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A Version of Fundamental Theorem for the Itô-McShane Integral of an Operator-Valued Stochastic Process

Jeffer Dave A. Cagubcob[1], Mhelmar A. Labendia[1,*]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, Mindanao Sate University-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. In this paper, we formulate a descriptive definition or a version of fundamental theorem for the Itô-McShane integral of an operator-valued stochastic process with respect to a Hilbert space-valued Wiener process. For this reason, we introduce the concept of belated Mcshane differentiability and a version of absolute continuity of a Hilbert space-valued stochastic process.

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1. Introduction

The Henstock integral, which was studied independently by Henstock and Kurzweil in the 1950s and later known as the Henstock-Kurzweil integral, is one of the notable integrals that was introduced which in some sense is more general than the Lebesgue integral. To avoid an extensive study of measure theory, Henstock-Kurzweil integration had been deeply studied and investigated by numerous authors, see [2–4, 7–9]. The Henstock-Kurzweil integral is a Riemann-type definition of an integral which is more explicit and minimizes the technicalities in the classical approach of the Lebesgue integral. This approach to integration is known as the generalized Riemann approach or Henstock approach.

In the classical approach to stochastic integration, the Itô integral of a real-valued stochastic process, which is adapted to a filtration, is attained from a limit of Itô integrals of simple processes. To give a more explicit definition and reduce the technicalities in the classical way of defining the Itô integral in the real-valued case, Henstock approach to stochastic integration had already been studied in several papers, see [10, 11, 15–17].

*Corresponding author.
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Email addresses: `jdacagubcob@gmail.com` (J.D. Cagubcob), `mhelmar.labendia@g.msuiit.edu.ph` (M. Labendia)

In infinite dimensional spaces, the Itô integral of an operator-valued stochastic process, adapted to a normal filtration, is obtained by extending an isometry from the space of elementary processes to the space of continuous square-integrable martingales. In this case, the value of the integrand is an operator and the integrator is a Q-Wiener process, a Hilbert space-valued Wiener process which is dependent on a symmetric nonnegative definite trace-class operator Q.

In this paper, we formulate a version of *Fundamental Theorem* for the Itô-McShane integral, a Henstock approach integral, for the operator-valued stochastic process with respect to a Q-Wiener process.

2. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a *filtered probability space*, $\mathcal{B}(H)$ be the Borel σ-field of a separable Banach space H, and $\mathcal{L}(h)$ be the *probability distribution* or the *law* of a random variable $h : \Omega \to H$.

A *stochastic process* $f : [0,T] \times \Omega \to H$, or simply a *process* $\{f_t\}_{0 \le t \le T}$, is said to be *adapted* to a filtration $\{\mathcal{F}_t\}$ if f_t is $\mathcal{F}_t$-measurable for all $t \in [0,T]$. When no confusion arises, we may refer to a process adapted to $\{\mathcal{F}_t\}$ as simply an *adapted process.*

Let U and V be separable Hilbert spaces. Denote by $L(U,V)$ the space of all bounded linear operators from U to V, $L(U) := L(U,U)$, $Qu := Q(u)$ for $Q \in L(U,V)$, and $L^2(\Omega, V)$ the space of all square-integrable random variables from Ω to V. An operator $Q \in L(U)$ is said to be *self-adjoint* or *symmetric* if for all $u, u' \in U$, $\langle Qu, u' \rangle_U = \langle u, Qu' \rangle_U$ and is said to be *nonnegative definite* if for every $u \in U$, $\langle Qu, u \rangle_U \ge 0$. Using the Square-root Lemma [14, p.196], if $Q \in L(U)$ is nonnegative definite, then there exists a unique operator $Q^{\frac{1}{2}} \in L(U)$ such that $Q^{\frac{1}{2}}$ is nonnegative definite and $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$.

Let $\{e_j\}_{j=1}^{\infty}$, or simply $\{e_j\}$, be an orthonormal basis (abbrev. as ONB) in U. If $Q \in L(U)$ is nonnegative definite, then the trace of Q is defined by tr $Q = \sum_{j=1}^{\infty} \langle Qe_j, e_j \rangle_U$. It is shown in [14, p.206] that tr Q is well-defined and may be defined in terms of an arbitrary ONB. An operator $Q : U \to U$ is said to be *trace-class* if tr $[Q] :=$ tr $(QQ^*)^{\frac{1}{2}} < \infty$. Denote by $L_1(U)$ the space of all trace-class operators on U, which is known [14, p.209] to be a Banach space with norm $\|Q\|_1 =$ tr $[Q]$. If $Q \in L(U)$ is a symmetric nonnegative definite trace-class operator, then there exists an ONB $\{e_j\} \subset U$ and a sequence of nonnegative real numbers $\{\lambda_j\}$ such that $Qe_j = \lambda_j e_j$ for all $j \in \mathbb{N}$, $\{\lambda_j\} \in \ell^1$, and $\lambda_j \to 0$ as $j \to \infty$ [14, p.203]. We shall call the sequence of pairs $\{\lambda_j, e_j\}$ an *eigensequence defined by* Q.

Let $Q : U \to U$ be a symmetric nonnegative definite trace-class operator. Let $\{\lambda_j, e_j\}$ be an eigensequence defined by Q. Then the subspace $U_Q := Q^{\frac{1}{2}}U$ of U equipped with the inner product $\langle u, v \rangle_{U_Q} = \left\langle Q^{-1/2}u, Q^{-1/2}v \right\rangle_U$, where $Q^{1/2}$ is being restricted to $[\mathrm{K}erQ^{1/2}]^{\perp}$ is a separable Hilbert space with $\left\{\sqrt{\lambda_j} e_j\right\}$ as its ONB, see [13, p.90], [1, p.23].

Let $\{f_j\}$ be an ONB in U_Q. An operator $S \in L(U_Q, V)$ is said to be *Hilbert-Schmidt* if $\sum_{j=1}^{\infty} \|Sf_j\|_V^2 = \sum_{j=1}^{\infty} \langle Sf_j, Sf_j \rangle_V < \infty$. Denote by $L_2(U_Q, V)$ the space of all Hilbert-Schmidt operators from U_Q to V, which is known [12, p.112] to be a separable Hilbert space with norm $\|S\|_{L_2(U_Q,V)} = \sqrt{\sum_{j=1}^{\infty} \|Sf_j\|_V^2}$. The Hilbert-Schmidt operator $S \in L_2(U_Q, V)$

and the norm $\|S\|_{L_2(U_Q,V)}$ may be defined in terms of an arbitrary ONB, see [13, p.418], [12, p.111]. It is shown in [1, p.25] that $L(U,V)$ is properly contained in $L_2(U_Q,V)$. We also note that $L_2(U_Q,V)$ contains genuinely unbounded linear operators from U to V.

Let $Q : U \to U$ be a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ be an eigensequence defined by Q, and $\{B_j\}$ be a sequence of independent *Brownian motions* (abbrev. as *BM*) defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. The process

$$\tilde{W}_t := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) e_j \tag{1}$$

is called a *Q-Wiener process in U*. The series in (1) converges in $L^2(\Omega, U)$. For each $u \in U$, denote $\tilde{W}_t(u) := \sum_{j=1}^{\infty} \sqrt{\lambda_j} B_j(t) \langle e_j, u\rangle_U$, with the series converging in $L^2(\Omega, \mathbb{R})$. Since the operator Q is assumed to be symmetric nonnegative definite trace-class, there exists a U-valued process W such that

$$\tilde{W}_t(u)(\omega) = \langle W_t(\omega), u\rangle_U \quad \mathbb{P}\text{-almost surely (abbrev. as } \mathbb{P}\text{-a.s.).} \tag{2}$$

We call the process W a *U-valued Q-Wiener process.* This process is a multidimentional *BM*. It should be noted that if we assume that $\lambda_j > 0$ for all j, $\frac{W_t(e_j)}{\sqrt{\lambda_j}}$, $j = 1, 2, \ldots$, is a sequence of real-valued *BM* defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, see [13, p.87].

A filtration $\{\mathcal{F}_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *normal* if (i) $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$, and (ii) $\mathcal{F}_t = \mathcal{F}_{t+} := \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0,T]$. A Q-Wiener process W_t, $t \in [0,T]$ is called a *Q-Wiener process with respect to a filtration* $\{\mathcal{F}_t\}$ if (i) W_t is adapted to $\{\mathcal{F}_t\}$, $t \in [0,T]$ and (ii) $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $0 \le s \le t \le T$. It is shown in [12, p.16] that a U-valued Q-Wiener process $W(t)$, $t \in [0,T]$, is a Q-Wiener process with respect to a normal filtration. From now onwards, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ shall mean a probability space equipped with a normal filtration.

3. Itô-McShane Integral and Belated McShane Derivative

In this section, we introduce the Itô-McShane integral of a process $f : [0,T] \times \Omega \to L(U,V)$ with respect to a U-valued Q-Wiener process W and the belated McShane derivative of a Hilbert space-valued function.

Throughout, assume that U and V are separable Hilberts spaces, $Q : U \to U$ is a symmetric nonnegative definite trace-class operator, $\{\lambda_j, e_j\}$ is an eigensequence defined by Q, and W is a U-valued Q-Wiener process. A stochastic process $f : [0,T] \times \Omega \to L(U,V)$ means a process measurable as mappings from $[0,T] \times \Omega, \mathcal{B}([0,T]) \otimes \mathcal{F})$ to $(L_2(U_Q,V), \mathcal{B}(L_2(U_Q,V)))$. Also, the given closed interval $[0,T]$ is *nondegenerate*, i.e. $0 < T$ and can be replaced with any closed interval $[a,b]$. If no confusion arises, we may write $(D)\sum$ instead of $\sum_{i=1}^{n}$ for the given finite collection D.

Definition 1. Let δ be a positive function defined on $[0,T]$. A finite collection $D = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ of interval-point pairs is a

(iii) *δ-fine belated McShane division* of $[0,T]$ if $\{[u_i,v_i]\}_{i=1}^n$ is a collection of non-overlapping intervals on $[0,T]$ with $\bigcup_{i=1}^n [u_i,v_i] = [0,T]$ and each $[u_i,v_i]$ is δ-fine belated McShane, that is, $[u_i,v_i] \subset [\xi_i, \xi_i + \delta(\xi_i))$

(iv) *δ-fine belated McShane partial division* of $[0,T]$ if $\{[u_i,v_i]\}_{i=1}^n$ is a collection of non-overlapping intervals on $[0,T]$ and each $[u_i,v_i]$ is δ-fine belated McShane.

We note that each ξ_i in Definition 1 does not necessarily belong to $[u_i,v_i]$. The term *partial division* is used in Definition 1 since the finite collection of non-overlapping intervals of $[0,T]$ may not cover the entire interval $[0,T]$.

Definition 2. Given $\eta > 0$, a given δ-fine belated McShane partial division $D = \{([u,v],\xi)\}$ is said to be a *(δ,η)-fine belated McShane partial division* of $[0,T]$ if it fails to cover $[0,T]$ by at most length η, that is,

$$\left| T - (D)\sum (v-u) \right| \le \eta.$$

To define the Itô-McShane integral, we shall use the definition of belated partial division in Definition 1, employed by the authors in [17, p.499].

Definition 3. Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process. Then f is said to be *Itô-McShane integrable*, or $\mathcal{IM}$-integrable, on $[0,T]$ with respect to W if there exists $A \in L^2(\Omega, V)$ such that for every $\epsilon > 0$, there is a positive function δ on $[0,T]$ and a number $\eta > 0$ such that for any (δ,η)-fine belated McShane partial division $D = \{([u_i,v_i],\xi_i)\}_{i=1}^n$ of $[0,T]$, we have

$$\mathbb{E}\left[\|S(f,D,\delta,\eta) - A\|_V^2 \right] < \epsilon,$$

where

$$S(f,D,\delta,\eta) := (D)\sum f_\xi (W_v - W_u) := \sum_{i=1}^n f_{\xi_i}(W_{v_i} - W_{u_i}).$$

In this case, f is $\mathcal{IM}$-integrable to A on $[0,T]$ and A is called the $\mathcal{IM}$-integral of f which will be denoted by $(\mathcal{IM})\int_0^T f_t \, dW_t$ or $(\mathcal{IM})\int_0^T f \, dW$. We shall denote $(\mathcal{IM})\int_0^0 f \, dW$ by the zero random variable $\mathbf{0}$ from Ω to V and denote by $\Lambda_{\mathcal{IM}}$, the collection of all Itô-McShane integrable processes on $[0,T]$.

Refer to [6, Lemma 3.5 and Lemma 3.6] for the proofs of the following two lemmas. Denote by $\mathcal{J}$, the collection of all closed intervals $[u,v] \subset [0,T]$.

Lemma 1. *Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process and $\{([u_i,v_i],\xi_i)\}_{i=1}^n$ be a finite collection such that $\{[u_i,v_i]\}$ is a collection of non-overlapping intervals in $\mathcal{J}$, $\xi_1 < \xi_2 < \cdots < \xi_n$, and $\xi_i \le u_i$ for each $i = 1,2,\ldots,n$. Then*

$$\mathbb{E}\left[\sum_{i<j} \left\langle f_{\xi_i}(W_{v_i} - W_{u_i}), f_{\xi_j}(W_{v_j} - W_{u_j})\right\rangle_V\right] = 0.$$

Lemma 2. *Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process and $\{([u_i,v_i],\xi_i)\}_{i=1}^n$ be a finite collection such that $\{(u_i,v_i]\}$ is a collection of non-overlapping intervals in $\mathcal{J}$, $\xi_1 < \xi_2 < \cdots < \xi_n$, and $\xi_i \le u_i$ for each $i = 1,2,\ldots,n$. Then*

$$\begin{aligned}\mathbb{E}\left[\left\|\sum_{i=1}^n f_{\xi_i}(W_{v_i} - W_{u_i})\right\|_V^2\right] &= \sum_{i=1}^n \mathbb{E}\left[\|f_{\xi_i}(W_{v_i} - W_{u_i})\|_V^2\right] \\ &= \sum_{i=1}^n (v_i - u_i)\mathbb{E}\left[\|f_{\xi_i}\|_{L_2(U_Q,V)}^2\right].\end{aligned}$$

Refer to [6, Example 3.7] for the proof of the following example.

Example 1. *Let $g : \Omega \to L(U,V)$ be a random variable bounded in $L_2(U_Q,V)$, that is, there exists $M > 0$ such that $\|g(\omega)\|_{L_2(U_Q,V)} \le M$ for all $\omega \in \Omega$ and let $\hat{\mathbf{0}} : \Omega \to L(U,V)$ be a random variable such that for all $\omega \in \Omega$, $\hat{\mathbf{0}}(\omega)$ is the zero operator in $L(U,V)$. Let $s \in [0,T]$ be fixed. Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ such that for $t \in [0,T]$,*

$$f_t = \begin{cases} g & \text{if } t = s \\ \hat{\mathbf{0}} & \text{if } t \neq s. \end{cases}$$

Then f is $\mathcal{IM}$-integrable to the zero random variable $\mathbf{0} \in L^2(\Omega, V)$ on $[0,T]$.

In the following proofs, denote by *Leb** and *Leb*, the Lebesgue outer measure and Lebesgue measure, respectively.

Example 2. *Let $f : [0,T]\times\Omega \to L(U,V)$ be an adapted process such that $\mathbb{E}\left[\|f_t\|_{L_2(U_Q,V)}^2\right] = 0$ almost everywhere (abbrev. as a.e.) on $[0,T]$. Then f is $\mathcal{IM}$-integrable to $\mathbf{0}$ on $[0,T]$.*

Proof. Let $\epsilon > 0$ be given. Let $G = \{t \in [0,T] : \mathbb{E}\left[\|f_t\|_{L_2(U_Q,V)^2}\right] \neq 0\}$. Then, $Leb(G) = 0$ and so G is measurable. Let $\xi \in G$. For any $[u,v] \subset [\xi, T]$,

$$\mathbb{E}\left[\|f_\xi(W_v - W_u)\|_V^2\right] = (v-u)\mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right].$$

Let $A_m = \sum_{k=1}^{m} \langle f_\xi(W_v - W_u), g_k\rangle^2$, where $\{g_k\}$ is an ONB in V. Since $A_m \to G := \sum_{k=1}^{\infty} \langle f_\xi(W_v - W_u), g_k\rangle^2$ as $m \to \infty$ and $A_m \le A_{m+1}$, by the monotone convergence theorem for Lebesgue integral,

$$\begin{aligned}\lim_{m\to\infty} \mathbb{E}\left[\sum_{k=1}^{m} \langle f_\xi(W_v - W_u), g_k\rangle^2\right] &= \mathbb{E}\left[\sum_{k=1}^{\infty} \langle f_\xi(W_v - W_u), g_k\rangle^2\right] \\ &= \mathbb{E}\left[\|f_\xi(W_v - W_u)\|_V^2\right] < \infty.\end{aligned} \tag{3}$$

Thus, there exists $N \in \mathbb{N}$ such that $\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right] < N$. Now, since G is measurable, there exists an open set O containing G such that $Leb(O) < \frac{\epsilon}{2N}$. Thus, for all $\xi \in G$, there exists $\delta_1(\xi) > 0$ such that $[\xi, \xi + \delta_1(\xi)) \subset O$. Let $D' = \{([u,v],\xi)\}$ be a δ_1-fine belated McShane partial division such that each $\xi \in G$. Then, $(D')\sum(v-u) < \frac{\epsilon}{2N}$ and so

$$\begin{aligned}\mathbb{E}\left[\left\|(D')\sum f_\xi(W_v - W_u) - \mathbf{0}\right\|_V^2\right] &\le 2(D')\sum(v-u)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right] \\ &< 2N \cdot \frac{\epsilon}{2N} = \epsilon.\end{aligned} \tag{4}$$

Thus, for any δ-fine belated McShane partial division $D = \{([u,v],\xi)\}$ where $\delta(\cdot) > 0$ on $[0,T]$ and $\delta(\xi) \ge \delta_1(\xi)$ for $\xi \in G$, we have

$$\begin{aligned}&\mathbb{E}\left[\left\|(D)\sum f_\xi(W_v - W_u) - \mathbf{0}\right\|_V^2\right] \\ &\qquad \le 2(D_{\xi\in G})\sum(v-u)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right] \\ &\qquad\qquad + 2(D_{\xi\in[0,T]\setminus G})\sum(v-u)\mathbb{E}\left[\|f_\xi\|^2_{L_2(U_Q,V)}\right] \\ &\qquad < \epsilon.\end{aligned} \tag{5}$$

The above inequality also holds for (δ,η)-fine belated McShane partial division of $[0,T]$. Thus, f is $\mathcal{IM}$-integrable to $\mathbf{0}$ on $[0,T]$.

It is worth noting that the Itô-McShane integral possesses some of the standard properties of an integral namely, uniqueness of an integral, linearity, integrability on every subinterval of $[0,T]$, the Cauchy criterion, and the Saks-Henstock Lemma. The proofs of these results are standard in Henstock-Kurzweil integration, hence omitted.

(i) The $\mathcal{IM}$ integral is uniquely determined, in the sense that if A_1 and A_2 are two $\mathcal{IM}$ integrals of f, then $\|A_1 - A_2\|_{L^2(U,V)} = 0$.

(ii) Let $f, g \in \Lambda_{\mathcal{IM}}$ and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g \in \Lambda_{\mathcal{IM}}$ and

$$(\mathcal{IM}) \int_0^T (\alpha f + \beta g)\ dW = \alpha \cdot (\mathcal{IM}) \int_0^T f\ dW + \beta \cdot (\mathcal{IM}) \int_0^T g\ dW.$$

(iii) *Cauchy criterion.* A process f is $\mathcal{IM}$-integrable on $[0, T]$ if and only if for every $\epsilon > 0$, there exist a positive function δ on $[0, T]$ and a number $\eta > 0$ such that for any two (δ, η)-fine belated McShane partial divisions D_1 and D_2 of $[0, T]$, we have

$$\mathbb{E}\left[\|S(f, D_1, \delta, \eta) - S(f, D_2, \delta, \eta)\|_V^2\right] < \epsilon.$$

(iv) If f is $\mathcal{IM}$-integrable on $[0, T]$, then f is $\mathcal{IM}$-integrable on $[c, d] \subset [0, T]$.

(v) If f is $\mathcal{IM}$-integrable on $[0, c]$ and $[c, T]$ where $c \in (0, T)$, then f is $\mathcal{IM}$-integrable on $[0, T]$ and

$$(\mathcal{IM}) \int_0^T f\ dW = (\mathcal{IM}) \int_0^c f\ dW + (\mathcal{IM}) \int_c^T f\ dW.$$

(vi) *Sequential definition.* A process f is $\mathcal{IM}$-integrable on $[0, T]$ if and only if there exist $A \in L^2(\Omega, V)$, a decreasing sequence $\{\delta_n\}$ of positive functions defined on $[0, T]$, and a decreasing sequence of positive numbers η_n such that for any (δ_n, η_n)-fine belated McShane partial division D_n of $[0, T]$, we have

$$\mathbb{E}\left[\|S(f, D_n, \delta_n, \eta_n) - A\|_V^2\right] \to 0 \text{ as } n \to \infty.$$

In this case, $A := (\mathcal{IM}) \displaystyle\int_0^T f_t\ dW_t.$

(vii) *Saks-Henstock lemma (Weak version).* Let f be $\mathcal{IM}$-integrable on $[0, T]$ and $F[u, v] := (\mathcal{IM}) \displaystyle\int_u^v f\ dW$ for any $[u, v] \subset [0, T]$. Then for every $\epsilon > 0$, there exists a positive function δ on $[0, T]$ such that for any δ-fine belated McShane partial division $D = \{([u, v], \xi)\}$ of $[0, T]$, we have

$$\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_v - W_\xi) - F[u, v]\}\right\|_V^2\right] < \epsilon.$$

Next, we define the concept of $AC^2[0, T]$-property, a version of absolute continuity.

Definition 4. *A function $F : \mathcal{J} \times \Omega \to V$ is said to be belated McShane differentiable at $\xi \in [0, T)$ if there exists a random variable $f_\xi : \Omega \to L(U, V)$ such that for all $\epsilon > 0$, there exists a positive function δ on $[0, T]$ such that for all δ-fine belated McShane interval-point pair $([u, v], \xi)$ of $[0, T]$,*

$$\mathbb{E}\left[\|f_\xi(W_v - W_u) - F[u, v]\|_V^2\right] < \epsilon(v - u).$$

The random variable f_ξ is called the belated McShane derivative of F at the point $\xi \in [0, T)$ and is denoted by DF_ξ.

We note that we write $F[u,v]$ instead of $F([u,v])$.

Definition 5. A function $F : \mathcal{J} \times \Omega \to V$

(i) is said to be $AC^2[0,T]$ if for every $\epsilon > 0$, there exists $\eta > 0$ such that for any finite collection $D = \{[u,v]\}$ of non-overlapping intervals $[u,v] \in \mathcal{J}$ with $(D)\sum(v-u) \leq \eta$, we have $\mathbb{E}\left[\left\|(D)\sum F[u,v]\right\|_V^2\right] < \epsilon$;

(ii) has the *orthogonal increment property* if for all non-overlapping intervals $[a,b], [u,v] \subset [0,T]$, $\mathbb{E}\left[\langle F[a,b], F[u,v]\rangle\right] = 0$.

The proof of the following theorem is parallel to the proof in [5].

Theorem 1. [5] *Let f be $\mathcal{IM}$-integrable on $[0,T]$ and define*

$$F[u,v] := (\mathcal{IM})\int_u^v f_s dW_s \text{ for all } [u,v] \subset [0,T].$$

Then F is $AC^2[0,T]$ and has the orthogonal increment property.

Lemma 3. [5] *Let $f : [0,T] \times \Omega \to L(U,V)$ be an adapted process, $F : \mathcal{J} \times \Omega \to V$ with orthogonal increment property and $\{[u_i,v_i]\}_{i=1}^n$ be a finite collection of non-overlapping subintervals of $[0,T]$. Then*

$$\mathbb{E}\left[\left\|\sum_{i=1}^n \{f_{\xi_i}(W_{v_i} - W_{u_i}) - F[u_i,v_i]\}\right\|_V^2\right] = \sum_{i=1}^n \mathbb{E}\left[\|f_{\xi_i}(W_{v_i} - W_{u_i}) - F[u_i,v_i]\|_V^2\right].$$

Lemma 4. *Let $f \in \Lambda_{\mathcal{IM}}$. Then for every $\epsilon > 0$, there exist a positive function δ on $[0,T]$ and a positive number η such that*

$$\mathbb{E}\left[\left\|(D)\sum f_\xi(W_v - W_u)\right\|_V^2\right] < \epsilon$$

for any δ-fine belated McShane partial division $D = \{([u,v],\xi)\}$ of $[0,T]$ with

$$(D)\sum(v-u) \leq \eta.$$

Proof. Let $\epsilon > 0$ be given. Then there exist a positive function δ on $[0,T]$ and a number $\eta > 0$ such that for any (δ,η)-fone belated McShane partial division P of $[0,T]$, wehave

$$\mathbb{E}\left[\left\|S(f,P,\delta,\eta) - (\mathcal{IM})\int_0^T f_t dW_t\right\|_V^2\right] < \frac{\epsilon}{4}.$$

Let $D = \{([u, v], \xi)\}$ be a δ-fine belated McShane partial division of $[0, T]$ with $(D) \sum(v - u) \leq \eta$. Construct a (δ, η)-fine belated McShane partial division of $[0, T]$. By assumption,

$$\mathbb{E}\left[\left\|(D \cup D_1) \sum f_\xi(W_v - W_u) - (\mathcal{IM}) \int_0^T f_t dW_t\right\|_V^2\right] < \frac{\epsilon}{4}.$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left[\left\|(D) \sum f_\xi(W_v - W_u)\right\|_V^2\right] \\
&\quad \leq 2\mathbb{E}\left[\left\|(D \cup D_1) \sum f_\xi(W_v - W_u) - (\mathcal{IM}) \int_0^T f_t dW_t\right\|_V^2\right] \\
&\qquad + 2\mathbb{E}\left[\left\|(\mathcal{IM}) \int_0^T f_t dW_t - (D_1) \sum f_\xi(W_v - W_u)\right\|_V^2\right] \\
&\quad 2\left(\frac{\epsilon}{4}\right) + 2\left(\frac{\epsilon}{4}\right) = \epsilon.
\end{aligned} \tag{6}$$

This proves the lemma.

Theorem 2. *A process $f : [0, T] \times \Omega \to L(U, V)$ is $\mathcal{IM}$-integrable if and only if*

(i) there exists an $AC^2[0, T]$ function $F : \mathcal{J} \times \Omega \to V$ and

(ii) for every $\epsilon > 0$, there exist a positive function δ on $[0, T]$ such that whenever $D = \{([u, v], \xi)\}$ is a δ-fine belated McShane partial division of $[0, T]$, we have

$$\mathbb{E}\left[\left\|(D) \sum\{f_\xi(W_v - W_u) - F[u, v]\}\right\|_V^2\right] < \epsilon.$$

Proof. Suppose that $f \in \Lambda_{\mathcal{IM}}$. By the Saks-Henstock lemma for $\mathcal{IM}$ integral, (ii) holds. Next we show that F is $AC^2[0, T]$. Let $\epsilon > 0$ be given. By Lemma 4, there exist a positive function δ on $[0, T]$ and a number $\eta > 0$ such that

$$\mathbb{E}\left[\left\|(D) \sum f_\xi(W_v - W_u)\right\|_V^2\right] < \frac{\epsilon}{4}$$

for any δ-fine belated McShane partial division $D = \{([u, v], \xi)\}$ of $[0, T]$ with $(D \sum(v - u)) \leq \eta$. Let $\{[a_j, b_j]\}_{j=1}^m$ be a finite collection of disjoint subintervals $[a_j, b_j] \in \mathcal{J}$ with $\sum_{j=1}^m (b_j - a_j) \leq \eta$. Note that f is also $\mathcal{IM}$-integrable on$[a_j, b_j]$ for all j. This means that for all j, there exist a positive function δ_j on $[a_j, b_j]$ and a number $\eta_j > 0$ such that for any (δ_j, η_j)-fine belated McShane partial division D_j of $[a_j, b_j]$, we have

$$\mathbb{E}\left[\|S(f, D_j, \delta_j, \eta_j) - F[a_j, b_j]\|_V^2\right] < \frac{\epsilon}{4 \cdot 2^{2j}}.$$

We can choose $\{\delta_j\}_{j=1}^m$ and $\{\eta_j\}_{j=1}^m$ such that $\delta_j(\xi) \leq \delta(\xi)$ for all j and $\sum_{j=1}^m \eta_j \leq \eta$. Let $P = D_1 \cup D_2 \cup \cdots \cup D_m$, which is a δ-fine belated partial division of $[0,T]$ with

$$(P)\sum (v-u) \leq \sum_{j=1}^m (b_j - a_j) \leq \eta.$$

This implies that

$$\mathbb{E}\left[\left\|(P)\sum f_\xi (W_v - W_u)\right\|_V^2\right] < \frac{\epsilon}{4}.$$

Hence,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{j=1}^m F[a_j,b_j]\right\|_v^2\right] \\
&\quad \leq 2\mathbb{E}\left[\left\|\sum_{j=1}^m \{F[a_j,b_j] - S(f,D_j,\delta_j,\eta_j)\}\right\|_V^2\right] \\
&\qquad\quad + 2\mathbb{E}\left[\left\|\sum_{j=1}^m S(f,D_j,\delta_j,\eta_j)\right\|_V^2\right] \\
&\quad \leq 2\left(\sum_{j=1}^m \sqrt{\mathbb{E}\left[\|F[a_j,b_j] - S(f,D_j,\delta_j,\eta_j)\|_V^2\right]}\right)^2 \\
&\qquad\quad 2\mathbb{E}\left[\left\|(P)\sum f_\xi (W_v - W_u)\right\|_V^2\right] \\
&\quad < 2\left(\sum_{j=1}^\infty \frac{\sqrt{\epsilon}}{2\cdot 2^j}\right)^2 + 2\left(\frac{\epsilon}{4}\right) \leq \epsilon.
\end{aligned} \tag{7}$$

Thus, F is $AC^2[0,T]$.

Conversely, assume that (i) and (ii) hold. Let $\epsilon > 0$ be given. Since F is $AC^2[0,T]$, choose $\eta > 0$ such that whenever $\{[u_j,v_j]\}_{j=1}^m$ is a finite collection of subintervals $[u_j,v_j] \in \mathcal{J}$ with $\sum_{j=1}^m (v_j - u_j) \leq \eta$, we have

$$\mathbb{E}\left[\left\|\sum_{j=1}^m F[u_j,v_j]\right\|_V^2\right] < \frac{\epsilon}{4}.$$

Let $D = \{([u,v],\xi)\}$ be a (δ,η)-fine belated McShane partial division of $[0,T]$ and let $D^c = \{[u,v]\}$ be the collection of all subintervals $[u,v] \subset [0,T]$ which are not included in the set D. Since F is $AC^2[0,T]$,

$$\mathbb{E}\left[\left\|(D^c)\sum F[u,v]\right\|_v^2\right] < \frac{\epsilon}{4}.$$

Hence,

$$\begin{aligned} \mathbb{E}\left[\left\|(D)\sum f_\xi(W_v - W_u) - F[0,T]\right\|_V^2\right] \\ &\leq 2\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_v - W_u) - F[u,v]\}\right\|_V^2\right] \\ &\quad + 2\mathbb{E}\left[\left\|(D^c)\sum F[u,v]\right\|_V^2\right] \\ &< 2\left(\frac{\epsilon}{4}\right) + 2\left(\frac{\epsilon}{4}\right) = \epsilon. \end{aligned} \tag{8}$$

Thus, f is $\mathcal{IM}$-integrable to $F[0,T]$.

Lemma 5. *Let $f \in \Lambda_{\mathcal{IM}}$ and define $F : \mathcal{J} \times \Omega \to V$ by*

$$F[u,v] := (\mathcal{IM})\int_u^v f_t \; dW_t.$$

(i) F has the orthogonal increment property,

(ii) $\mathbb{E}\left[\langle f_c(W_b - W_a), F[u,v]\rangle_V\right] = 0$, where $c \leq a$.

Proof. We shall only prove (i) since (ii) follows the same arguments in (i). By the sequential definition of $\mathcal{IM}$ integral, there exists a decreasing sequence $\{\delta_n\}$ of positive functions defined on $[0,T]$ and a decreasing sequence $\{\eta_n\}$ of positive numbers such that for any (δ_n, η_n)-fine belated McShane partial division $D_n[a,b] = \{([u_i^{(n)}, v_i^{(n)}, \xi_i^{(n)}])\}_{i=1}^m$ and $D_n[u,v] = \{([u_j^{(n)}, v_j^{(n)}, \xi_j^{(n)}])\}_{j=1}^p$ of $[a,b]$ and $[u,v]$, respectively, we have

$$\mathbb{E}\left[\|S(f, D_n[a,b], \delta_n, \eta_n) - F[a,b]\|_V^2\right] \to 0 \text{ as } n \to \infty$$

and

$$\mathbb{E}\left[\|S(f, D_n[u,v], \delta_n, \eta_n) - F[u,v]\|_V^2\right] \to 0 \text{ as } n \to \infty.$$

By Lemma 2, for every $n \in \mathbb{N}$

$$\mathbb{E}\left[\sum_{i=1}^m \sum_{j=1}^p \left\langle f_{\xi_i^{(n)}}(W_{v_i^{(n)}} - W_{u_i^{(n)}}), f_{\xi_j^{(n)}}(W_{v_j^{(n)}} - W_{u_j^{(n)}})\right\rangle_V\right] = 0.$$

Since $S(f, D_n[a,b], \delta_n, \eta_m) \to F[a,b]$ and $S(f, D_n[u,v], \delta_n, \eta_m) \to F[u,v]$ in $L^2(\Omega, V)$ as $n \to \infty$, it follows that

$$\mathbb{E}\left[\langle S(f, D_n[a,b], \delta_n, \eta_m), S(f, D_n[u,v], \delta_n, \eta_n)\rangle_V\right] \to \mathbb{E}\left[\langle F[a,b], F[u,v]\rangle_V\right]$$

as $n \to \infty$. Thus, $\mathbb{E}\left[\langle F[a,b], F[u,v]\rangle_V\right] = 0$.

In view of Lemma 5, we have the following lemma.

Lemma 6. *Let* $f \in \Lambda_{\mathcal{IM}}$ *and define* $F : \mathcal{J} \times \Omega \to V$ *by*

$$F[u,v] := (\mathcal{IM}) \int_u^v f_t \; dW_t.$$

Let $\{([u_i, v_i], \xi_i)\}_{i=1}^n$ *be a finite collection such that* $\{[u_i, v_i]\}$ *is a collection of non-overlapping intervals in* $\mathcal{J}$, $\xi_1 < \xi_2 < \cdots < \xi_n$, *and* $\xi_i \le u_i$ *for each* $i = 1, 2, \ldots, n$. *Then*

(i) $$\mathbb{E}\left[\left\|\sum_{i=1}^n \{f_{\xi_i}(W_{v_i} - W_{u_i}) - F(u_i, v_i)\}\right\|_V^2\right] = \sum_{i=1}^n \mathbb{E}\left[\|f_{\xi_i}(W_{v_i} - W_{u_i}) - F(u_i, v_i)\|_V^2\right];$$

(ii) $$\mathbb{E}\left[\left\|\sum_{i=1}^n F(u_i, v_i)\right\|_V^2\right] = \sum_{i=1}^n \mathbb{E}\left[\|F(u_i, v_i)\|_V^2\right].$$

The immediate consequence of Lemma 5.(i) is the strong version of Saks-Henstock lemma.

Lemma 7. (Saks-Henstock Lemma (Strong Version)). *Let* $f \in \Lambda_{\mathcal{IM}}$ *and let* $F : \mathcal{J} \times \Omega \to V$ *be defined by* $F(u,v) := (\mathcal{IM}) \int_u^v f_t \; dW_t$. *Then for every* $\epsilon > 0$, *there exists a positive function* δ *on* $[0,T]$ *such that for any* δ*-fine belated McShane partial division* $D = \{([u,v], \xi)\}$ *of* $[0,T]$, *we have*

$$(D)\sum \mathbb{E}\left[\|f_\xi(W_v - W_u) - F(u,v)\|_V^2\right] < \epsilon.$$

4. Descriptive Definition of Itô-McShane Integral

In this section, we present a version of *Fundamental Theorem* for the Itô-McShane integral of an operator-valued stochastic process.

Theorem 3. *Let* $f \in \Lambda_{\mathcal{IM}}$ *and let*

$$F[u,v] := (\mathcal{IM}) \int_u^v f \; dW \text{ for all } [u,v] \subset [0,T].$$

Then $DF_t = f_t$ *a.e. on* $[0,T)$.

Proof. Let $A = \{t \in [0,T) : DF_t$ does not exist or $DF_t \neq f_t\}$. Let $\xi \in A$. Then there exists $\gamma(\xi) > 0$ such that for every positive function δ on $[0,T]$, there exists a δ-fine belated McShane interval-point pair $([u,v], \xi)$ of $[0,T]$ with

$$\mathbb{E}\left[\|f_\xi(W_v - W_u) - F[u,v]\|_V^2\right] \ge \gamma(\xi)(v-u). \tag{9}$$

Let $\epsilon > 0$. By the strong version of Saks-Henstock lemma (Lemma 7), there exists a positive function δ_1 on $[0, T]$ such that for any δ_1-fine belated McShane partial division $D = \{([u, v], \xi)\}$ of $[0, T]$, we have

$$(D) \sum \mathbb{E}\left[\| f_\xi (W_v - W_u) - F[u, v] \|_V^2 \right] < \epsilon. \tag{10}$$

Let $\xi_1, \xi_2, \ldots, \xi_n \in A$. By (9), each ξ_i corresponds a δ_1-fine belated interval $[u_i, v_i]$, for $i = 1, 2, \ldots, n$. Thus,

$$\sum_{i=1}^{n} \gamma(\xi_i)(v_i - u_i) \quad \leq \quad \sum_{i=1}^{n} \mathbb{E}\left[\| f_{\xi_i}(W_{v_i} - W_{u_i}) - F[u_i, v_i] \|_V^2 \right] < \epsilon.$$

For $m \in \mathbb{N}$, let $A_m = \left\{ t \in A : \gamma(t) \geq \dfrac{1}{m} \right\}$. Then $A = \displaystyle\bigcup_{m \in \mathbb{N}} A_m$. Hence, for all $i \in \{1, 2, \ldots, n\}$, $\xi_i \in A_{m_i}$ for some $m_i \in \mathbb{N}$. Let $m : \max\{m_i : i = 1, 2, \ldots, n\}$. Then

$$\sum_{i=1}^{n} \frac{1}{m}(v_i - u_i) \leq \sum_{i=1}^{n} \gamma(\xi_i)(v_i - u_i) < \epsilon$$

which implies that $\displaystyle\sum_{i=1}^{n}(v_i - u_i) < m\epsilon$. Let $\mathcal{V}$ be the family of interval-point pairs $\{([u, v], \xi)\}$ induced from all δ_1-fine belated McShane partial division of $[0, T]$ such that $\xi \in A_m$. Then $\mathcal{V}$ is a Vitali cover of A_m. Applying the Vitali Covering Lemma, there exists a finite collection $\{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ in $\mathcal{V}$ such that

$$Leb^*(A_m) < \sum_{i=1}^{n}(v_i - u_i) + \epsilon < (m + 1)\epsilon.$$

Since ϵ is arbitrary, $Leb(A_m) = 0$. Thus, $Leb(A) = 0$.

Theorem 4. *Let $f : [0, T] \times \Omega \to L(U, V)$ be an adapted process and let $F : \mathcal{J} \times \Omega \to V$ be $AC^2[0, T]$, has the orthogonal increment property, and $DF_t = f_t$ a.e. on $[0, T)$. Then $f \in \Lambda_{\mathcal{IM}}$ and*

$$F[u, v] := (\mathcal{IM}) \int_u^v f_s \, dW_s \text{ for all } [u, v] \subset [0, T].$$

Proof. Let $A = \{t \in [0, T) : DF_t \text{ does not exist or } DF_t \neq f_t\}$. Then $Leb(A) = 0$. Let $\xi \in A^c := [0, T] \backslash A$. Then for every $\epsilon > 0$, there exists a positive function δ_1 on $[0, T]$ such that for any δ_1-fine belated McShane interval-point pair $([u, v], \xi)$ of $[0, T]$, we have

$$\mathbb{E}\left[\| f_\xi (W_v - W_u) - F[u, v] \|_V^2 \right] < \frac{\epsilon}{4T}(v - u).$$

Let $D_1 = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ be a δ_1-fine belated McShane partial division on $[0, T]$ with $\xi_i \in A^c$. Then by Lemma 3,

$$
\begin{aligned}
\mathbb{E}&\left[\left\|\sum_{i=1}^{n}\{f_{\xi_i}(W_{v_i}-W_{u_i})-F[u_i,v_i]\}\right\|_V^2\right] \\
&= \sum_{i=1}^{n}\mathbb{E}\left[\|f_{\xi_i}(W_{v_i}-W_{u_i})-F[u_i,v_i]\|_V^2\right] \\
&< \frac{\epsilon}{4T}\sum_{i=1}^{n}(v_i-u_i) \\
&\leq \frac{\epsilon}{4}.
\end{aligned} \tag{11}
$$

If $A=\varnothing$, then we are done. Suppose that $A\neq\varnothing$. Let $\xi\in A$. Then for any $[u,v]\subset[\xi,T]$,

$$
\mathbb{E}\left[\|f_\xi(W_v-W_u)\|_V^2\right]=(v-u)\mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right].
$$

Let $G_m=\displaystyle\sum_{k=1}^{m}\langle f_\xi(W_v-W_u),g_k\rangle^2$, where $\{g_k\}$ is an ONB in V. Since $G_m\to G:=\displaystyle\sum_{k=1}^{\infty}\langle f_\xi(W_v-W_u),g_k\rangle^2$ as $m\to\infty$ and $G_m\leq G_{m+1}$, by the monotone convergence theorem for Lebesgue integral,

$$
\begin{aligned}
\lim_{m\to\infty}\mathbb{E}\left[\sum_{k=1}^{m}\langle f_\xi(W_v-W_u),g_k\rangle^2\right] &= \mathbb{E}\left[\sum_{k=1}^{\infty}\langle f_\xi(W_v-W_u),g_k\rangle^2\right] \\
&= \mathbb{E}\left[\|f_\xi(W_v-W_u)\|_V^2\right]<\infty.
\end{aligned}
$$

It follows that there exists $N\in\mathbb{N}$ such that $N-1\leq\mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right]<N$.

Since F is $AC^2[0,T]$, there exists $\eta>0$ with $\eta<\frac{\epsilon}{4N}$ such that for all finite collection $\{[u_j,v_j]\}_{j=1}^{p}$ of non-overlapping intervals of $[0,T]$ with $\sum_{j=1}^{p}(v_j-u_j)<\eta$, we have

$$
\mathbb{E}\left[\left\|\sum_{j=1}^{p}F[u_j,v_j]\right\|_V^2\right]<\frac{\epsilon}{4}.
$$

Since $Leb(A)=0$, there exists an open set O containing A such that $Leb(O)<\eta$. Hence, for all $\xi\in A\subseteq O$. Thus, there exists $\delta_2(\xi)>0$ such that $[\xi_i,\xi_i+\delta_2(\xi)]\subset O$. Let $D_2=\{([u,v],\xi)\}$ be a δ-fine belated McShane partial division such that $\xi\in A$. Then, $(D_2)\sum(v-u)\leq Leb(O)<\eta$. Then by Lemma 3,

$$\begin{aligned}
&\mathbb{E}\left[\left\|(D_2)\sum\{f_\xi(W_v-W_u)-F[u,v]\}\right\|_V^2\right] \\
&\qquad= (D_2)\sum\mathbb{E}\left[\|f_\xi(W_v-W_u)-F[u,v]\|_V^2\right] \\
&\qquad\leq 2(D_2)\sum\mathbb{E}\left[\|f_\xi(W_v-W_u)\|_V^2\right]+2(D_2)\sum\mathbb{E}\left[\|F[u,v]\|_V^2\right] \\
&\qquad= 2(D_2)\sum(v-u)\mathbb{E}\left[\|f_\xi\|_{L_2(U_Q,V)}^2\right]+2(D_2)\sum\mathbb{E}\left[\|F[u,v]\|_V^2\right] \\
&\qquad< 2N(D_2)\sum(v-u)+2(D_2)\sum\mathbb{E}\left[\|F[u,v]\|_V^2\right] \\
&\qquad\leq 2N\cdot\frac{\epsilon}{4N}+2\cdot\frac{\epsilon}{4}=\epsilon.
\end{aligned} \tag{12}$$

Let $D=\{([u,v],\xi)\}$ be a δ-fine belated McShane partial division of $[0,T]$. Then using (11) and (12), we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|(D)\sum\{f_\xi(W_v-W_u)-F[u,v]\}\right\|_V^2\right] \\
&\qquad\leq\ 2\mathbb{E}\left[\left\|\sum_{\xi\in A^c}\{f_\xi(W_v-W_u)-F[u,v]\}\right\|_V^2\right] \\
&\qquad\qquad+2\mathbb{E}\left[\left\|\sum_{\xi\in A}\{f_\xi(W_v-W_u)-F[u,v]\}\right\|_V^2\right] \\
&\qquad<\ 2\left(\frac{\epsilon}{4}\right)+2\left(\frac{\epsilon}{4}\right)=\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

By Theorem 2, $f\in\Lambda_{\mathcal{IM}}$ and

$$F[u,v]:=(\mathcal{IM})\int_u^v f_s\,dW_s \text{ for all } [u,v]\subset[0,T].$$

Combining Theorem 1, Theorem 3, and Theorem 4, we get the following result, which is referred to as the *Fundamental Theorem* or the descriptive definition of the Itô-McShane integral for the Hilbert-Schmidt-valued stochastic process.

Theorem 5. *Let $f:[0,T]\times\Omega\to L(U,V)$ be an adapted process. Then $f\in\Lambda_{\mathcal{IM}}$ if and only if there exists an $AC^2[0,T]$ function $F:\mathcal{J}\times\Omega\to V$ that satisfies the orthogonal increment property and $DF_t=f_t$ a.e. on $[0,T)$.*

5. Conclusion and Recommendation

In this paper, we formulate an equivalent definition of the Itô-McShane integral of a operator-valued stochastic process with respect to a Hilbert space-valued Q-Wiener process using the concept of belated McShane derivative and $AC^2[0,T]$-property, a version of absolute continuity. A worthwhile direction for further investigation is to use Henstock-Kurzweil approach to define the stochastic integral with respect to a cylindrical Wiener process.

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