq f(I) \), which is an \( I \)-open cover of \( f(I) \). Since \( f \) is \( \tau \)-open and \( \mathcal{V}_1 \) is \( \tau \)-locally finite, \( \mathcal{V} = \{ f(V_\lambda) : \lambda \in \Lambda \} \) is \( \sigma \)-locally finite by Lemma 3. Let \( f(V_\lambda) \in \mathcal{V} \). Then \( V_\lambda \in \mathcal{V}_1 \). Since \( \mathcal{V}_1 \) refines \( U_1 \), there exists \( f^{-1}(U_\alpha) \in U_1 \) such that \( V_\lambda \subseteq f^{-1}(U_\alpha) \). Thus \( f(V_\lambda) \subseteq f(f^{-1}(U_\alpha)) \) implies that \( f(V_\lambda) \subseteq U_\alpha \) for some \( U_\alpha \in \mathcal{U} \). Hence \( \mathcal{V} \) refines \( \mathcal{U} \). Therefore, \((Y, \sigma, f(I))\) is \( \beta_1 f(I) \)-paracompact.

Recall that a function \( f : (X, \tau) \to (Y, \sigma) \) is said to be pre \( \beta \)-open\([15]\) if for every \( \beta \)-open set \( V \) of \((X, \tau)\), \( f(V) \) is \( \beta \)-open in \((Y, \sigma)\).

**Theorem 8.** Let \( f : (X, \tau) \to (Y, \sigma, J) \) be a pre \( \beta \)-open, continuous, bijective function. If \((Y, \sigma, J)\) is a \( \beta_1 J \)-paracompact, then \((X, \tau, f^{-1}(J))\) is \( \beta_1 f^{-1}(J) \)-paracompact.

**Proof.** Let \( U = \{ U_\alpha : \alpha \in \Delta \} \) be a \( \beta \)-open cover of \( X \). Since \( f \) is a pre \( \beta \)-open, \( f(U) = \{ f(U_\alpha) : \alpha \in \Delta \} \) is a \( \beta \)-open cover of \( Y \) and so it has a \( \sigma \)-locally finite \( \sigma \)-open refinement \( W = \{ W_\lambda : \lambda \in \Lambda \} \) of \( f(U) \) such that \( Y \setminus \bigcup \{ W_\lambda : \lambda \in \Lambda \} \in J \). Let \( Y \setminus \bigcup \{ W_\lambda : \lambda \in \Lambda \} = J \subseteq \bigcup \{ W_\lambda : \lambda \in \Lambda \} \subseteq f^{-1}(J) \). This implies \( Y = (\bigcup \{ W_\lambda : \lambda \in \Lambda \}) \cup J \). Then \( f^{-1}(Y) = (\bigcup \{ f^{-1}(W_\lambda) : \lambda \in \Lambda \}) \cup f^{-1}(J) \) which implies \( X = (\bigcup \{ f^{-1}(W_\lambda) : \lambda \in \Lambda \}) \cup f^{-1}(J) \). It follows that \( X \setminus \bigcup \{ f^{-1}(W_\lambda) : \lambda \in \Lambda \} \subseteq f^{-1}(J) \). Since \( f \) is continuous, by Lemma 2, \( V = \{ f^{-1}(W_\lambda) : \lambda \in \Lambda \} \) is is \( \tau \)-open, \( \tau \)-locally finite. Let \( f^{-1}(W_\lambda) \in \mathcal{V} \). Then \( f^{-1}(W_\lambda) \subseteq \mathcal{V} \). Since \( \mathcal{V} \) refines \( f(U) \), there exists \( f(U_\alpha) \in f(U) \) such that \( W_\lambda \subseteq f(U_\alpha) \). Thus \( f^{-1}(W_\lambda) \subseteq f^{-1}(f(U_\alpha)) \) implies that \( f^{-1}(W_\lambda) \subseteq U_\alpha \) for some \( U_\alpha \in \mathcal{U} \). Hence \( \mathcal{V} \) refines \( \mathcal{U} \). Therefore, \((X, \tau, f^{-1}(J))\) is \( \beta_1 f^{-1}(J) \)-paracompact.

If \( I = \emptyset \) in Theorem 8, then we have the following corollary.

**Corollary 6.** Let \( f : (X, \tau) \to (Y, \sigma) \) be a pre \( \beta \)-open, continuous, bijective function. If \((Y, \sigma)\) is \( \beta_1 \)-paracompact, then \((X, \tau)\) is \( \beta_1 \)-paracompact.

### 3. \( \beta_1 I \)-paracompact subsets

In this section, we define the subsets and subspaces of \( \beta_1 I \)-paracompact and study some of their properties.

**Definition 3.** A subset \( A \) of an ideal space \((X, \tau, I)\) is said to be \( \beta_1 I \)-paracompact relative to \( X \) \((\beta_1 I \)-paracompact subset\) if each cover \( \mathcal{U} \) of \( A \) by \( \beta \)-open sets of \( X \), there exists a
locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $A \setminus \bigcup \{V : V \in \mathcal{V}\} \in \mathcal{I}$. $A$ is said to be $\beta_1\mathcal{I}_A$-paracompact ($\beta_1\mathcal{I}_A$-paracompact subspace) if $(A, \tau_A, \mathcal{I}_A)$ is $\beta_1\mathcal{I}_A$-$\mathcal{I}_A$-paracompact as a subspace, where $\tau_A$ is the usual subspace topology and $\mathcal{I}_A = \{A \cap I : I \in \mathcal{I}\}$.

A subset $A$ of a space $(X, \tau)$ is said to be $\beta g$-closed [6] if $\beta cl(A) \subseteq U$ whenever $A \subset U$ and $U$ is any $\beta$-open set in $(X, \tau)$.

**Theorem 9.** Every $\beta g$-closed subset of a $\beta_1\mathcal{I}$-$\mathcal{I}$-paracompact is $\beta_1\mathcal{I}$-$\mathcal{I}$-paracompact.

**Proof.** Let $A$ be a $\beta g$-closed subset of $(X, \tau, \mathcal{I})$ and $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A$ by $\beta$-open sets of $X$. Since $A \subseteq \cup_{\alpha \in \Delta} U_\alpha$ and $A$ is a $\beta g$-closed, we have $\beta cl(A) \subseteq \cup_{\alpha \in \Delta} U_\alpha$. Then $\mathcal{U}_1 = \mathcal{U} \cup \{X \setminus \beta cl(A)\}$ is a $\beta$-open cover of $X$. By hypothesis, there exist a locally finite open family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}$ which refines $\mathcal{U}_1$ ( $V_\lambda \subset U_\alpha$ for some $\alpha \in \Delta$ and $V \subset X \setminus \beta cl(A)$) such that $X \setminus \bigcup \{V_\lambda : \lambda \in \Lambda\} \cup \{V\} \in \mathcal{I}$. Then $\beta cl(A) \setminus \{V_\lambda : \lambda \in \Lambda\} = \beta cl(A) \setminus \{V \cup \bigcup \{V_\lambda : \lambda \in \Lambda\}\} \subset X \setminus \bigcup \{V_\lambda : \lambda \in \Lambda\} \cup \{V\} \in \mathcal{I}$. Since $A \setminus \{V_\lambda : \lambda \in \Lambda\} \subset \beta cl(A) \setminus \{V_\lambda : \lambda \in \Lambda\}$, $A \setminus \{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$, by heredity property of $\mathcal{I}$. Since $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}$ is a locally finite, the family $\mathcal{V}_1 = \{V_\lambda : \lambda \in \Lambda\}$ is locally finite. Thus, the family $\mathcal{V}_1$ is locally finite open and $\mathcal{V}_1$ refines $\mathcal{U}$. Therefore, $A$ is $\beta\mathcal{I}$-paracompact.

**Theorem 10.** Every regular open subset of a $\beta_1\mathcal{I}$-$\mathcal{I}$-paracompact is $\beta_1\mathcal{I}_A$-$\mathcal{I}_A$-paracompact.

**Proof.** Let $A$ be a regular open in $(X, \tau)$ and $\mathcal{W} = \{W_\alpha : \alpha \in \Delta\}$ be a $\beta$-open cover of $A$ in $(A, \tau_A, \mathcal{I}_A)$. Since $A$ is open in $(X, \tau, \mathcal{I})$, $W_\alpha$ is a $\beta$-open set in $(X, \tau, \mathcal{I})$ for each $\alpha \in \Delta$, by Theorem 1. Then $\mathcal{U} = \{W_\alpha : \alpha \in \Delta\} \cup \{X \setminus A\}$ is a $\beta$-open cover of the $\beta_1\mathcal{I}$-$\mathcal{I}$-paracompact $(X, \tau, \mathcal{I})$ and so it has a locally finite open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $X \setminus \{V_\lambda : \lambda \in \Lambda\} = \mathcal{I} \in \mathcal{I}$. Then $A \subset A \cap \{\bigcup \{V_\lambda : \lambda \in \Lambda\}\} \cup \mathcal{I} = \{\bigcup \{V_\lambda \cap A : \lambda \in \Lambda\}\} \cup \mathcal{I}$ which implies that $A \setminus \{V_\lambda \cap A : \lambda \in \Lambda\} \in \mathcal{I}_A$. Let $x \in A$. Since $\mathcal{V}$ is locally finite, there exists $V \in \tau(x)$ such that $V_\lambda \cap V = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$. Then $(V_\lambda \cap V) \cap A = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, ..., \lambda_n$ and so $(V_\lambda \cap A) \cap (V \cap A) = \emptyset$. Therefore, $V_\lambda = \{V_\lambda \cap A : \lambda \in \Lambda\}$ is $\tau_A$-locally finite. Let $V_\lambda \cap A \in \mathcal{V}_A$. Since $\mathcal{V}$ refines $\mathcal{U}$, there is some $W_\alpha \in \mathcal{U}$ such that $V_\lambda \subset W_\alpha$ which implies $V_\lambda \cap A \subset W_\alpha$. Therefore, $V_\lambda$ refines $\mathcal{W}$. Hence $A$ is $\beta_1\mathcal{I}_A$-$\mathcal{I}_A$-paracompact.

**Corollary 7.** Every clopen subset of a $\beta_1\mathcal{I}$-$\mathcal{I}$-paracompact is $\beta_1\mathcal{I}_A$-$\mathcal{I}_A$-paracompact.

If $\mathcal{I} = \{\emptyset\}$ in Theorem 9 and Theorem 10, then we have the following corollary.

**Corollary 8.** [1, Theorem 3.5] Let $(X, \tau)$ be a $\beta_1$-$\mathcal{I}$-paracompact space. Then:

(i) If $A$ is regular open subset of $(X, \tau)$, then $(A, \tau_A)$ is $\beta_1\mathcal{I}_A$-paracompact;

(ii) If $A$ is a $\beta g$-closed subset of $(X, \tau)$, then $A$ is a $\beta_1\mathcal{I}$-paracompact.

**Theorem 11.** Let $A$ and $B$ be subsets of an ideal space $(X, \tau, \mathcal{I})$ such that $A \subset B \subset X$. Then the following conditions hold.

(i) If $A$ is $\beta_1\mathcal{I}$-$\mathcal{I}$-paracompact and $B$ is $\beta$-open in $(X, \tau)$, then $A$ is $\beta_1\mathcal{I}_B$-$\mathcal{I}_B$-paracompact.

(ii) If $A$ is $\beta_1\mathcal{I}_B$-$\mathcal{I}_B$-paracompact and $B$ is open in $(X, \tau)$, then $A$ is $\beta_1\mathcal{I}$-$\mathcal{I}$-paracompact.
Proof. (i) Let $U = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A$ such that $U_\alpha \in \beta \mathcal{O}(B, \tau_B)$. Since $B \in \beta \mathcal{O}(X, \tau)$, $U$ is a $\beta$-open cover of $A$ in $(X, \tau)$, by Theorem 1. By hypothesis, there exists a locally finite open family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ refines $U$ such that $A \setminus \bigcup\{V_\lambda : \lambda \in \Lambda\} = I \in \mathcal{I}$. Then $A \subseteq \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I$ and $A = A \cap B \subseteq \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I \cap B = \bigcup\{V_\lambda \cap B : \lambda \in \Lambda\} \cup (I \cap B)$ implies $A \setminus \bigcup\{V_\lambda \cap B : \lambda \in \Lambda\} \in \mathcal{I}_B$. Let $x \in B$. Since $\mathcal{V}$ is locally finite, there exists $U \in \tau(x)$ such that $U \cap V_\lambda = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n$. This implies $(U \cap V_\lambda) \cap B = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n$. Hence $V_\lambda \cap B \in \mathcal{V}_B$. Since $\mathcal{V}$ refines $U$ there exists $U_\alpha \in U$ such that $V_\lambda \subset U_\alpha$ and so $V_\lambda \cap B \subset U_\alpha$. Hence $V_\lambda$ refines $U$. Therefore $A$ is $\beta_1 \mathcal{I}_B$-paracompact.

(ii) Let $U = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A$ by $\beta$-open subsets of $X$. Then the family $U_1 = \{B \cap U_\alpha : \alpha \in \Delta\}$ is a $\beta$-open cover of $A$ in $(B, \tau_B, \mathcal{I}_B)$. By hypothesis, exists $\tau_B$-locally finite $\tau_B$-open family $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ refines $U_1$ such that $A \setminus \bigcup\{\lambda \in \Lambda\} \in \mathcal{I}_B$, where $\mathcal{I}_B = I \cap B$. It follows that $A \setminus \bigcup\{V_\lambda : \lambda \in \Lambda\} \in I \cap B$. Since $B$ is open in $X$. Then by Theorem 1, $\mathcal{V}$ is a locally finite open refinement of $U$. Therefore, $A$ is $\beta_1 I$-paracompact.

If $\mathcal{I} = \{\emptyset\}$ in Theorem 11, then we have the following corollary.

Corollary 9. [1, Theorem 3.6] Let $A$ and $B$ be subsets of an ideal space $(X, \tau)$ such that $A \subset B \subset X$. Then:

(i) If $A$ is $\beta$-paracompact and $B$ is $\beta$-open in $(X, \tau)$, then $A$ is $\beta_1 B$-paracompact.

(ii) If $A$ is $\beta_1 B$-paracompact and $B$ is open in $(X, \tau)$, then $A$ is $\beta_1 B$-paracompact.

Theorem 12. Let $A$ be a clopen subspace of an ideal space $(X, \tau, \mathcal{I})$. Then $A$ is $\beta_1 A \mathcal{I}_A$-paracompact if and only if it is $\beta_1 \mathcal{I}$-paracompact.

Proof. To prove necessity, let $U = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A$ by $\beta$-open subsets of the ideal subspace $(A, \tau_A, \mathcal{I}_A)$. Since $A$ is open, $U$ is a cover of $A$ by $\beta$-open subsets of $X$ and so it has a locally finite open refinement, say $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $A \setminus \bigcup\{V_\lambda : \lambda \in \Lambda\} = I \in \mathcal{I}$. Then $A \subseteq \bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I$. Now $A \subseteq A \cap \bigcup\{\lambda \in \Lambda\} \cup (A \cap I)$. It follows that $A \setminus \bigcup\{V_\lambda \cap A : \lambda \in \Lambda\} \in \mathcal{I}_A$. Let $x \in A$. Since $\mathcal{V}$ is $\tau_A$-locally finite, there exists $W \in \tau(x)$ such that $V_\lambda \cap W = \emptyset$ for $\lambda \neq \lambda_1, \lambda_2, \ldots, \lambda_n$. Thus the family $\mathcal{V}_A = \{V_\lambda \cap A : \lambda \in \Lambda\}$ is $\tau_A$-locally finite $\tau_A$-open refinement of $U$. Hence $A$ is $\beta_1 A \mathcal{I}_A$-paracompact.

To prove sufficiency, let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a cover of $A$ by $\beta$-open subsets of an ideal space $(X, \tau, \mathcal{I})$. Then $\mathcal{U}_1 = \{A \cap U_\alpha : \alpha \in \Delta\}$ is a $\beta$-open cover of the $\beta_1 A \mathcal{I}_A$-paracompact ideal subspace $(A, \tau_A, \mathcal{I}_A)$ and so it has a $\tau_A$-locally finite $\tau_A$-open refinement $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ such that $A \setminus \bigcup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}_A$. Then $A \setminus \bigcup\{V_\lambda : \lambda \in \Lambda\} \in \mathcal{I}$. But $A$ is an open set in $X$, so $V_\lambda$ is an open set for every $\lambda \in \Lambda$. Now $\tau_A \subseteq \tau$ and $X \setminus A$ is an open set in $X$ which intersects no member of $\mathcal{V}$. Therefore $\mathcal{V}$ is locally finite and refines $\mathcal{U}$. Thus $A$ is $\beta_1 B$-paracompact.

If $\mathcal{I} = \{\emptyset\}$ in Theorem 12, then we have the following corollary.

Corollary 10. [1, Theorem 3.8] Let $A$ be a clopen subspace of a space $(X, \tau)$. Then $A$ is a $\beta_1 A$-paracompact if and only if it is $\beta_1$-paracompact.
Theorem 13. If \((X, \tau, I)\) is a T_2 space and \(A\) is \(\beta_I\)-paracompact relative to \(X\), then \(A\) is closed in \((X, \tau^*)\).

Proof. Let \(x \in X \setminus A\). For each \(y \in A\), there exists \(U \in \tau\) such that \(y \in U_y\) and \(x \notin \text{cl}(U_y)\). Therefore, the family \(\mathcal{U} = \{U_y : y \in A\}\) is an open cover of \(A\) which is \(\beta_I\)-paracompact relative to \(X\). Since \(U\) is a \(\beta\)-open cover of \(A\) and so it has a locally finite open refinement \(\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}\) of \(\mathcal{U}\) such that \(A \setminus \bigcup\{V_\lambda : \lambda \in \Lambda\} \in I\). Now \(x \notin \text{cl}(V_\lambda)\) for each \(\lambda\) implies that \(x \notin \bigcup\{\text{cl}(V_\lambda) : \lambda \in \Lambda\}\). Since the locally finite family \(\mathcal{V}\) is closure-preserving, \(x \notin \bigcup\{\text{cl}(V_\lambda) : \lambda \in \Lambda\} = \text{cl}(\bigcup\{V_\lambda : \lambda \in \Lambda\})\). Let \(U = X \setminus \text{cl}(\bigcup\{V_\lambda : \lambda \in \Lambda\})\) and \(J = A \setminus \text{cl}(\bigcup\{V_\lambda : \lambda \in \Lambda\}) \subseteq A \setminus \bigcup\{V_\lambda : \lambda \in \Lambda\} = I_1\), where \(I_1 \in I\). Then \(U \setminus J \in \tau^*(x)\) and \((U \setminus J) \cap A = \emptyset\) which implies \(x \notin A^*\). Hence \(A^* \subseteq A\). This shows that \(A\) is closed in \((X, \tau^*)\).

If \(I = \{\emptyset\}\) in Theorem 13, then we conclude the following corollary.

Corollary 11. Let \(A\) be a \(\beta_1\)-paracompact relative subset of a T_2 space \((X, \tau)\). Then \(A\) is closed in \((X, \tau)\).

Theorem 14. In an ideal space \((X, \tau, I)\), if \(A\) and \(B\) are \(\beta_I\)-paracompact, then \(A \cup B\) is \(\beta_I\)-paracompact.

Proof. Let \(\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}\) be a cover of \(A \cup B\) by \(\beta\)-open sets in \(X\). Then \(\mathcal{U}\) is a \(\beta\)-open cover of \(A\) and \(B\). By hypothesis, there exist locally finite families \(\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}\) of \(A\) and \(\mathcal{W} = \{W_\gamma : \gamma \in \Lambda_0\}\) of \(B\) which refines \(\mathcal{U}\) such that \(A \setminus \bigcup\{V_\lambda : \lambda \in \Lambda\} \in I\) and \(B \setminus \bigcup\{W_\gamma : \gamma \in \Lambda_0\} \in I\). Then \(A \cup B \subseteq (\bigcup\{V_\lambda : \lambda \in \Lambda\} \cup I_1) \cup (\bigcup\{W_\gamma : \gamma \in \Lambda_0\} \cup I_2)\), where \(I_1, I_2 \in I\) which implies that \(A \cup B \subseteq (\bigcup\{V_\lambda \cup W_\gamma : \lambda \in \Lambda, \gamma \in \Lambda_0\}) \cup (I_1 \cup I_2)\). It follows that \((A \cup B) \setminus \bigcup\{V_\lambda \cup W_\gamma : \lambda \in \Lambda, \gamma \in \Lambda_0\} \in I\). Since the families \(\mathcal{V}\) and \(\mathcal{W}\) are locally finite the family \(\mathcal{V}' = \{V_\lambda \cup W_\gamma : \lambda \in \Lambda, \gamma \in \Lambda_0\}\) is locally finite, by Lemma 1 which refines \(\mathcal{U}\). Therefore, \(A \cup B\) is \(\beta_I\)-paracompact.

Theorem 15. In an ideal space \((X, \tau, I)\), if \(A\) is \(\beta_I\)-paracompact and \(B\) is a \(\beta\)-closed subset of \(X\), then \(A \cap B\) is \(\beta_I\)-paracompact.

Proof. Let \(\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}\) be a cover of \(A \cap B\) by \(\beta\)-open subsets of \(X\). Then \(\mathcal{U}_A = \mathcal{U} \cup \{X \setminus B\}\) is a cover of \(A\) by \(\beta\)-open sets in \(X\). By hypothesis, there exists a locally finite open refinement \(\mathcal{V}_A = \{V_\lambda : \lambda \in \Lambda\} \cup \{V\}\) of \(\mathcal{U}_A\), where \(V \subseteq U_\alpha\) and \(V \subseteq X \setminus B\) such that \(A \setminus [(\bigcup\{V_\lambda : \lambda \in \Lambda\}) \cup \{V\}] \in I\). Let \(A \setminus [(\bigcup\{V_\lambda : \lambda \in \Lambda\}) \cup \{V\}] = I_1\). Then \(I \cap B = A \setminus [(\bigcup\{V_\lambda : \lambda \in \Lambda\}) \cup \{V\}] \subseteq B = A \cap (X \setminus [(\bigcup\{V_\lambda : \lambda \in \Lambda\}) \cup \{V\}]) \cap B\) implies that \(I \cap B = A \cap [X \setminus (\bigcup\{V_\lambda : \lambda \in \Lambda\}) \cup \{V\}] \cap B\). It follows that \(I \cap B = (A \cap B) \setminus \{V_\lambda : \lambda \in \Lambda\} \in I\). Since \(V \subseteq V \cup V, \mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}\) is locally finite open by Theorem 2 which refines \(\mathcal{U}\). Hence \(A \cap B\) is \(\beta_I\)-paracompact.

Corollary 12. Let \(f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{J})\) be a pre \(\beta\)-open, continuous, bijective function. If \(A\) is \(\beta_1\mathcal{J}\)-paracompact relative to \(Y\), then \(f^{-1}(A)\) is \(\beta_1 f^{-1}(\mathcal{J})\)-paracompact relative to \(X\).
Acknowledgements

The author would like to thank the referees for their helpful suggestions.

References


