Some estimates below the modulus of integrals of some polynomials in the complex plane

Todor Stoyanov Stoyanov

1 University of Economics, Department of Mathematics bul. Knyaz Boris I 77, Varna 9002, Bulgaria

Abstract. In this paper, we make some estimates below the modulus of some integrals in the complex plane. Our aim is to prove the Conjecture 1, which we could see in [2–4]. The proof of the conjecture appears the Corollary. 

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1. Introduction

In papers [2–4], we consider the Conjecture 1: If \(a_k \geq 0, a_k \in \mathbb{R}\), Then we assert

\[
\left| \int_0^{e^{i\varphi}} \prod_{k=1}^{n} (x + a_k) \, dx \right| \geq \frac{1}{n+1},
\]

for arbitrary natural \(n\), \(\varphi \in [0, \pi/2]\). There exists a connection between this conjecture and Conjecture 2: If \(\phi_k \in [\pi/2, \pi]\), then

\[
\left| \int_{-1}^{0} (x + 1) \prod_{k=1}^{n} \left( x - e^{i\phi_k} \right) \, dx \right| \geq \frac{1}{n+2}.
\]

Both conjectures are very important for the proofs of some famous conjectures, like Sendov’s and Obreshkoff’s ones. A possible connection between both conjectures appears [5]. Here we shall extend this problem (Conjecture 1): what kind of set \(L\) satisfies this assertion, i.e. if \(a_k\) belongs to the set \(L\), then the upper inequality is true. The results related with the Conjecture 1, we observe in Theorem 1, Theorem 2. In Theorem 4 we generalize and prove the extended conjecture. We can see the results of Theorem 1 in

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Email address: todstoyanov@yahoo.com (T. S. Stoyanov)
[2, 4]. Such one of Theorem 2 could be seen in [3]. Many authors use some modulus of some integrals in the complex plane for various estimates in their works. For example we can see how Bojanov and Rahman in [1] use this method. These estimates are explored for the localization of the zeros of some polynomials. The results are useful in the (open) problems of [6–9].

2. Related Results

**Theorem 1.** Let $k = 1, 2, \ldots, n, n \in \mathbb{N}, a_k \in [0, 1], \varphi \in \left[0, \frac{\pi}{2}\right]$. Then the function
\[
\left| e^{i \varphi} \int_{-1}^{n} x \prod_{k=1}^{n} (x + a_k) \, dx \right| \geq \frac{1}{n + 2}
\]
for $n = 1, 2, 3$.

**Theorem 2.** let $k \in \mathbb{N}, a \in \mathbb{R}, a \in [0, 1]$. Then the function
\[
\left| \int_{0}^{i} x (x + a)^k \, dx \right| \geq \frac{1}{n + 2}.
\]

3. Preliminaries

We note:
\[
D(a, r) = \{ z \in \mathbb{C} : |z - a| < r \} \text{ is the open disk with center } a \text{ and radius } r.
\]
\[
\overline{D}(0, r) = \{ z \in \mathbb{C} : |z - a| \leq r \} \text{ is the closed disk with center } a \text{ and radius } r.
\]
\[
A = \{ z \in \mathbb{C}, \text{Re} z \leq 0 \} \text{ is the left semiplane.}
\]

4. Main Results

**Theorem 3.** We consider a polynomial $r(z) = z^{n-1} + r_{n-1}z^{n-1} + \ldots + r_1z + r_0$. where $r_k \in \mathbb{R}, n \geq 1, n \in \mathbb{N}, k = 0, n - 1$. The zeros $z_k$ of $r(z)$ satisfy the condition $\text{Re} z_k \leq 0$. If $a \geq 0, \text{ then } I = n \int_{0}^{a} r(z) \, dz \geq a^n$.

**Proof.** Let $r(z) = (z + a_1)(z + a_2) \ldots (z + a_1)(z - b_1)(z - b_1) \ldots (z - b_m)(z - b_m)$, where $1 + 2s = n - 1, a_k \geq 0, b_m \in \mathbb{C}, k = 1, l, m = 1, s, a_k \in \mathbb{R}, k, m \in \mathbb{N},$ and $b_m = \rho_m e^{i \varphi_m}, \rho_m \geq 0, \varphi_m \in \left[\frac{\pi}{2}, \pi\right], (z - b_m)(z - b_m) = z^2 - 2\rho_m \cos \varphi_m z + \rho_m^2 \geq z^2.$ Then
\[
n \int_{0}^{a} r(z) \, dz = n \int_{0}^{a} (z + a_1)(z + a_2) \ldots (z + a_1)(z - b_1)(z - b_1) \ldots (z - b_m)(z - b_m) \, dz \geq
\]
\[
n \int_{0}^{a} z^{n-1} \, dz = a^n.
\]
**Theorem 4.** We consider a polynomial $r(z) = z^n + r_{n-1}z^{n-2} + \ldots + r_1z + r_0$, where $r_k \in \mathbb{R}, n \geq 1, n \in \mathbb{N}, k = 0, n-1$. The zeros $z_k$ of $r(z)$ satisfy the condition $z_k \in A \setminus D(z_0, a) \setminus D(\overline{z_0}, a), z_0 = ae^{i\theta_0}$, where $\alpha \geq 0, \theta_0 \in [0, \frac{\pi}{2}]$. Then

$$I = \left| \int_0^{z_0} r(z) \, dz \right| \geq a^n.$$

**Proof.** Let us put $v(\theta) = ae^{i\theta}, \theta \in [0, \theta_0], I + 2s = n - 1,$

$$r(z) = \prod_{p=1}^{l} (z + a_p) \prod_{p=1}^{s} (z - b_p)(z - \overline{b_p}),$$

$l, s \in \mathbb{N}$ (one of the factors could be not existing, i.e., $l = 0$ or $s = 0$).

We put $f(\theta) = n \int_0^{v(\theta)} r(z) \, dz, g(\theta) = f(\theta), \overline{f}(\theta).$

Let us calculate

$$\frac{dq}{d\theta} = n \left[ r(v(\theta)) \frac{dv}{d\theta} \overline{f}(\theta) + r(v(\theta)) \frac{df}{d\theta} f(\theta) \right],$$

and if we put

$$U_0 = v(\theta) = ae^{i\theta}, U_p = v(\theta) + a_p, p = 1, l, U_{l+2p+1} = v(\theta) - b_p,$$

$$U_{l+2p+2} = v(\theta) - \overline{b_p}, p = 0, s - 1.$$ Knowing

$$\frac{df}{d\theta} = i n \left[ \overline{f}(\theta) \prod_{p=0}^{n-1} U_p - f(\theta) \prod_{p=0}^{n-1} U_p \right],$$

we have

$$\frac{dg}{d\theta} = i n \left[ 2 \frac{df}{d\theta} \prod_{p=0}^{n-1} U_p + \frac{d\prod_{p=0}^{n-1} U_p}{d\theta} \overline{f}(\theta) - i \frac{d\prod_{p=0}^{n-1} U_p}{d\theta} f(\theta) \right],$$

$$\frac{d^2g}{d\theta^2} = n \left[ 2n \prod_{p=0}^{n-1} |U_p|^2 - \left( U_0 \sum_{p=0, j \neq p}^{n-1} U_j \right) \overline{f}(\theta) - \left( U_0 \sum_{p=0, j \neq p}^{n-1} U_j \right) f(\theta) \right],$$

$$\frac{d^2g}{d\theta^2} = 2n \left[ n \prod_{p=0}^{n-1} |U_p|^2 - \text{Re} \left( U_0 \sum_{p=0, j \neq p}^{n-1} U_j \right) \overline{f}(\theta) \right].$$
and consequently

$$\frac{d^2 g}{d\theta^2} \geq 2n \prod_{p=0}^{n-1} |U_p| \left[ \sum_{p=0}^{n-1} \frac{\Pi_{j=p+1}^{n-1} |U_j|}{P_{p=0}^{n-1} |U_p|} \right].$$

If we note

$$B = \{ A \setminus D(z, a) \setminus D(\bar{z}, a) \}, B_0 = \{ A \setminus D(z_0, a) \setminus D(\bar{z}_0, a) \}$$

and since

$$\theta \in [0, \theta_0] \implies B_0 \subset B, \text{ i.e., } |U_p(\theta)| \geq a, p = 1, n - 1.$$ 

If we assume

$$|f(\theta)| = |\mathcal{F}(\theta)| \leq a^n,$$

then

$$\frac{d^2 g}{d\theta^2} \geq 2n \prod_{p=0}^{n-1} |U_p| \left[ naa^{n-1} - \left( 1 + \frac{U_0}{U_1} + ... + \frac{U_0}{U_{n-1}} \right) . a^n \right]$$

$$\geq 2na.\Pi_{p=0}^{n-1} |U_p| \left[ na^{n-1} - \left( 1 + \frac{a(n-1)}{a} \right) . a^{n-1} \right] = 0.$$ 

Then

$$\frac{d^2 g}{d\theta^2} \geq 0.$$ 

Hence

$$\frac{dg}{d\theta}(\theta) \geq \frac{dg}{d\theta}(0) = 0.$$ 

Consequently $g(\theta_0) > g(0), \text{ i.e., } |f(\theta_0)| > a^n$, according to the proof of Theorem 3. Therefore $a^n < |f(\theta_0)| \leq a^n$, which is impossible. The contradiction proves the Theorem 4.

Corollary. If in the condition of Theorem 4, we put $a=1, \text{ and } s=0, \text{ i.e., all the zeros of } r(z) \text{ are real and negative, then we get that the Conjecture 1 is true.}$

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